SOME FIBONACCI AND LUCAS IDENTITIES

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1. In the usual notation, put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n \tag{1.1}$$

where

$$\begin{gathered} \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = \tfrac{1}{2}(1 - \sqrt{5}), \\ \alpha^2 = \alpha + 1, \qquad \beta^2 = \beta + 1, \qquad \alpha\beta = -1 . \end{gathered} \tag{1.2}$$

It is rather obvious that polynomial identities can be used in conjunction with (1.1) and (1.2) to produce Fibonacci and Lucas identities. For example, by the binomial theorem,

$$\alpha^{2n} = (\alpha + 1)^n = \sum_{s=0}^{n} \binom{n}{s} \alpha^s$$

which gives

$$F_{2n} = \sum_{s=0}^{n} \binom{n}{s} F_s, \qquad L_{2n} = \sum_{s=0}^{n} \binom{n}{s} L_s \tag{1.3}$$

and indeed

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$$F_{2n+k} = \sum_{s=0}^{n} \binom{n}{s} F_{s+k}, \quad L_{2n+k} = \sum_{s=0}^{n} \binom{n}{s} L_{s+k}, \tag{1.4}$$

where k is an arbitrary integer.

Again, since

$$\alpha^3 = \alpha(\alpha + 1) = 2\alpha + 1,$$

we get in the same way

$$F_{3n+k} = \sum_{s=0}^{n} \binom{n}{s} 2^s F_{s+k}, \quad L_{3n+k} = \sum_{s=0}^{n} \binom{n}{s} 2^s L_{s+k}. \tag{1.5}$$

More generally, since

$$\alpha^r = F_r\alpha + F_{r-1},$$

it follows that

$$F_{rn+k} = \sum_{s=0}^{n} \binom{n}{s} F_r^s F_{r-1}^{n-s} F_{s+k}, \quad L_{rn+k} = \sum_{s=0}^{n} \binom{n}{s} F_r^s F_{r-1}^{n-s} L_{s+k}. \tag{1.6}$$

This can be carried further. For example, since

$$\alpha^{2m} = L_m\alpha^m - (-1)^m, \tag{1.7}$$

we get

$$\begin{aligned} F_{2mn+k} &= \sum_{s=0}^{n} (-1)^{(n-s)(m+1)} \binom{n}{s} L_m^s F_{ms+k}, \\ L_{2mn+k} &= \sum_{s=0}^{n} (-1)^{(n-s)(m+1)} \binom{n}{s} L_m^s L_{ms+k}. \end{aligned} \tag{1.8}$$

The identity (1.7) generalizes to

$$\alpha^{rm} = \frac{F_{rm}}{F_m}\alpha^m - (-1)^m \frac{F_{(r-1)m}}{F_m} \quad . \tag{1.9}$$

Indeed (1.9) is equivalent to

$$\alpha^{rm}(\alpha^m - \beta^m) = (\alpha^{rm} - \beta^{rm})\alpha^m - \alpha^m\beta^m(\alpha^{(r-1)m} - \beta^{(r-1)m}) \ ,$$

which is obviously true. From (1.9), we obtain

$$\begin{aligned} F_m^n F_{rmn+k} &= \sum_{s=0}^{n} (-1)^{(n-s)(m+1)} F_{(r-1)m}^{n-s} F_{rm}^s F_{ms+k} \ , \\ F_m^n L_{rmn+k} &= \sum_{s=0}^{n} (-1)^{(n-s)(m+1)} F_{(r-1)m}^{n-s} F_{rm}^s L_{ms+k} \ . \end{aligned} \tag{1.10}$$

With each of the above identities is associated a number of related identities. For example, we may rewrite

$$\alpha^r = F_r\alpha + F_{r-1}$$

as either

$$\alpha^r - F_r\alpha = F_{r-1} \qquad \text{or} \qquad \alpha^r - F_{r-1} = F_r\alpha \ .$$

Hence we obtain

$$\begin{aligned} \sum_{s=0}^{n} (-1)^s \binom{n}{s} F_r^s F_{r(n-s)+s+k} &= F_{r-1}^n F_k \ , \\ \sum_{s=0}^{n} (-1)^s \binom{n}{s} F_r^s L_{r(n-s)+s+k} &= F_{r-1}^n L_k \ , \end{aligned} \tag{1.11}$$

and

(1.12)
$$\sum_{s=0}^{n} (-1)^{n-s}\binom{n}{s} F_{r-1}^{n-s} F_{rs+k} = F_r^n F_{n+k} ,$$
$$\sum_{s=0}^{n} (-1)^{n-s}\binom{n}{s} F_{r-1}^{n-s} L_{rs+k} = F_r^n L_{n+k} .$$

We remark that (1.9) can be generalized even further, namely to

(1.13)
$$F_{rm}\alpha^{sm} - F_{sm}\alpha^{rm} = (-1)^{sm} F_{(r-s)m} ,$$

and (1.10) can now be extended in an obvious way.

