SOME GENERALIZED FIBONACCI IDENTITIES

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1. Let

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, \; F_1$$

and define

$$(1.1) \qquad f_n(x) = \sum_{k=0}^{\infty} F_{n+k}\, x^k/k! \quad (n = 0,1,2,\cdots) \;,$$

so that

$$f'_n(x) = f_{n+1}(x), \quad f_{n+1}(x) = f_n(x) + f_{n-1}(x) \;.$$

Note that $f_n(0) = F_n$.

In a recent paper [1] in this Quarterly, Elmore has pointed out that many of the familiar formulas involving the Fibonacci numbers F_n can be extended to the functions $f_n(x)$. For example, the identities

$$F_{m-1}F_{m+1} - F_m^2 = (-1)^m, \quad F_{2m-1} = F_{m-1}^2 + F_m^2$$

become

$$f_{m-1}(x)f_{m+1}(x) - f_m^2(x) = (-1)^m\, e^x$$

and

$$f_{2m-1}(2x) = f_{m-1}^2(x) + f_m^2(x) \;,$$

respectively; the identity

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$$F_{m+n} = F_{m-1}F_n + F_mF_{n-1}$$

becomes

$$f_{m+n}(u + v) = f_{m-1}(u)f_n(v) + f_m(u)f_{n+1}(v) .$$

The formulas

$$f_m(u)f_n(v) = (-1)^r[f_{m+r}(u)f_{n+r}(v) - F_rf_{m+n+r}(u + v)] , \tag{1.2}$$

$$F_mf_m(v - u)e^u = (-1)^r[f_{m+r}(u)f_{n+r}(v) - f_r(u)f_{m+n+r}(v)] , \tag{1.3}$$

seem particularly striking. Elmore remarks that they may be special cases of a more general formula in which no capital F's appear. This is indeed the case, as we shall show below. The formula

$$f_{m+r}(u)f_{n+r}(v) - f_r(x)f_{m+n+r}(y) = (-1)^r e^x f_m(u - x)f_n(v - x) , \tag{1.4}$$

where $x + y = u + v$, reduces to (1.2) when $x = 0$ and reduces to (1.3) when $u = x$, $v = y$.

2. Since it is no more difficult, we consider the following slightly more general situation. Let

$$H_{n+1} = pH_n - qH_{n-1}, \quad H_0 = 0, \quad H_1 = 1 , \tag{2.1}$$

and define

$$h_n(x) = \sum_{k=0}^{\infty} H_{n+k}\, x^k/k! \quad (n = 0,1,2,\cdots) , \tag{2.2}$$

so that

$$h'_n(x) = h_{n+1}(x), \quad h_{n+1}(x) = ph_n(x) - qh_{n-1}(x) .$$

Corresponding to (1.4), we shall show that

$$(2.3)\qquad h_{m+r}(u)h_{n+r}(v) - h_r(x)h_{m+n+r}(y) = q^r e^{px} h_m(u - x)h_n(v - x) ,$$

provided $x + y = u + v$.

Let α, β denote the roots of $x^2 - px + q = 0$. Then

$$(2.4)\qquad H_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} ,$$

and (2.2) implies

$$(2.5)\qquad h_n(x) = \frac{1}{\alpha - \beta} (\alpha^n e^{\alpha x} - \beta^n e^{\beta x}) .$$

It follows at once from (2.5) that

$$(2.6)\qquad \sum_{k=0}^{\infty} h_{n+k}(x)\, y^k/k! = h_n(x + y) .$$

Consider

$$\sum_{m,n=0}^{\infty} \{h_{m+r}(u)h_{n+r}(v) - h_r(x)h_{m+n+r}(y)\} \frac{z^m w^n}{m!\,n!}$$

$$\begin{aligned}
&= h_r(u + z)h_r(v + w) - h_r(x)h_r(y + z + w)\\
&= (\alpha - \beta)^{-2} \{(\alpha^r e^{\alpha(u+z)} - \beta^r e^{\beta(u+z)})(\alpha^r e^{\alpha(v+w)} - \beta^r e^{\beta(v+w)}\\
&\qquad - (\alpha^r e^{\alpha x} - \beta^r e^{\beta x})(\alpha^r e^{\alpha(y+z+w)} - \beta^r e^{\beta(y+z+w)})\}\\
&= (\alpha - \beta)^{-2}\{\alpha^{2r} e^{\alpha(u+v+z+w)} + \beta^{2r} e^{\beta(u+v+z+w)}\\
&\qquad - q^r(e^{\alpha(u+z)+\beta(v+w)} + e^{\beta(u+z)+\alpha(v+w)})\}\\
&\qquad - (\alpha - \beta)^{-2}\{\alpha^{2r} e^{\alpha(x+y+z+w)} + \beta^{2r} e^{\beta(x+y+z+w)}\\
&\qquad - q^r(e^{\alpha x+\beta(y+z+w)} + e^{\beta x+\alpha(y+z+w)})\} .
\end{aligned}$$

