A GENERALIZED FIBONACCI SEQUENCE OVER AN ARBITRARY RING

D. J. DeCARLI
Rosary Hill College, Buffalo, New York

Let S be a ring with identity I. Consider the sequence $\{M_n\}$ of elements of S, recursively defined by:

$$M_{n+2} = A_1M_{n+1} + A_0M_n \quad \text{for} \quad n \geq 0, \tag{1}$$

where M_0, M_1, A_0, and A_1 are arbitrary elements of S.

Special cases of (1) have been considered by Buschman [1], Horadam [2], and Vorobyov [3] where S was taken to be the set of integers. Wyler [4] also worked with such a sequence over a particular commutative ring with identity. In this note, we establish several results for such sequences over S (not necessarily commutative) which are analogues of results derived for similarly defined sequences of integers.

We begin by considering a special case of (1), denoted by $\{F_n\}$ and defined by:

$$F_{n+2} = A_1F_{n+1} + A_0F_n \quad \text{for} \quad n \geq 0, \tag{2}$$

where $F_0 = 0$, $F_1 = I$ and A_0, A_1 are arbitrary elements of S.

The fact that S need not be commutative causes difficulty in trying to derive results for the $\{F_n\}$ sequence. However, we note that the terms of this sequence possess an internal symmetry which enables us to make a start at deriving identities.

Theorem 1. If $F_{n+2} = A_1F_{n+1} + A_0F_n$, then

$$F_{n+2} = F_{n+1}A_1 + F_nA_0 . \tag{3}$$

Proof: The proof is straightforward by induction.

Corollary 1:

(i) $$F_{n+1}F_{n-1} - F_n^2 = F_{n-1}A_0F_{n-1} - F_nA_0F_{n-2},$$

(ii) $$F_{n-1}F_{n+1} - F_n^2 = F_{n-1}A_0F_{n-1} - F_{n-2}A_0F_n, \qquad n \geq 1.$$

Proof of (i): From (3), we have

$$F_{n+1}F_{n-1} - F_n^2 = (F_nA_1 + F_{n-1}A_0)F_{n-1} - F_n(A_1F_{n-1} + A_0F_{n-2})$$
$$= F_{n-1}A_0F_{n-1} - F_nA_0F_{n-2}.$$

The second result can be obtained in a similar manner. We note that the results of Corollary 1 are analogues of Equation (11) of Horadam's paper [2].

The $\{M_n\}$ sequence does not, in general, possess the symmetry of the $\{F_n\}$ sequence and consequently it is even more difficult to work with. There is, however, a relation between the $\{M_n\}$ sequence and the $\{F_n\}$ sequence.

Theorem 2:

$$M_{n+r} = F_rA_0M_{n-1} + F_{r+1}M_n, \qquad n \geq 1, \quad r \geq 0.$$

Proof: The result is easily established by induction.

Corollary 2:

$$M_n = F_nM_1 + F_{n-1}A_0M_0, \qquad n \geq 1.$$

Proof: Interchange r and n, replace n by $n - 1$ and set $r = 1$ in Theorem 2.

We note that the result of Theorem 2 is identical with Equation (12) of Buschman's paper [1] which was derived for a similarly defined sequence of integers.

For the $\{F_n\}$ sequence, Theorem 2 becomes

(4) $$F_{n+r} = F_rA_0F_{n-1} + F_{r+1}F_n, \qquad n \geq 1.$$

If we replace n by $n + 1$ and r by n in (4), then we have

(5) $$F_{n+1}^2 + F_nA_0F_n = F_{2n+1}.$$

The commutator of the $\{F_n\}$ sequence is characterized by

Theorem 3:

$$F_nF_{n+r} - F_{n+r}F_n$$
$$= F_nF_rA_oF_{n-1} - F_{n-1}A_0F_rF_n, \qquad n \geq 1, \quad r \geq 1 .$$

Proof: If we replace n by $r + 1$ and r by $n - 1$ in (4), we have

$$F_{n+r} = F_{n-1}A_0F_r + F_nF_{r+1} . \tag{6}$$

From (4), (6), and the fact that S satisfies the associative law for multiplication, we have:

$$F_n(F_{r+1}F_n) = (F_nF_{r+1})F_n \quad .$$

$$\therefore F_n(F_rA_0F_{n-1} + F_{r+1}F_n - F_rA_0F_{n-1})$$
$$= (F_{n-1}A_0F_r + F_nF_{r+1} - F_{n-1}A_0F_r)F_n .$$

$$\therefore F_n(F_{n+r} - F_rA_0F_{n-1}) = (F_{n+r} - F_{n-1}A_0F_r)F_n \quad .$$

$$\therefore F_nF_{n+r} - F_{n+r}F_n = F_nF_rA_0F_{n-1} - F_{n-1}A_0F_rF_n .$$

The $\{M_n\}$ sequence appears to be very difficult to work with directly. Investigations indicate that the best that can be done is to concentrate effort on the $\{F_n\}$ sequence and use Theorem 2 and Corollary 2 to derive analogous results for the $\{M_n\}$ sequence.

As a final remark, we note that the sequence obtained from (1) by setting $M_0 = R$, $M_1 = P + Q$, P,Q arbitrary elements of S, and $A_0 = A_1 = I$, yields a nice set of identities which are analogues of those derived by Horadam [2] for a similarly defined sequence of integers.

REFERENCES

1. R. G. Buschman, "Fibonacci Numbers, Chebyschev Polynomials, Generalizations and Difference Equations," Fibonacci Quarterly, Vol. 1, No. 4, (1963), pp. 1-7.

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