ON THE ENUMERATION OF CERTAIN TRIANGULAR ARRAYS

C. A. CHURCH, JR.
University of North Carolina, Greensboro, North Carolina

1. In [2], this quarterly, D. P. Roselle considered the enumeration of certain triangular arrays of integers. He obtained recurrences for these which had a Fibonacci character. In this paper, we obtain explicit formulae for the enumeration of these arrays, with a slight change of notation, and some generalizations. Although difficult to state in its full generality, it will be seen that the method of enumeration can be applied to a rather general class of arrays in a given instance.

By a lattice point in the plane is meant a point with integral coordinates, non-negative unless stated otherwise. By a path (lattice path) is meant a minimal path via lattice points, taking unit horizontal and vertical steps.

It is well known that the number of paths from $(0,0)$ to (p,q) is

$$\binom{p+q}{p},$$

and there are

$$\binom{p+q-1}{p-1}$$

which start with a unit horizontal step.

With $[x]$ the greatest integer $\leq x$, note that

$$\left[\frac{n-1}{m}\right] + 1 = \left[\frac{n+m-1}{m}\right] = \begin{cases} \left[\frac{n}{m}\right] & \text{if } m \mid n, \\ \left[\frac{n}{m}\right] + 1 & \text{if } m \nmid n. \end{cases}$$

2. To fix the idea, we take the simplest case first.

For integral $n \geq 1$, $m \geq 1$, consider the triangular array of integers $a_{ij} = 0$ or 1, $i = 1, 2, \cdots, [(n-1)/m] + 1$ and $j = (i-1)m + 1, \cdots, n$, with the restrictions $1 \geq a_{i,j} \geq a_{i+1,j} \geq 0$ and $1 \geq a_{i,j} \geq a_{i,j+1} \geq 0$, [2, §2]. Let $f(n;m)$ denote the number of these arrays.

For example, with $m = 3$ and $n = 11$, the arrays have the shape

$$\begin{array}{r} xxxxxxxxxxx \\ xxxxxxxx \\ xxxxx \\ xx \end{array}$$

$f(11;3) = 88$, and a typical array is

$$\begin{array}{ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ & & & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 \end{array} .$$

It follows from the restrictions on the a_{ij} that

$$f(n;m) = f(n-1;m) + f(n-m;m), \qquad n > m, \tag{2.1}$$

according as $a_{1,n} = 0$ or 1 with the initial conditions

$$f(n;m) = n + 1, \qquad 1 \leq n \leq m,$$

We adjoin the conventional value $f(0;m) = 1$. Compare [2, (1.1) and (1.3)].

We show directly that

$$f(n;m) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \binom{n-(m-1)(k-1)}{k} . \tag{2.2}$$

It is easy to show that (2.2) satisfies (2.1) and the initial conditions.

As in [2], we note in passing that $f(n;1) = 2^n$; and $f(n;2) = F_{n+2}$, the Fibonacci numbers.

To get (2.2) directly, note first that there is only one array, consisting of all zeros, if $a_{1,1} = 0$. For each $k \geq 1$, we get a new set of arrays in each case where at least $a_{k,(k-1)m+1} = 1$ and all the $a_{k+1,j} = 0$. This adjoins an artificial row of zeros in the case of the last row, but it does not change the count. In view of the restrictions on the a_{ij}, we need only consider the rectangular arrays

$$\begin{matrix} a_{1,(k-1)m+1} & \cdots & a_{1,n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdots & \cdot \\ a_{k,(k-1)m+1} & \cdots & a_{k,n} \end{matrix}$$

with

$$a_{i,(k-1)m+1} = 1, \quad i = 1,2,\cdots,k \ .$$

These arrays correspond in a one-one way with the

$$\binom{n - (k - 1)(m - 1)}{k} \tag{2.3}$$

paths from $((k - 1)m, 0)$ to (n, k) which start with a unit horizontal step as follows: for each path, place 1's in the unit squares above and to the **left** (northwest side) of the path and 0's in the unit squares below and to the right (southeast) of the path. For example, see the blocked out section of the preceding example. Sum (2.3) over

$$k = 0, 1, \cdots, [(n - 1)/m] + 1$$

to get (2.2).

The preceding result also enumerates one-line arrays

$$n_1 n_2 \cdots n_s, \qquad s = [(n - 1)/m] + 1 \ , \tag{2.4}$$

where $n_1 = n$ and $0 \le n_{j+1} \le n_j - m$. Compare [2, §4]. This is seen by taking row sums of the a_{ij} in the case that all the $a_{1,j} = 1$. This is precisely the original problem with n replaced by $n - m$. That is, there are

$f(n - m; m)$ such one-line arrays. These may also be thought of as combinations of the first n natural numbers written in descending order; compare [1, p. 222, problem 1].

