ON SOLVING $C_{n+2} = C_{n+1} + C_n + n^m$ BY EXPANSIONS AND OPERATORS

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1. INTRODUCTION

It was the purpose of this paper to derive the general solution of the non-homogeneous difference equation

$$C_{n+2} = C_{n+1} + C_n + n^m$$

In so doing, two distinct approaches were employed to derive the particular solution associated with this equation, 1) a polynomial expansion method and 2) an operator method. The latter approach is believed to be a unique combined application of E, Δ, and the Fibonacci generating function.

The general solution of the non-homogeneous difference equation

$$C_{n+2} = C_{n+1} + C_n + n^m \tag{1}$$

is composed of the solution to the homogeneous equation

$$C_{n+2} = C_{n+1} + C_n \tag{2}$$

and a solution of the particular equation. See [4,5]. Since the polynomial term, n^m, is of degree m, a particular solution to (1) can be expected to be of the form

$$(C_n)_p = \sum_{i=0}^{m} a_{im} n^{m-i} , \tag{3}$$

from considerations produced in Section 4. A related problem appears in [8].

If

$$P_m(n) = (C_n)_p \quad , \tag{4}$$

the general solution of Eq. (1) can be expressed as

$$C_n = A_m F_{n+1} + B_m F_n - P_m(n) \tag{5}$$

where $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$, since F_{n+1} and F_n are linearly independent (Fibonacci numbers), and therefore span the space of solutions of the homogeneous part (e).

2. THE PARTICULAR SOLUTION

Since the particular solution of Eq. (1) is of the form

$$(C_n)_p = \sum_{i=0}^{m} a_{im} n^{m-i} \quad , \tag{6}$$

substitution of (6) into Eq. (1) yields

$$\sum_{i=0}^{m} a_{im}(n+2)^{m-i} = \sum_{i=0}^{m} a_{im}(n+1)^{m-i} + \sum_{i=0}^{m} a_{im} n^{m-i} + n^m \ . \tag{7}$$

By transposing and expanding these terms and then equating coefficients of, say, the n^{m-j} terms for $j \neq 0$, the general term of (7) becomes

$$\begin{aligned} &\left(a_{jm}n^{m-j} + \frac{2(m-j+1)n^{m-j}}{1!} a_{(j-1)m} + \cdots + \frac{2^j m(m-1)\cdots(m-j+1)n^{m-j}}{j!} a_{0m}\right) \\ &- \left(n^{m-j}a_{jm} + \frac{(m-j+1)n^{m-j}}{1!} a_{(j-1)m} + \cdots + \frac{m(m-1)\cdots(m-j+1)n^{m-j}}{j!} a_{0m}\right) \\ &- a_{jm}n^{m-j} = 0 \qquad j \neq 0 \text{ and all non-negative integers, } n. \end{aligned} \tag{8}$$

Since this equation is satisfied for all non-negative integer values of n, it is, in particular, satisfied for $n \neq 0$. Therefore, combining like subscripted a_{im} terms, this equation becomes

$$(9)\qquad -a_{jm} + \frac{(2^1 - 1)(m - j + 1)}{1!} a_{(j-1)m} + \cdots + \frac{(2^j - 1)m(m - 1)(m - 2)\cdots(m - j + 1)}{j!} a_{0m} = 0 .$$

Solving (9) for a_{jm} immediately yields

$$(10)\qquad a_{jm} = \sum_{i=0}^{j-1} a_{im} \frac{(2^{j-i} - 1)(m - i)!}{(j - i)!\,(m - j)!} = \sum_{i=0}^{j-1} a_{im}(2^{j-i} - 1)\binom{m - i}{j - i} .$$

Consequently, from Eq. (10), an expression for each a_{im} of Eq. (6) has been obtained in terms of the previous a_{im}.

For the particular case in which $j = 1$, Eq. (10) reduces to

$$(11)\qquad a_{1m} = a_{0m}\, m .$$

But Eq. (8) and those terms of the type $a_{0m}n^m$ yield the expression

$$(12)\qquad -a_{0m}n^m = n^m ,$$

which is valid for all non-negative integers n. Consequently, this equation immediately yields the result

$$(13)\qquad a_{0m} = -1$$

for all non-negative integers m. From Eq. (11), therefore, it is evident that

$$(14)\qquad a_{1m} = -m$$

for all non-negative integers m.