If we take $x + y = u + v$, this reduces to

$$(\alpha - \beta)^{-2} q^r \{ e^{\alpha x + \beta(y+z+w)} + e^{\beta x + \alpha(y+z+w)} - e^{\alpha(u+z)+\beta(v+w)} - e^{\beta(u+z)+\alpha(v+w)} \}$$

$$= (\alpha - \beta)^{-2} q^r e^{px} \{ e^{\alpha(-x+y+z+w)} + e^{\beta(-x+y+z+w)} - e^{\alpha(-x+u+z)+\beta(-x+v+w)} - e^{\alpha(-x+v+w)+\beta(-x+u+z)} \}$$

$$= (\alpha - \beta)^{-2} q^r e^{px} (e^{\alpha(-x+v+w)} - e^{\beta(-x+v+w)}) \cdot (e^{\alpha(-x+u+z)} - e^{\beta(-x+u+z)}) .$$

In view of (2.5), we have therefore proved

$$(2.7) \qquad \sum_{m,n=0}^{\infty} \{ h_{m+r}(u) h_{n+r}(v) - h_r(x) h_{m+n+r}(y) \} \frac{z^m w^n}{m!\, n!} = q^r e^{px} h_0(-x + u + z) h_0(-x + v + w) .$$

But by (2.6),

$$h_0(-x + u + z) h_0(-x + v + w) = \sum_{m,n=0}^{\infty} h_m(-x + u) h_n(-x + v) \frac{z^m w^n}{m!\, n!} .$$

Equating coefficients of $z^m w^m$ we immediately get (2.3).

3. Analogous to (2.2), we may define

$$(3.1) \qquad h_n^{\star}(x) = h_n^{\star}(x, \lambda) = \sum_{k=0}^{\infty} H_{n+k} \binom{x}{k} \lambda^k .$$

Then

$$h_n^{\star}(0) = H_n$$

and

$$h^\star_{n+1}(x) = h^\star_n(x) + h^\star_{n-1}(x) .$$

Moreover,

$$\mathop{\Delta}_x h^\star_n(x) = h^\star_n(x+1) - h^\star_n(x) = \sum_{k=1}^{\infty} H_{n+k}\binom{x}{k-1}\lambda^k ,$$

so that

$$\mathop{\Delta}_x h^\star_n(x) = \lambda h^\star_{n+1}(x) .$$

Clearly the series in the right member of (3.1) converges for sufficiently small $|\lambda|$.

It follows at once from (2.4) and (3.1) that

$$h^\star_n(x) = \frac{1}{\alpha - \beta}\,[\alpha^n(1 + \lambda\alpha)^x - \beta^n(1 + \lambda\beta)^x]. \tag{3.2}$$

We have also

$$h^\star_n(x + y) = \sum_{k=0}^{\infty} h^\star_{n+k}(x)\binom{y}{k} . \tag{3.3}$$

Now by (3.2),

$$\begin{aligned}
&-qh^\star_{m-1}(u)h^\star_n(v) + h^\star_m(u)h^\star_{n+1}(v)\\
&= (\alpha - \beta)^{-2}\{-q[\alpha^{m-1}(1 + \alpha\lambda)^u - \beta^{m-1}(1 + \beta\lambda)^v][\alpha^n(1 + \alpha\lambda)^u - \beta^n(1 + \beta\lambda)^u]\\
&\qquad + [\alpha^m(1 + \alpha\lambda)^u - \beta^m(1 + \beta\lambda)^v][\alpha^{n+1}(1 + \alpha\lambda)^v - \beta^{n+1}(1 + \beta\lambda)^v]\}\\
&= (\alpha - \beta)^{-2}\{(-q\alpha^{m+n-1} + \alpha^{m+n+1})(1 + \alpha\lambda)^{u+v}\\
&\qquad + (-q\beta^{m+n-1} + \beta^{m+n+1})(1 + \beta\lambda)^{u+v}\\
&\qquad - (q\alpha^{m-1}\beta^n + \alpha^m\beta^{n+1})(1 + \alpha\lambda)^u(1 + \beta\lambda)^v\\
&\qquad - q(\alpha^n\beta^{m-1} + \alpha^{n+1}\beta^m)(1 + \alpha\lambda)^v(1 + \beta\lambda)^u\} .
\end{aligned}$$

Since $\alpha\beta = q$ and

$$\alpha^2 - q = \alpha(\alpha - \beta), \quad \beta^2 - q = -\beta(\alpha - \beta),$$

this reduces to

$$(\alpha - \beta)^{-1} \; \alpha^{m+n}(1 + \alpha\lambda)^{u+v} - \beta^{m+n}(1 + \beta\lambda)^{u+v} .$$

We have therefore,

$$h^\star_{m+n}(u + v) = -qh^*_{m-1}(u)h^\star_n(v) + h^\star_m(u)h^\star_{n+1}(v) . \tag{3.4}$$

Similarly, we have

$$h^\star_{m-1}(u)h^\star_{m+1}(u) - h^{\star 2}_m(u) = -q^{m-1}(1 + p\lambda + q\lambda^2)^u . \tag{3.5}$$

Finally, corresponding to (2.3), we have

$$\begin{aligned} h^\star_{m+r}(u)h^\star_{n+r}(v) - h^\star_r(x)h^\star_{m+n+r}(y) \\ = q^r h_m(u - x)h_n(v - x)(1 + p\lambda + q\lambda^2)^x , \end{aligned} \tag{3.6}$$

provided

$$x + y = u + v .$$

REFERENCE

1. M. Elmore, "Fibonacci Functions," Fibonacci Quarterly, Vol. 5 (1967), pp. 371-382.