SPIRALS, CHECKERBOARDS, POLYOMINOES, AND THE FIBONACCI SEQUENCE

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Any number N may be written as the sum of powers of two; that is,

$$N = a_{n-1} \cdot 2^{n-1} + a_{n-2} \cdot 2^{n-2} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0 ,$$

where the coefficients are each either 1 or 0. Thus, for example,

$$\begin{aligned} 37 &= 32 + 4 + 1 \\ &= 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 100101 \text{ in binary notation.} \end{aligned}$$

Suppose that, beginning at the right-hand side, the whole expression for N is coiled up and fitted onto a checkerboard with $a_0 \cdot 2^0$ at the lower-left (red) of the four central cells and the other terms proceeding in a counterclockwise manner. Then the upper left (black) cell of the board coincides with $a_{63} \cdot 2^{63}$. We can say that a vacant cell represents a 0 coefficient and a cell containing a pawn represents a 1 coefficient. Thus any number N (smaller than 2^{64}) may be indicated by a unique arrangement of pawns on the checkerboard; conversely, any particular arrangement of pawns corresponds to a unique number N.

The following observations may be made:

1. Since the spiral can be continued indefinitely or terminated at any integer, checkerboards of any size may be so covered.

2. Cells can be labeled with the corresponding exponent of two. Pawns placed on 0, 4, 16, and 36, for example — along the upper left diagonal — correspond to the binary number 1000000000000000000010000000000010001, or 68,719,542,289 in the customary base ten notation.

3. A k-by-k board can be used to represent any number K less than $2^{(k)^2}$.

4. Any cell has a numerical value greater than the sum of all lesser cells.

5. With the usual placement of the checkerboard having the lower left corner colored red, all red cells are labeled with even numbers — including 0 — and all black cells are labeled with odd numbers.

6. The "parity" of a number N is defined as the absolute value of the difference between the number of pawns on black cells and the number of pawns on red cells in its checkerboard representation. With an even number of pawns the parity is always even, while an odd number of pawns has an odd parity.

7. Any cell n lies rookwise adjacent to at most four cells, with no cell greater than $(k)^2$, the size of the board. These cells are:

(1) If n is a square, the adjacent cells are $n + 1$, $n - 1$, $n + 4\sqrt{n} + 3$, and $n + 4\sqrt{n} + 5$. These values of n lie at the corners of the spiral which fall along the diagonal going upward to the left and passing through cells 0 and 1. If $n - 1$ is negative (for the 0 cell only), replace $n - 1$ by 7.

(2) For cells lying along the diagonal upward to the right, that is, if n is of the form $a \cdot (a + 1)$, then the adjacent cells are $n + 1$, $n - 1$, $n + 4[\sqrt{n}] + 5$, and $n + 4[\sqrt{n}] + 7$, where the brackets [ ] indicate $n - 1$ is negative (for the 0 cell only), replace $n - 1$ by 3.

(3) For all other cells, the adjacent cells are $n + 1$, $n - 1$, $n + 4[\sqrt{n}] + 6 + j$, and $n - 4[\sqrt{n}] + 2 - j$, where $j = +1$ if $n > [\sqrt{n}] \cdot [\sqrt{n} + 1]$, and $j = -1$ if $n < [\sqrt{n}] \cdot [\sqrt{n} + 1]$. If $n = [\sqrt{n}] \cdot [\sqrt{n} + 1]$, the formulae in (2) should be used; if $\sqrt{n} = [\sqrt{n}]$, the formulae in (1) should be used.

The above rules enable us to travel from any cell to any other cell on any size checkerboard, without even seeing the board, simply by repeated applications of algebraic formulae. The only limitation is that the board be either square — $(k)^2$ — or square plus one extra row — $(k) \cdot (k + 1)$ — for any k.