Using the previously derived expressions, it is now possible to generate all of the coefficients, a_{im}, of Eq. (6). In fact, the following theorem provides an expression for the a_{im} which is independent of any summation.

Theorem:

$$a_{jm} = a_{jj}\binom{m}{j} \quad \text{for all} \quad j < m . \tag{15}$$

Proof by mathematical induction:

From Eq. (10), it is evident that

$$a_{(j+1)(j+1)} = \sum_{i=0}^{j} a_{i(j+1)}(2^{j+1-i} - 1) \tag{16}$$

By the use of mathematical induction it can be easily verified that for $j = 0$ and $j = 1$,

$$a_{jm} = a_{jj}\binom{m}{j} \quad \text{for} \quad 0 < m < j . \tag{17}$$

Therefore, assume

$$a_{jm} = a_{jj}\binom{m}{j} \quad \text{for some particular } j . \tag{18}$$

It must be shown that

$$a_{(j+1)m} = a_{(j+1)(j+1)}\binom{m}{j+1} . \tag{19}$$

From Eq. (18) it is immediate that

$$a_{i(j+1)} = a_{ii}\binom{j+1}{i} \quad \text{for} \quad i \leq j . \tag{20}$$

Substituting expression (20) into (16) yields

(21) $$a_{(j+1)(j+1)} = \sum_{i=0}^{j} a_{ii}(2^{j+1-i} - 1)\binom{j+1}{i} .$$

Multiplying both sides of (21) by

(22) $$\binom{m}{j+1}$$

transforms the above equation into

(23) $$a_{(j+1)(j+1)}\binom{m}{j+1} = \sum_{i=0}^{j} a_{ii}(2^{j+1-i} - 1)\frac{m!}{1!(m-j-1)!} \cdot \frac{1}{(j+1+i)!} .$$

But from Eq. (10),

(24) $$a_{(j+1)m} = \sum_{i=0}^{j} a_{im}(2^{j+1-i} - 1)\binom{m-i}{j+1-i} .$$

By substituting (18) into (24), one obtains the equation

(25) $$a_{(j+1)m} = \sum_{i=0}^{j} a_{ii}(2^{j+1-i} - 1)\frac{m!}{1!(m-j-1)!(j+1-i)} .$$

Consequently, equating expressions (23) and (25) yields the desired result that

(26) $$a_{(j+1)(j+1)}\binom{m}{j+1} = a_{(j+1)m} .$$ Q. E. D.

From this theorem and Eqs. (4) and (6), the particular solution of expression (1) can now be written in the final form

(27) $$P_m(n) = \sum_{i=0}^{m} a_{ii}\binom{m}{i} n^{m-i} ,$$

where, from (10) and (26), it is seen that

(28) $$a_{kk} = \sum_{p=0}^{k-1} (2^{k-p} - 1)a_{pk} = \sum_{p=0}^{k-1} (2^{k-p} - 1)\binom{k}{k-p} a_{pp}$$

where $a_{0k} = -1$ and $a_{1k} = -k$.

3. THE GENERAL SOLUTION

Using the expressions for the particular solution of (5) which were derived in the previous section, the general solution of Eq. (1) can now be found. Assuming the general solution is of the form

(29) $$C_n = A_m F_{n+1} + B_m F_n - P_m(n) ,$$

an expression for A_m and B_m will now be derived.

From Eq. (29), it is clear that

(30) $$C_0 = A_m - P_m(0) .$$

But from Eq. (27), it is immediately evident that

(31) $$P_m(0) = a_{mm} ,$$

and, consequently, Eq. (30) becomes

(32) $$A_m = C_0 + a_{mm} .$$

Substituting this expression into Eq. (29) and solving for B_m by setting $n = 1$ yields

$$B_m = C_1 - C_0 - a_{mm} + P_m(1) \quad . \tag{33}$$

But since from Eq. (3)

$$P_m(1) = \sum_{i=0}^{m} a_{im} \quad , \tag{34}$$

Equation (33) now becomes

$$B_m = C_1 - C_0 + \sum_{i=0}^{m-1} a_{im} \quad . \tag{35}$$

The final expression for the general solution, Eq. (29), can be written

$$C_n = (C_0 + a_{mm})F_{n+1} + \left(C_1 - C_0 + \sum_{i=0}^{m-1} a_{im} \right) F_n - P_m(n) \quad . \tag{36}$$