2. Additional identities are obtained by making use of formulas such as

(2.1)
$$\alpha^2 + 1 = \alpha\sqrt{5} .$$

Note that

(2.2)
$$\beta^2 + 1 = -\beta\sqrt{5} .$$

Thus we get

$$\sum_{s=0}^{n}\binom{n}{s}\alpha^{2s} = 5^{n/2}\alpha^n , \quad \sum_{s=0}^{n}\binom{n}{s}\beta^{2s} = (-1)^n 5^{n/2}\beta^n ,$$

so that

$$\sum_{s=0}^{n}\binom{n}{s} F_{2s+k} = 5^{n/2}\frac{\alpha^{m+k} - (-1)^n \beta^{n+k}}{\alpha - \beta} .$$

It follows that

$$\sum_{s=0}^{n}\binom{n}{s}F_{2s+k} = \begin{cases} 5^{n/2}\,F_{n+k} & (n \text{ even}) \\ 5^{(n-1)/2}\,L_{n+k} & (n \text{ odd}) \end{cases} \tag{2.3}$$

Similarly

$$\sum_{s=0}^{n}\binom{n}{s}L_{2s+k} = \begin{cases} 5^{n/2}\,L_{n+k} & (n \text{ even}) \\ 5^{(n+1)/2}\,F_{n+k} & (n \text{ odd}) \end{cases} \tag{2.4}$$

We omit the variants of (2.3) and (2.4).

We can generalize (2.1) as follows:

$$\begin{cases} \alpha^{2m} = F_{2m}\,\alpha\sqrt{5} - L_{2m-1} \\ \alpha^{2m+1} = L_{2m+1}\,\alpha - L_{2m}\sqrt{5} \end{cases} \tag{2.5}$$

The corresponding formulas for β^m are

$$\begin{cases} \beta^{2m} = -F_{2m}\,\beta\sqrt{5} - L_{2m-1} \\ \beta^{2m+1} = L_{2m+1}\,\beta + F_{2m}\sqrt{5} \end{cases} \tag{2.6}$$

We therefore get the following generalizations of (2.3) and (2.4):

$$\sum_{s=0}^{n}\binom{n}{s}L_{2m-1}^{n-s}\,F_{2ms+k} = \begin{cases} 5^{n/2}\,F_{2m}^{n}\,F_{n+k} & (n \text{ even}) \\ 5^{(n-1)/2}\,F_{2m}^{n}\,L_{n+k} & (n \text{ odd}) \end{cases}, \tag{2.8}$$

$$\sum_{s=0}^{n}\binom{n}{s}L_{2m-1}^{n-s}\,L_{2ms+k} = \begin{cases} 5^{n/2}\,F_{2m}^{n}\,L_{n+k} & (n \text{ even}) \\ 5^{(m+1)/2}\,F_{2m}^{n}F_{n+k} & (n \text{ odd}) \end{cases}, \tag{2.9}$$

$$\sum_{s=0}^{n}(-1)^{n-s}\binom{n}{s}L_{2m+1}^{s}\,F_{2m(n-s)+n+k} = \begin{cases} 0 & (n \text{ even}) \\ 2\cdot 5^{(n-1)/2}\,F_{2m}^{n} & (n \text{ odd}), \end{cases} \tag{2.10}$$

$$(2.11)\qquad \sum_{s=0}^{n} (-1)^{n-s}\binom{n}{s} L_{2m+1}^{s}\, L_{2m(n-s)+n+k} = \begin{cases} 2 \cdot 5^{n/2}\, F_{2m}^{2} & (n \text{ even}) \\ 0 & (n \text{ odd}) \,. \end{cases}$$

We omit the variants of (2.8), $\cdots$, (2.11).