NUMBERS REPRESENTED BY p PAWNS

One can easily (in theory, anyhow) make a list N(p) of all numbers which can be expressed by exactly p pawns. For $p = 5$, for example, the list begins with $N(5)_{min} = 2^5 - 1 = 31$, and continues 47, 55, 59, 61, 62, 79, 87, 91, 94, 103, 107, 109, 110, 115, 117, 118, 121, $\cdots$.

The number of integers less than 2^z which can be represented by one pawn is obviously z; it is the number of ways of selecting one object from z identical objects. The number of integers less than z which can be represented by p pawns is

$$\binom{z}{p} = \frac{z!}{p!(z-p)!} \quad ,$$

or, for 5 pawns and a regulation checkerboard,

$$\binom{64}{5} = \frac{64 \cdot 63 \cdot 62 \cdot 61 \cdot 60}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7{,}624{,}512 \ ,$$

a not inconsequential number. For z = 16 the number of integers represented by 5 pawns drops to 4368.

<u>Question:</u> What is the largest integer (decimal notation) that can be represented by 5 pawns on a 4-by-4 checkerboard? (See Fig. 1.)

<u>Answer:</u> The board has $(4)^2 = 16$ cells. The cells giving the largest number using five pawns are cells 15, 14, 13, 12, and 11, corresponding to the number $2^{15} + 2^{14} + 2^{13} + 2^{12} + 2^{11}$, or $32768 + 16384 + 8192 + 4096 + 2048 = 63,448$.

THE FIBONACCI SEQUENCE

Some interesting patterns on the checkerboard are obtained by plotting the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, $F_3 = 2, \cdots, F_{k+2} = F_k + F_{k+1}$. (At each step simply add together the binary representations of the last two numbers. $F_1 = F_2$, F_3, F_4, F_7, F_{10}, F_{13}, F_{16}, and F_{22}, alone of all Fibonacci numbers less than F_{60} (= 1548008755920), possess the property that each pawn lies rookwise adjacent to at least one other pawn. This happens to be the property defining the polyominoes. Specifically, F_{10} (= 55) is the U-pentomino and F_{13} (= 233) is the P-pentomino. (See Solomon W. Golomb, <u>Polyominoes</u>, for an extensive discussion of polyomino properties and problems.)

The Lucas sequence ($L_1 = 1$, $L_2 = 2$, $L_{k+2} = L_k + L_{k+1}$) similarly produces several polyominoes at the beginning of the run, notably the pentominoes P (= 47) and W (= 199). Other sequences do the same. (See V. E. Hoggatt, Jr., <u>Fibonacci and Lucas Numbers.</u>)

<u>Question:</u> Given a large enough checkerboard, can any polyomino be so positioned as to result in a Fibonacci (or Lucas) number?

<u>Answer:</u> Unsolved.

65536 16	32768 15	16384 14	8192 13	4096 12
131072 17	16 4	8 3	4 2	2048 11
262144 18	32 5	1 0	2 1	1024 10
524288 19	64 6	128 7	256 8	512 9
1048576 20	2097152 21	4194304 22	8388608 23	16777216 24

Fig. 1

(The numbers in the center of the squares represent blue. The numbers in the corners represent red. The red cells of the checkerboard are the screened ones.)

POLYOMINOES

Finally we arrive at the focal point of this paper. We have shown that any set of pawns uniquely represents a particular number N. A particular configuration of pawns may be shifted up or down or sideways, or even rotated or reflected, thus generating an entire sequence of numbers describing the relative positions of the pawns within the set and differing only in the placement of the set on the board. For example, the X-pentomino can be described by 171, 1287, 10254, 163896. We can specify that a configuration of pawns is best described by the least number N.

Our purpose is to find the number P(p) of p-ominoes.

We observe first that, since all p-ominoes can be placed on a checkerboard having no more than p x p cells, there are at the very most

$$\binom{p^2}{p} = \frac{(p^2)!}{p!\,(p^2 - p)!}$$

different p-ominoes. Thus for $p = 5$, $P(5) \leq 53{,}130$. This is the number of ways of choosing <u>any</u> five cells of the 25, without specifying that they be rook-wise connected.

