A REMARKABLE LATTICE GENERATED BY FIBONACCI NUMBERS

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Functions which can be represented in the s-dimensional unit interval by rapidly convergent Fourier series of unit period in each coordinate can be integrated numerically over this interval with great efficiency by averaging their values over all the points obtained by reducing modulo 1 the coordinates of the multiples of a suitable vector $\bar{g} = \langle g_1/p, \cdots, g_s/p \rangle$, where $g_1, \cdots, g_s$, and p are integers. The crucial property of this vector can be described as follows: For any vector $\bar{h} = \langle h_1, \cdots, h_s \rangle$ put

$$R(\bar{h}) = \max(1, h_1) \cdots \max(1, h_s) ,$$

and denote by $\rho(\bar{g})$ the minimum of $R(\bar{h})$ for all the vectors having integral coordinates not all zero, and satisfying

$$\bar{g} \cdot \bar{h} \equiv 0 \pmod 1 ,$$

where the dot denotes, as usual, the scalar product. Hlawka [5] describes $p\bar{g}$ as a <u>good lattice point modulo p</u> if

$$(1) \qquad \rho(\bar{g}) \geq p(8 \log p)^{1-s} \quad ‡$$

because upper bounds for the error of integration can be expressed as rapidly decreasing functions of $(\bar{g})$, and he proves the existence of good lattice points modulo any prime for any number of dimensions. The requirement that p should be a prime was introduced only in order to facilitate the proof. Understandably, however, one assumes in any event $(g_1, \cdots, g_s, p) = 1$, so that $\bar{g}$ generates exactly p different multiples modulo 1. Of course, here and in

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‡As a result of a misprint, the exponent of $8 \log p$ appears to be $-s$ in Hlawka's paper, but his proof applies to lattice points satisfying (1). Thus his results are sharper than those of Korobov ([7], [8]).

what follows, by a multiple modulo 1 of any vector, we understand the result of reducing modulo 1 each coordinate of the multiple of the given vector.

In the case of more than two dimensions no recipe other than trial and error is known for finding good lattice points, and indeed such a recipe seems unlikely to exist. However, in two dimensions, the best lattice points in the sense of maximizing the ratio $(\bar{g})$:p are obtained by putting

$$p = F_n, \quad g_1 = 1, \quad g_2 = F_{n-1},$$

where $\langle F_j \rangle$ are the Fibonacci numbers [9]. One finds, then, $\rho(\bar{g}) = F_{n-2}$, which is of a better order of magnitude than (1).

The case when the integrand has not the required properties of periodicity can be reduced to the periodic case. In the case of two dimensions, denoting the coordinates by x and y, we add to the integrand a suitable polynomial in x with coefficients depending on y, and a polynomial in y with coefficients depending on x. The precise upper bounds for the error ([9], [12]) are too complicated to be discussed here in detail. Let it suffice to say that if the integrand f has partial derivatives up to

$$\frac{\partial^{2r} f}{\partial x^r \partial y^r}$$

of bounded variation in the sense of Hardy and Krause (for a precise definition see, for instance [5] or [9]), and if we add to it suitable polynomials of degree r, this allows us to obtain the value of the integral with an error of the order $F_n^{-(r+1)} \log F_n$ by averaging f over the F_n points defined above. Trial computations carried out by this method [12] gave a very high degree of accuracy. For instance, taking r = 3, the value of the integral over the unit square of $\exp(-x^2 - 2y^2)$ (true value 0.446708379 to nine decimals) was obtained with eight correct decimals from n = 7 onwards, i.e., using 13 or more points.

The sets of points corresponding to n = 5, 6, and 7 are shown in Figs. 1, 2, and 3. It will be seen that they define regular grids, and indeed square grids when n is odd. The importance of these grids lies not so much in the fact that they may be thought to be picturesque, as rather in conclusions of a far-reaching nature which can be drawn from their existence.

