# LINEAR RECURSION RELATIONS — LESSON SIX
# COMBINING LINEAR RECURSION RELATIONS

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Suppose we have two sequences $P_i(1, 5, 25, 125, 625, 3125, \cdots)$ with a recursion relation:

(1) $$P_{n+1} = 5P_n ,$$

and $Q_i(3, 10, 13, 23, 36, 59, \cdots)$, A a Fibonacci sequence with recursion relation:

(2) $$Q_{n+1} = Q_n + Q_{n-1} .$$

Let

(3) $$T_n = P_n + Q_n .$$

What is the recursion relation of $T_n$ and how can it be conveniently obtained from the recursion relations of $P_n$ and $Q_n$?

Proceeding in a straightforward manner, we could first eliminate $P_n$ as follows:

$$T_{n+1} = P_{n+1} + Q_{n+1}$$

$$5T_n = 5P_n + 5Q_n .$$

Subtracting and using relation (1),

$$T_{n+1} - 5T_n = Q_{n+1} - 5Q_n .$$

We can proceed likewise for Q. Thus

$$T_{n+1} - 5T_n = Q_{n+1} - 5Q_n$$
$$T_n - 5T_{n-1} = Q_n - 5Q_{n-1}$$
$$T_{n-1} - 5T_{n-2} = Q_{n-1} - 5Q_{n-2} \quad .$$

Now subtract the sum of the last two equations from the first and use relation (2). The result is:

$$T_{n+1} - 6T_n + 4T_{n-1} + 5T_{n-2} = 0 \quad ,$$

a recursion relation involving only $T_i$.

A much simpler approach is by means of an operator E, such that

(3) $$(E)T_n = T_{n+1}.$$

The effect of E is to increase the subscript by 1. A relation

$$P_{n+1} - 5P_n = 0 \quad ,$$

can be written

$$(E - 5)P_n = 0,$$

and a relation

$$Q_{n+1} - Q_n - Q_{n-1} = 0 \quad ,$$

can be written

$$(E^2 - E - 1)Q_{n-1} = 0 \quad .$$

It is not difficult to convince oneself that these operators obey the usual algebraic laws. As a result, if

$$T_n = P_n + Q_n \quad ,$$

$(E - 5)(E^2 - E - 1)T_n = (E - 5)(E^2 - E - 1)P_n + (E - 5)(E^2 - E - 1)Q_n.$ But $(E - 5)P_n = 0$ and $(E^2 - E - 1)Q_n = 0$, so that

$$(E - 5)(E^2 - E - 1)T_n = 0$$

or

$$(E^3 - 6E^2 + 4E + 5)T_n = 0 \;,$$

which is ·valent to the recursion relation

$$T_{n+3} = 6T_{n+2} - 4T_{n+1} - 5T_n \;.$$

In general, if we have linear operators such that:

$$f(E)P_n = 0 \qquad \text{and} \qquad g(E)Q_n = 0 \qquad \text{and} \qquad T_n = AP_n + BQ_n,$$

where A and B are constants, then

$$f(E)g(E)T_n = Af(E)g(E)P_n + Bf(E)g(E)Q_n = 0 \;,$$

since $f(E)P_n = 0$, and $g(E)Q_n = 0$. Thus when $T_n$ is the sum of terms of two sequences with different recursion relations, the recursion relation for $T_n$ is found by multiplying $T_n$ by the two recursion operators for the two sequences.

<u>Example.</u> What is the recursion relation for $T_n = 2 \times 5^n + n^2 - n + 4$? The recursion relation for $2 \times 5^n$ is $(E - 5)P_n = 0$, and that for $n^2 - 4 + 4$ is $(E^3 - 3E^2 + 3E - 1)Q_n = 0$. Thus the recursion relation for the given sequence is

$$(E - 5)(E^3 - 3E^2 + 3E - 1)T_n = 0,$$

which is equivalent to:

$$T_{n+4} = 8T_{n+3} - 18T_{n+2} + 16T_{n+1} - 5T_n.$$

Example. Find the recursion relation corresponding to $T_n$ if

$$P_{n+1} = P_n + P_{n-1} + P_{n-2} \quad \text{and} \quad Q_n = 3n^2 - 4n + 5 \quad \text{and} \quad T_n = P_n + Q_n .$$

The operator expressions for these recursion relations are:

$$(E^3 - E^2 - E - 1)P_{n-2} = 0 \quad \text{and} \quad (E^3 - 3E^2 + 3E - 1)Q_{n-2} = 0 .$$