3. In the next place,

$$\begin{aligned}(1 + \alpha x)^{n}(1 + x)^{n} &= \sum_{r=0}^{n} \binom{n}{r} \alpha^{r} x^{r} \sum_{s=0}^{n} \binom{n}{s} x^{s} \\ &= \sum_{k=0}^{2n} x^{k} \sum_{r=0}^{k} \binom{n}{r}\binom{n}{k-r} \alpha^{r} \,.\end{aligned}$$

On the other hand,

$$\begin{aligned}(1 + \alpha x)^{n}(1 + x)^{n} &= (1 + \alpha^{2}x + \alpha x^{2})^{n} \\ &= \sum_{r=0}^{n} \binom{n}{r} \alpha^{r} x^{r} (\alpha + x)^{r} \\ &= \sum_{r=0}^{n} \binom{n}{r} \alpha^{r} x^{r} \sum_{s=0}^{r} \binom{r}{s} \alpha^{r-s} x^{s} \\ &= \sum_{k=0}^{2n} x^{k} \sum_{3s \le 2k} \binom{n}{k-x}\binom{k-s}{s} \alpha^{2k-3s} \,.\end{aligned}$$

Comparing coefficients of x^{k} we get

$$(3.1)\qquad \sum_{r=0}^{k} \binom{n}{r}\binom{n}{k-r} \alpha^{r} = \sum_{3s-2k} \binom{n}{k-s}\binom{k-s}{s} \alpha^{2k-3s} \,.$$

It therefore follows that

$$\sum_{r=0}^{k} \binom{n}{r}\binom{n}{k-r} F_{r+j} = \sum_{3s \le 2k} \binom{n}{k-s}\binom{k-s}{s} F_{2k-3s+j} \tag{3.2}$$

and

$$\sum_{r=0}^{k} \binom{n}{r}\binom{n}{k-r} L_{r+j} = \sum_{3s \le 2k} \binom{n}{k-s}\binom{k-s}{s} L_{2k-3s+j} \tag{3.3}$$

for all j. In particular, for $k = n$, these formulas reduce to

$$\sum_{r=0}^{n} \binom{n}{r}^2 F_{r+j} = \sum_{3s \le 2n} \binom{n}{2s}\binom{2s}{s} F_{2n-3s+j} \tag{3.4}$$

and

$$\sum_{r=0}^{n} \binom{n}{r}^2 L_{r+j} = \sum_{3s \le 2n} \binom{n}{2s}\binom{2s}{s} L_{2n-3s+j} , \tag{3.5}$$

respectively.

We have similarly

$$\begin{aligned}(1 + \alpha^2 x)^n (1 - x)^n &= \sum_{r=0}^{n} \binom{n}{r} \alpha^{2r} x^r \sum_{s=0}^{n} (-1)^s \binom{n}{s} x^s \\ &= \sum_{k=0}^{2n} x^k \sum_{r=0}^{k} (-1)^{k-r} \binom{n}{r}\binom{n}{k-r} \alpha^{2r} ;\end{aligned}$$

$$
\begin{aligned}
(1+\alpha^2 x)^n(1-x)^n &= (1+\alpha x-\alpha^2 x^2)^n \\
&= \sum_{r=0}^{n}\binom{n}{r}\alpha^r x^r (1-\alpha x)^r \\
&= \sum_{r=0}^{n}\binom{n}{r}\alpha^r x^r \sum_{s=0}^{r}(-1)^s\binom{r}{s}\alpha^s x^s \\
&= \sum_{k=0}^{2n}\alpha^k x^k \sum_{2s\le k}(-1)^s\binom{n}{k-s}\binom{k-s}{s} .
\end{aligned}
$$

Comparing coefficients of x^k we get

$$
(3.6)\qquad \sum_{r=0}^{k}(-1)^{k-r}\binom{n}{r}\binom{n}{k-r}\alpha^{2r} = \alpha^k \sum_{2s\le k}(-1)^s\binom{n}{k-s}\binom{k-s}{s} .
$$

It follows that

$$
(3.7)\qquad \sum_{r=0}^{k}(-1)^{k-r}\binom{n}{r}\binom{n}{k-r}F_{2r+j} = F_{k+j}\sum_{2s\le k}(-1)^s\binom{n}{k-s}\binom{k-s}{s} ,
$$

$$
(3.8)\qquad \sum_{r=0}^{k}(-1)^{k-r}\binom{n}{r}\binom{n}{k-r}L_{2r+j} = L_{k+j}\sum_{2s\le k}(-1)^s\binom{n}{k-s}\binom{k-s}{s} .
$$

In particular, for $k = n$, we get

$$
(3.9)\qquad \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}^2 F_{2r+j} = F_{n+j}\sum_{2s\le n}(-1)^s\binom{n}{2s}\binom{2s}{s} ,
$$

$$
(3.10)\qquad \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}^2 L_{2r+j} = L_{n+j}\sum_{2s\le n}(-1)^s\binom{n}{2s}\binom{2s}{s} .
$$

More general results can be obtained by using (1.7). For brevity, we shall omit the statement of the formulas in question.