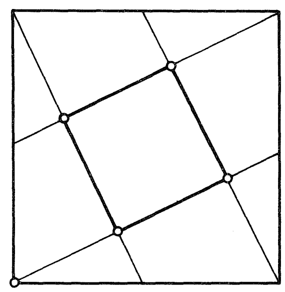

Fig. 1 : n = 5

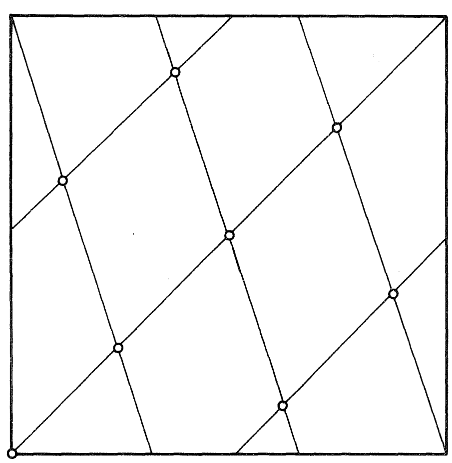

Fig. 2 : n = 6

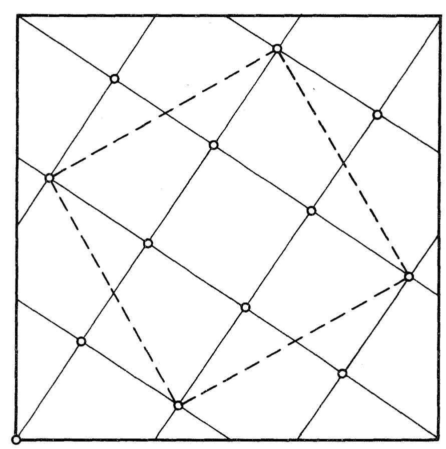

Fig. 3 : n = 7

We begin, however, with a description of the grids themselves. It is easily seen that the sets of points in question form lattices. The lattice generated modulo 1 by the vector

$$\overline{V} = \langle F_n^{-1}, F_{n-1}F_n^{-1}\rangle$$

will be denoted by L_n. It obviously has a base formed by the vectors $\overline{V}$ and $\overline{e_2} = \langle 0, 1\rangle$. The more detailed nature of L_n depends on the parity of n. In its investigation, we shall repeatedly use the identities

(2) $$F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}$$

(see, for instance, I_{26} in [6]), and

(3) $$F_{-n} = (-1)^{n+1}F_n .$$

When $n = 2\mu + 1$, an alternative basis of L_n is formed by the vectors

$$\overline{V_1} = \langle F_\mu F_{2\mu+1}^{-1}, -F_{-\mu-1}F_{2\mu+1}^{-1}\rangle \quad \text{and} \quad \overline{V_2} = \langle F_{\mu+1}F_{2\mu+1}^{-1}, -F_{-\mu}F_{2\mu+1}^{-1}\rangle.$$

Indeed, from (3) and from (2) with $n = -2\mu - 1$ and with $m = \mu - 1$ and $m = \mu$, respectively, we deduce

$$F_\mu F_{2\mu} \equiv -F_{-\mu-1} \pmod{F_{2\mu+1}}$$

and

$$F_{\mu+1}F_{2\mu} \equiv -F_{-\mu} \pmod{F_{2\mu+1}},$$

so that

$$F_\mu \overline{V} \equiv \overline{V_1} \pmod{1},$$

and

$$F_{\mu+1}\overline{V} \equiv V_2 \pmod 1 .$$

Thus $L_{2\mu+1}$ contains the lattice generated by $\overline{V_1}$ and $\overline{V_2}$.