Thus the recursion relation for $T_n$ is:

$$(E^3 - E^2 - E - 1)(E^3 - 3E^2 + 3E - 1)T_n = 0 ,$$

which is equivalent to:

$$T_{n+6} = 4T_{n+5} - 5T_{n+4} + 2T_{n+3} - T_{n+2} + 2T_{n+1} - T_n .$$

It may be noted that two apprently different recursion relations may conceal the fact that they embody partly identical recursion relations. For example, if

$$P_n = 4P_{n-1} - 3P_{n-2} - 2P_{n-3} + P_{n-4}$$
$$Q_n = 3Q_{n-1} - 2Q_{n-2} - Q_{n-3} + Q_{n-4} ,$$

and we proceed directly to find the recursion operator and corresponding recursion relation for $T_n = P_n + Q_n$, we arrive at a recursion relation of order eight. However, in factored form, we have:

$$(E^2 - E - 1)(E^2 - 3E + 1)P_{n-4} = 0 ,$$

and

$$(E^2 - E - 1)(E^2 - 2E + 1)Q_{n-4} = 0 .$$

The recursion relation for $T_n$ in simpler form would thus be:

$$(E^2 - E - 1)(E^2 - 3E + 1)(E^2 - 2E + 1)T_n = 0,$$

which is only of order six.

If the terms of the two sequences are given explicitly, a slightly different but equivalent procedure using the auxiliary equation is possible. Thus if

$$P_n = 5n + 2 + 2 \times 3^n + F_n$$
$$Q_n = n^2 - 3n + 5 - 6 \times 2^n + L_n ,$$

the roots of the auxiliary equation for $P_n$ are 1, 1, 3, r, and s, while those of the auxiliary equation for $Q_n$ are 1, 1, 1, 2, r, s. Hence the roots for the auxiliary equation of $T_n$ would be 1, 1, 1, 2, 3, r, s, where r and s are the roots of the equation $x^2 - x - 1 = 0$. Thus the auxiliary equation for $T_n$ would be:

$$(x - 1)^3(x - 2)(x^2 - x - 1) = 0$$

which leads equivalently to the recursion relation

$$T_{n+7} = 9T_{n+6} - 31T_{n+5} + 50T_{n+4} - 33T_{n+3} - 5T_{n+2} + 17T_{n+1} - 6T_n .$$

## PROBLEMS

1. If $P_n$ is the geometric progression 3, 15, 75, 375, 1875, $\cdots$ and

$$Q_n = 5F_n + 2(-1)^n ,$$

what is the recursion relation for $T_n = P_n + Q_n$?

2. Given recursion relations

$$P_{n+1} = 4P_n - P_{n-1} - 6P_{n-2} \quad \text{and} \quad Q_{n+1} = 6Q_n - 10Q_{n-1} + Q_{n-2} + 6Q_{n-3} ,$$

with $T_{n+1} = P_{n+1} + Q_{n+1}$, determine the recursion relation of lowest order satisfied by $T_{n+1}$.

3. Determine the recursion relation for $T_n = P_n + Q_n$ where $P_n$ is the arithmetic progression 3, 7, 11, 15, 19, ⋯ and $Q_n$ is the geometric progression 2, 6, 18, 54, ⋯ .

4. Determine the recursion relation for $T_n = 2^n + F_n^2$ given that the recursion relation for $F_n^2$ is

$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 .$$

5. Determine the recursion relation for $T_n = 5L_n^2 + (-1)^{n-1} + 4F_n$ .

(See page 544 for solutions to problems.)

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## A SHORTER PROOF

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In his article (April 1967) on 1967 as the sum of squares, Brother Brousseau proves that 1967 is not the sum of three squares. This fact canbe proved more briefly as follows:

If $1967 = a^2 + b^2 + c^2$, where a, b and c are positive integers, then, as Brother Brousseau has shown, a, b and c are all odd. Then $a = 2x + 1$, $b = 2y + 1$, and $c = 2z + 1$, where x, y and z are integers.

Consequently,

$$\begin{aligned} 1967 &= (2x + 1)^2 + (2y + 1)^2 + (2z + 1)^2 \\ &= 4x^2 + 4x + 4y^2 + 4y + 4z^2 + 4z + 3. \end{aligned}$$

Then

$$1964 = 4x^2 + 4x + 4y^2 + 4y + 4z^2 + 4z .$$

Dividing by 4, we get

$$491 = x^2 + x + y^2 + y + z^2 + z ,$$

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