But only the straight p-omino needs such a large board; in fact, it requires only one cell more than a $(p - 1) \times (p - 1)$ board. All other p-ominoes can be fitted onto a $(p - 1) \times (p - 1)$ board and in fact require only one cell more than a $(p - 2) \times (p - 2)$ board. For $p = 5$, $P(5)$ thus becomes no more than 1 plus

$$\binom{(p - 2)^2 + 1}{p}$$

or $P(5) \leq 1 + 252$. Actually, only t_1 pieces require such a large board; all the rest can be fitted onto a $(p - 3) \times (p - 3)$ board plus one cell. For $p = 5$, then, $P(5) \leq 1 + t_1 + 1$. The argument can be generalized for any p.

A candidate for t_1 has at least one pawn which lies in the strip $(p - 2)^2 = (p - 3)^2$; that is, the decimal representation of a t_1 polyomino lies in the range $2^{(p-2)^2+1}$ down to $2^{(p-3)^2+1}$. For pentominoes this range is 1024 to 32.

Going back to the list $N(5)$ of numbers having five pawns in their plots, we can see that for connected cells the parity of $N(5)$ is no more than $(p + 2)/4$, that is, either 3 or 1 for $p = 5$. This reduces the number of candidates for polyominoes; specifically, a parity of 5 means that all pawns lie on cells of the same color. A 4x4 board with one additional cell has 9 red cells and 8 black cells, producing

$$\binom{9}{5} + \binom{8}{5} = 126 + 56 = 182$$

numbers of parity 5. A 3x3 board with one extra cell has 5 red cells and 5 black sells, together yielding two numbers of parity 5.

Now at last we start counting polyominoes. We count one straight p-omino first. Then we examine each number in the range $2^{(p-2)^2+1}$ down to 2^p (1024 to 32 for the pentominoes), and finally count one for the p-omino formed by the first p cells of the spiral. Certain restrictions in the range can often be developed. Within the range, an acceptable number must have exactly p one's in its binary representation, and must have a parity of no more than $(p + 2)/4$. Then we look at the p exponents $n_1, n_2, \cdots, n_p$ associated with 1 coefficients (in other words, the labels on the cells occupied by pawns). We calculate the rookwise adjacent cells associated with n_1 (from paragraph 7) and see if at least one of these is included in the set of p exponents, say n_3. If so, we calculate the neighbors of n_3 and see if at least one of these is included in the set of exponents. If any one of the exponents cannot be reached by a series of steps from n_1, the number being tested does not represent a polyomino.

Finally having excluded all numbers which do not correspond to polyominoes, we are of necessity left with the list of numbers which do. We do not yet, however, have $P(p)$, the total number of p-ominoes, for we have not yet excluded rotations, reflections, and translations. Methods of algebraically excluding these duplications can obviously easily be developed.

The general expression for $N(p)_{max}$, corresponding to the straight p-omino, is $2^{(p-1)(p-2)} \cdot (2^p - 1)$, and for $N(p)_{min}$ it is $2^p - 1$, corresponding to an occupation of the first p cells of the spiral.

Question: What are the "best" decimal and binary representations and parities of the twelve pentominoes?

Answer:

	Binary	Decimal	Parity
I	11 11100 00000 00000	126976	1
L	11111 00000	992	1
Y	11110 00001	963	1
N	11001 00011	803	1
X	101 01011	171	3
V	11 11100	124	1
T	11 10011	115	1
W	11 01101	109	1
F	11 01011	107	1
Z	11 00111	103	1
U	1 01111	55	1
P	11111	31	1

Bonus Question: The reader who determines the parity of 16760865 deserves an "E" for effort, with a "well done!" for replotting it into its "best" configuration and decimal representation.

Answer: Parity 1, 13 pawns forming a W, 991177.

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