4. The formulas

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_k = -\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{k} F_{n-k}\,, \tag{4.1}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 L_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{k} L_{n-k} \tag{4.2}$$

were proposed as a problem in this Quarterly (Vol. 4 (1966), p. 332, H-97). The formulas

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{n-k}\,, \tag{4.3}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} L_{n-k} \tag{4.4}$$

were also proposed as a problem (Vol. 5 (1967), p. 70, H-106). They can be proved rapidly by making use of known formulas for the Legendre polynomial.

We recall that [1, 162, 166]

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{k}\left(\frac{x+1}{2}\right)^k. \end{aligned} \tag{4.5}$$

If we take $u = (x + 1)/(x - 1)$, we get

$$\sum_{k=0}^{n} \binom{n}{k}^2 u^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(u-1)^{n-k}, \tag{4.6}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 u^k = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k}u^k(u-1)^{n-k}. \tag{4.7}$$

Multiplying both sides of (4.6) by u^j and then take $u = \alpha, \beta$. Since $\alpha - 1 = \alpha^{-1}$, we get

$$\sum_{k=0}^{n} \binom{n}{k}^2 \alpha^{k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\alpha^{j+k-n},$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \beta^{k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\beta^{j+k-n}.$$

It follows that

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}F_{j+k-n}, \tag{4.8}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}L_{j+k-n}. \tag{4.9}$$

For $j = 0$, (4.8) and (4.9) reduce to (4.1) and (4.2), respectively.

If in the next place we replace $u = \alpha^2, \beta^2$ in (4.6), we get

$$\sum_{k=0}^{n} \binom{n}{k}^2 \alpha^{2k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\alpha^{n-k+j},$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \beta^{2k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \beta^{n-k+j} ,$$

so that

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{2k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{n-k+j} , \tag{4.10}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{2k+j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} L_{n-k+j} . \tag{4.11}$$

These formulas evidently include (4.3) and (4.4).

In exactly the same way (4.7) yields

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{k+j} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k} F_{2k+j-n} , \tag{4.12}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{k+j} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k} L_{2k+j-n} , \tag{4.13}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{2k+j} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k} F_{k+j+n} , \tag{4.14}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{2k+j} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k} L_{k+j+n} . \tag{4.15}$$

The identities (4.8), $\cdots$, (4.15) can be generalized further by employing, in place of (4.5), the following formulas for Jacobi polynomials 1, 255 :

$$P_n^{(\lambda,\mu)}(x) = \sum_{k=0}^{n} \binom{n+\lambda}{n-k}\binom{n+\mu}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}$$

$$= \binom{\lambda+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}\left(\frac{x-1}{2}\right)^k$$

$$= \binom{\mu+n}{n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\mu+1)_k}\left(\frac{x+1}{2}\right)^k ,$$

where

$$(\lambda+1)_n = (\lambda+1)(\lambda+2)\cdots(\lambda+n), \qquad (\lambda+1)_0 = 1 .$$

The final results are

$$(4.16) \qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}F_{k+j} = \binom{\lambda+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}F_{j+k-n},$$

$$(4.17) \qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}L_{k+j} = \binom{\lambda+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}L_{j+k-n},$$

$$(4.18) \qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}F_{2k+j} = \binom{\lambda+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}F_{n-k+j},$$

$$(4.19) \qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}L_{2k+j} = \binom{\lambda+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}L_{n-k+j},$$

$$(4.20)\qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}F_{k+j} = \binom{\mu+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}F_{j+2k-n},$$

$$(4.21)\qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}L_{k+j} = \binom{\mu+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}L_{j+2k-n},$$

$$(4.22)\qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}F_{2k+j} = \binom{\mu+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}F_{j+k+n},$$

$$(4.23)\qquad \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\mu}{n-k}L_{2k+j} = \binom{\mu+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\lambda+\mu+1)_k}{(\lambda+1)_k}L_{j+k+n}.$$

We remark that taking $u = -\alpha$ in (4.6) leads to

$$(4.24)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 F_{k+j} = \sum_{n=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}F_{2n-2k+j},$$

$$(4.25)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 L_{n+j} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}L_{2n-2k+j},$$

and so on.

Some of the formulas in the earlier part of the paper are certainly not new. However, we have not attempted the rather hopeless task of finding where they first occurred. In any event, it may be of interest to derive them by the methods of the present paper. It would, of course, be possible to find many additional identities.

REFERENCE

1. E. D. Rainville, Special Functions, Macmillan, New York, 1960.

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