To prove that, conversely, $L_{2\mu+1}$ is contained in the lattice generated by $\overline{V_1}$ and $\overline{V_2}$, we note that by (3),

$$-F_{-\mu-1}F_{\mu} - F_{-\mu}F_{\mu+1} = 0,$$

while by (2) and (3),

$$F^2_{-\mu-1} + F^2_{-\mu} = F_{2\mu+1} .$$

Hence

$$-F_{-\mu-1}\overline{V_1} - F_{-\mu}\overline{V_2} = \overline{e_2} .$$

On the other hand, by (3) and by (2) with $m = -\mu$ and with $n = \mu$ and $n = -\mu - 1$, respectively, we find

$$F_{-\mu}F_{\mu} + F_{1-\mu}F_{\mu+1} = F_1 = 1$$

and

$$-F_{-\mu}F_{-\mu-1} = F_{1-\mu}F_{-\mu} = F_{2\mu} ,$$

so that

$$F_{-\mu}\overline{V_1} + F_{1-\mu}\overline{V_2} = \overline{V} .$$

Thus $\overline{V_1}$ and $\overline{V_2}$ generate the same lattice as $\overline{V}$ and $\overline{e_2}$, that is the lattice $L_{2\mu+1}$.

Since $\overline{V_1}$ and $\overline{V_2}$ are orthogonal and of equal length, $L_{2\mu+1}$ forms a grid of squares with sides inclined to the axes of coordinates. It will be seen that this grid is invariant with respect to rotations preserving the unit square.

Since clearly such rotations transform the grid into a parallel one, it suffices to show that a rotation by a right angle about the centre of the square transforms at least one lattice point into a lattice point. Now the point $\overline{V}$ of $L_{2\mu+1}$ is transformed by such a rotation into the point

$$<F_{2\mu-1}F_{2\mu+1}^{-1}, F_{2\mu+1}^{-1}> \equiv F_{2\mu-1}\overline{V} \pmod 1 ,$$

since, by (2) with $m = 1 - 2\mu$, $n = 2\mu$,

$$F_{2\mu-1}F_{2\mu} \equiv 1 \pmod{F_{2\mu+1}} .$$

Further rotations transform the point in question into

$$<1 - F_{2\mu-1}^{-1}, F_{2\mu-1}F_{2\mu+1}^{-1}> \text{ and } <F_{2\mu}F_{2\mu+1}^{-1}, 1 - F_{2\mu+1}^{-1}> .$$

These points form a square with vertices close to the sides of the unit square, but it does not follow that the sides of this new square are contained in the grid formed by $L_{2\mu+1}$. It is so if, and only if, μ is even, and this can best be seen as follows.

By (I_{25}) in [6] with $n = \mu$, $p = 1$, or by (I_{10}), we have

$$F_{\mu+1}^2 - F_{\mu-1}^2 = F_{2\mu}, \tag{4}$$

and by (I_{19}) in the same book, with $n = \mu - 1$, $k = 2$,

$$F_{\mu-1}^2 = F_{\mu+1}F_{\mu-3} + (-1)^{\mu+1} .$$

Hence

$$F_{\mu+1}(F_{\mu+1} - F_{\mu-3}) = F_{2\mu} - 1$$

when μ is even. It follows that in this case, the abscissa of the point

$$<F_{2\mu-1}F_{2\mu+1}^{-1},\ F_{2\mu+1}^{-1}> + (F_{\mu+1} - F_{\mu-3})\overline{V_2}$$

is

$$(F_{2\mu-1} + F_{2\mu} - 1)F_{2\mu+1}^{-1} = 1 - F_{2\mu+1}^{-1} .$$

Since no pair of points $L_{2\mu+1}$ in the unit square can have the same abscissa, this is necessarily the point

$$<1 - F_{2\mu+1}^{-1},\ F_{2\mu-1}F_{2\mu+1}^{-1}>$$

which was mentioned above as another vertex of the square in question. Let it be noted in passing that there are $4(F_{\mu+1} - F_{\mu-3})$ points of $L_{2\mu+1}$ on the perimeter of this square. In Fig. 1, this square is shown by thicker lines.