4. THE USE OF OPERATORS TO FIND THE PARTICULAR SOLUTION, $P_m(n)$

An interesting method for finding the particular solution of Eq. (1) without the necessity of solving a large system of linear equations will now be investigated. The material in this section is experimental and unrigorous. For the difference operator, Δ, the method is valid for polynomials but for the forward shifting operator, E, the limitations are less clear. This method uses the two operators E and Δ which are defined in the following manner:

$$E[f(n)] = f(n + 1) \tag{37}$$

and

$$\Delta f(n) = f(n + 1) - f(n) \tag{38}$$

Consequently,

$$E = \Delta + 1 . \tag{39}$$

From Eq. (37), it is possible to write (1) as

$$(E^2 - E - 1)C_n = n^m . \tag{40}$$

Therefore, the particular solution of this expression is the function generated by using the inverse operator $(E^2 - E - 1)^{-1}$, on n^m. That is,

$$P_m(n) = (E^2 - E - 1)^{-1} n^m . \tag{41}$$

But from Eq. (39), it is immediate that

$$\frac{1}{(E^2 - E - 1)} = \frac{1}{(\Delta^2 + \Delta - 1)} \tag{42}$$

From the definition of the Fibonacci generating function [1],

$$\frac{1}{1 - x - x^2} = \sum_{i=0}^{\infty} F_{i+1} x^i , \tag{43}$$

it is seen that

$$\frac{-1}{(1 - \Delta - \Delta^2)} = -\sum_{i=0}^{\infty} F_{i+1} \Delta^i . \tag{44}$$

Therefore, from Eqs. (41), (42), and (44),

$$P_m(n) = -\sum_{i=0}^{\infty} F_{i+1} \Delta^i (n^m) . \tag{45}$$

But from Eq. (38), it is clear that

(46) $$\Delta^i(n^m) = 0 \quad \text{for all } i > m.$$

Consequently, the final form of the particular solution can be written as

(47) $$P_m(n) = -\sum_{i=0}^{m} F_{i+1}\Delta^i(n^m) .$$

As an example, suppose $m = 2$. Then (47) reduces to

(48) $$P_2(n) = -(F_1n^2 + F_2[(n + 1)^2 - n^2] + F_3[(n + 2)^2 - 2(n + 1)^2 + n^2]) .$$

Combining terms reduces this equation to

(49) $$P_2(n) = -(n^2 + 2n + 5) .$$

Another expression for $P_m(n)$ can be derived solely in terms of E. Clearly,

(50) $$\frac{1}{(E^2 - E - 1)} = \frac{1}{E^2}\left(\frac{1}{1 - \frac{1}{E} - \frac{1}{E^2}}\right) .$$

But, once again, the right side of this expression can be written as

(51) $$\frac{1}{E^2}\sum_{i=0}^{\infty} F_{i+1}E^{-i} .$$

Consequently, the final expression for the particular solution can be written as

(52) $$P_m(n) = \sum_{i=0}^{\infty} F_{i+1} E^{-(i+2)}(n^m)$$

where

(53) $$E(n^m) = (n + 1)^m$$

and

(54) $$E^{-1}(n^m) = (n - 1)^m .$$

Here we stop when n is reduced to zero. These solutions are of a different form than those using Δ and include the homogeneous part, too. We note the equivalence in a paper by Ledin [6]. See Brother Alfred [9] and Zeitlin [10].

5. CONCLUSION

As far as the authors know, the conditions under which the methods in Section 4 remain valid is an open and interesting question. Douglas Lind has pointed out that if $C_{n+1} = C_n + n$ were to be solved by the method $(E - 1)C_n = n$, then

$$C_n = \frac{-1}{1 - E}(n) = -\left(\sum_{k=0}^{\infty} E^k\right)(n)$$

diverges unless some stopping rule is invoked.

REFERENCES

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