When μ is odd, the ordinate of $\overline{V_2}$ is negative. Consequently, it is along $\overline{V_1}$ that we should attempt to move from

$$<F_{2\mu-1}F_{2\mu+1}^{-1},\ F_{2\mu+1}^{-1}>$$

to

$$<1 - F_{2\mu+1}^{-1},\ F_{2\mu-1}F_{2\mu+1}^{-1}>.$$

But by (2) with $m = \mu$, $n = \mu - 1$,

$$F_{\mu}(F_{\mu+1} + F_{\mu-1}) = F_{2\mu} .$$

Hence if we add $(F_{\mu+1} + F_{\mu-1})\overline{V_1}$ to the starting point, we obtain a point of abscissa

$$(F_{2\mu-1} + F_{2\mu})F_{2\mu+1}^{-1} = 1 ,$$

which shows that, for $\mu > 1$, adding multiples of $\overline{V_1}$ to our starting point cannot produce a point of abscissa $1 - F_{2\mu+1}^{-1}$. Thus indeed the square in question is not formed by the grid; this is illustrated in Fig. 3, where this square is marked in dotted lines.

When n is even, say $n = 2\mu$, the vectors

$$\overline{V_1'} = \langle F_\mu F_{2\mu}^{-1}, F_{-\mu} F_{2\mu}^{-1} \rangle \quad \text{and} \quad \overline{V_2'} = \langle F_{\mu+1} F_{2\mu}^{-1}, F_{1-\mu} F_{2\mu}^{-1} \rangle$$

form a basis of $L_{2\mu}$. Indeed, writing $\overline{V}$ as

$$\langle F_{2\mu}^{-1}, F_{1-2\mu} F_{2\mu}^{-1} \rangle ,$$

we find, by easy applications of (2),

$$\text{(5)} \qquad F_\mu \overline{V} \equiv \overline{V_1'} \pmod 1 \quad \text{and} \quad F_{\mu+1} \overline{V} \equiv \overline{V_2'} \pmod 1 .$$

On the other hand, by (2) and (3), we find

$$\text{(6)} \qquad F_{-\mu} \overline{V_1'} + F_{1-\mu} \overline{V_2'} = \overline{V} \quad \text{and} \quad -F_{-\mu-1} \overline{V_1'} - F_{-\mu} \overline{V_2'} = \overline{e_2}$$

Now (5) and (6) show that the lattice generated by $\overline{V_1'}$ and $\overline{V_2'}$ is nothing else but $L_{2\mu}$.

However, $\overline{V_1'}$ and $\overline{V_2'}$ are not orthogonal, their scalar product being

$$(F_\mu F_{\mu+1} + F_{-\mu} F_{1-\mu}) F_{2\mu}^{-2} = F_\mu^2 F_{2\mu}^{-2} ,$$

since

$$F_\mu F_{\mu+1} + F_{-\mu} F_{1-\mu} = F_\mu (F_{\mu+1} - F_{\mu-1}) = F_\mu^2 .$$

When $n = 2\mu$, L_n does not form a square grid. The determinant of $L_{2\mu}$ being equal to $F_{2\mu}^{-1}$, this would indeed require a pair of orthogonal vectors of lengths $F_{2\mu}^{-\frac{1}{2}}$ each. The cases of $\mu = 2$ and $\mu = 3$ being trivial, assume $\mu > 3$. In our search for the required vectors, we can dismiss those which have a coordinate equal to, or bigger than, $F_{\mu+1}F_{2\mu}^{-1}$ in absolute value, since by (4), their length exceeds $F_{2\mu}^{-\frac{1}{2}}$. All linear combinations $\alpha V_1' + \beta V_2'$ in which $\beta \neq 0$ are thereby excluded because of their abscissa if $\alpha\beta \geq 0$ and because of their ordinate if $\alpha\beta < 0$. There remain the multiples of $\overline{V_1'}$. But ((78) in [1])

$$F_{2\mu} = F_\mu^2 + 2F_\mu F_{\mu+1} \; ; \tag{7}$$

This identity can also be deduced from (4) noting that

$$F_{\mu+1}^2 - F_{\mu-1}^2 = F_\mu^2 + 2F_\mu F_{\mu-1} \; .$$

It follows from (7) that when $\mu > 3$, we have $F_{2\mu} > 2F_\mu^2$, so that $\overline{V_1'}$ is too short for our purposes, while $2\overline{V_1'}$ has an abscissa exceeding $F_{\mu+1}$, so that it is too long.

The figures representing the lattices with $F_5 = 5$, $F_6 = 8$, and $F_7 = 13$ points show that in each case there is a relatively large number of lattice points on a straight line passing through the origin. In order to evaluate this number in general, we must again distinguish two cases according to the parity of n.

If $n = 2\mu + 1$, one of the vectors $\overline{V_1}$ and $\overline{V_2}$ has both its coordinates positive, one being equal to

$$F_\mu F_{2\mu+1}^{-1}$$

and the other to

$$F_{\mu+1}F_{2\mu+1}^{-1} \, .$$

The origin being a lattice point, it follows that the line passing through it and parallel to the vector in question contains

$$[F_{2\mu+1}F_{\mu+1}^{-1}] + 1$$

lattice points, where, as usual, $[x]$ denotes the biggest integer not exceeding x. This number is easily determined as follows: By (2) and by (I_{13}) in [6], we have

$$F_{\mu+1}^2 + F_\mu^2 = F_{2\mu+1}$$

and

$$F_{\mu+1}F_{\mu-1} - F_\mu^2 = (-1)^\mu \quad .$$

Hence

$$F_{\mu+1}(F_{\mu+1} + F_{\mu-1}) = F_{2\mu+1} + (-1)^\mu \quad ,$$

and consequently the number of lattice points on the line in question is

$$F_{\mu+1} + F_{\mu-1} + \frac{1}{2}(1 + (-1)^{\mu+1}) \quad .$$

When $n = 2\mu$, one of the vectors $\overline{V_1'}$ and $\overline{V_2'}$ has its coordinates equal to $F_\mu F_{2\mu}^{-1}$ and $F_{\mu+1}F_{2\mu}^{-1}$ in either order. But by (2),

$$F_{\mu+1}F_\mu + F_\mu F_{\mu-1} = F_{2\mu} ,$$

and

$$F_\mu F_{1-\mu} + F_{\mu+1}F_{2-\mu} = F_2 \quad ,$$

or

$$F_\mu F_{\mu-1} - F_{\mu+1}F_{\mu-2} = (-1)^\mu \quad .$$

Hence

$$F_{\mu+1}(F_{\mu} + F_{\mu-2}) = F_{2\mu} + (-1)^{\mu+1} ,$$

which shows that the line through the origin parallel to the vector mentioned above contains

$$F_{\mu} + F_{\mu-2} + \frac{1}{2}(1 + (-1)^{\mu})$$

points of L_2 .

Thus, in either case, there is a line, say 1, which contains a number of points of L_n in the unit square which is of the order $\sqrt{F_n}$. The importance of this fact is a consequence of the following considerations.

Let S be any finite set of, say p points of the unit square

$$0 \leq x < 1; \quad 0 \leq y < 1 ,$$

and denote by $\nu(x,y)$ the number of points of S with coordinates smaller than, or equal to, x and y, respectively. The function

$$g(x,y) = p^{-1}\nu(x,y) - xy$$

can be regarded as describing the equidistribution of S over Q^2. If a single number is required to characterize this equidistribution, it can be obtained by taking any of the plausible norms of g. In particular, it has been proposed ([1], [11]) to call

$$D(S) = \sup_{\langle x,y\rangle \in Q^2} |g(x,y)|$$

the extreme discrepancy of S in order to distinguish it from other possible norms of g; the previously used term is simply discrepancy. If f is any function of bounded variation in the sense of Hardy and Krause over the closure of Q^2, then its integral over Q^2 is approximately equal to the average value of f over the points of S, the absolute value of the error not exceeding VD(S), where V is the sum of the two-dimensional variation of f over Q^2 in the sense of Vitali and of the (one-dimensional) variations of f(x, 1) and f(1, y) over [0,1] ([3]; for a slight sharpening of this result, see [9]).

Thus sets of points with low extreme discrepancies provide us with a method of numerical integration over Q^2 even when the integrand cannot be expanded into a uniformly convergent Fourier series. In the case of the set of points determined by the multiples modulo 1 of $\overline{V}$, the extreme discrepancy has been shown to be smaller than $(7/6)F_n^{-1} \log (15 F_n)$ [9]. However, the integrals we may want evaluated numerically do not necessarily lend themselves to a reduction to integrals over Q^2. It might seem that, if the domain of integration, say D, is contained in Q^2, we could replace the integrand by a function equal to it in D, and to O outside, integrating this new function over the whole of Q^2; this is what would be likely to be done if the Monte Carlo method were applied. The difficulty lies in the fact that in general the new integrand will not be of bounded variation in the sense of Hardy and Krause over Q^2, however regular the initially given function might be, and indeed even if it is a constant, consequently Hlawka's theorem cannot be applied to this situation.

The sets of points which have been described above show that even when the integrand is a constant, say 1, and the domain of integration is, for instance, convex, the integration error can be of the order of $\sqrt{F_n}$ instead of that of $F_n^{-1} \log F_n$. To see this, it suffices to consider two lines, say l_1 and l_2, on opposite sides of l, parallel to it, and arbitrarily near to each other. Let D_i be the part of Q^2 above l_i $(i = 1, 2)$. Then the integrals over D_1 and D_2 will differ arbitrarily little, while the numerical values found for them will differ by the number of points of L_n on l divided by F_n, so that for one at least of the two integrals the error of the computation will indeed be of the order of $F_n^{-\frac{1}{2}}$.

When $n = 2\mu + 1$, μ being even, by slightly expanding, or contracting the previously discussed square formed by the grid, it is possible to obtain a similar example of an integration domain leading to errors of the order of $F_n^{-\frac{1}{2}}$ when applying the method in question; of course, many variations on this theme are possible.

All these considerations extend to an arbitrary number of dimensions, and the phenomenon illustrated by the lattice L_n in two dimensions becomes even more accentuated as the number of dimensions increases. The present author, impressed by the pattern of L_n, proved [10] that if a set of p points of the s-dimensional unit interval Q^s is generated by a good lattice point, then there exists an s-1-dimensional linear variety (or hyperplane, to use a rather old-fashioned terminology) which forms with Q^s an intersection containing more than $(4s)^{1-s}p^{1-1/s}$ points of the set. This leads again in an obvious way to an example of convex domains (actually simple polyhedral domains) having the property that by integrating over them, by Hlawka's method, arbitrarily regular functions, which could even reduce to constants, we are always liable to commit errors of the order of $p^{-1/s}$.

This contrasts sharply with the error committed when integrating over the whole of Q^s a function of bounded variation in the sense of Hardy and Krause and using the same set of points; the discrepancy is then $\overline{O}$ $((\log p)^s/p)$ 4 , and the integration error is of the same order of magnitude. With $s > 2$, in the former case the bound obtained for the error (and this is a sharp bound!) is much less favorable than that which is practically claimed by the Monte Carlo method, namely $\overline{O}(p^{-\frac{1}{2}})$.

The irrelevance of some traditional tests applied to so-called random numbers, or to pseudo-random numbers with a view to applications to Monte-Carlo integration was discussed in detail by the present author 11 ; the considerations adduced here show that when the domain of integration is not reduced to a multidimensional unit interval, even discrepancy tests in the appropriate number of dimensions do not guarantee the success of Monte Carlo, although, naturally, nobody can be denied the right of hoping for the best.

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