If, in (2.4), we only require $0 \le n_1 \le n$, we have the obvious additional one-line arrays.

The arrays above had a row depth of one. It is easy to expand the problem to the case of row depth $p \geq 1$. That is, let $f(n; m|p)$ denote the number of arrays of $a_{ij} = 0$ or 1, where $i = 1, 2, \cdots, p[(n + m - 1)/m])$ and for $i = (k - 1)p + s$ ($s = 1, 2, \cdots, p$ and $k = 1, 2, \cdots, [(n + m - 1)/m]$) $j = (k - 1)m + 1, \cdots, n$. We have the same restrictions as before.

For example, with $m = 3$, $n = 11$, and $p = 2$, the arrays have the shape

```
XXXXXXXXXXX
XXXXXXXXXXX
   XXXXXXXX
   XXXXXXXX
      XXXXX
      XXXXX
         XX
         XX  ,
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and $f(11; 3/2) = 871$.

We shall find in §3 that

$$(2.5) \qquad f(n;m|p) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \sum_{s=1}^{p} \binom{n - (k - 1)(m - p) + s - 1}{s + (k - 1)p}$$

is a special case of a more general class of arrays. With obvious notational changes, the case $p = m$ of (2.5) is Roselle's $N_k(n,k) = N_n(k)$ [2, (1.11) and (3.10)]. That is, $f(n;k\,k) = N_k(n,k)$. Roselle's (3.10) gives the representation

$$(2.6) \qquad f(n;m|m) = \frac{1}{m} \sum_{j=0}^{m-1} \left\{(\rho^{-j} + 1)^m - 1\right\}(\rho^j + 1)^n ,$$

where ρ is a primitive m^{th} root of unity. Now (2.6) and

(2.7)
$$f(n;m|m) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \sum_{s=1}^{m} \binom{n+s-1}{s+(k-1)m}$$

are the same. To see this first apply the binomial identity

(2.8)
$$\sum_{k=0}^{n} \binom{x+k}{r} = \binom{x+n+1}{r+1} - \binom{x}{r+1}$$

to the inner sum of (2.7) to get

(2.9)
$$f(n;m\ m) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \left\{ \binom{n+m}{km} - \binom{n}{(k-1)m} \right\} .$$

Next (2.6) can be rewritten as

$$f(n;m|m) = \frac{1}{m} \sum_{j=0}^{m-1} (\rho^j + 1)^{m+n} - \frac{1}{m} \sum_{j=0}^{m-1} (\rho^j + 1)^n .$$

But this is just another way of writing (2.9), c.f. [1, p. 41, problem 7].

Application of (2.8) to (2.5) yields the form

$$f(n;m|p) = \sum_{k=0}^{\left[\frac{n+m-1}{m}\right]} \left\{ \binom{n-(k-1)(m-p)+p}{kp} - \binom{n-(k-1)(m-p)}{(k-1)p} \right\} .$$

which can also be gotten by a direct combinatorial argument.

3. With the same restrictions as before on the a_{ij}, we consider a slightly more general array. In this case, the indentations will still be $m \geq 1$, the first block will have row depth $q \geq 1$, and the successive blocks will have row depth $p + (k - 1)$ respectively, $k = 2, 3, \cdots, [(n - 1)/m] + 1$, $p \geq 1$, $\alpha \geq 0$.

As before, the case $a_{11} = 0$ contributes only one array, all zeros. For each of the cases $a_{s,1} = 1$, $s = 1, 2, \cdots, q$, and $a_{s+1,1} = 0$ there are

$$\binom{n + s - 1}{s}$$

arrays — corresponding to the paths from $(0,0)$ to (n,s) with an initial horizontal step. Thus the q by n rectangle contributes

$$1 + \sum_{s=1}^{q} \binom{n + s - 1}{s} = \binom{n + q}{n} \tag{3.1}$$

arrays. Note that this rectangle always gives the initial conditions; compare [2, (1.4)]. For the count on the remaining blocks, we consider the case of

$$a_{q+(k-2)p+\binom{k-1}{2}\alpha+s,\ (k-1)m+1} = 1 \quad ,$$

where $k \geq 2$ and $s = 1, 2, \cdots, p + (k - 1)\alpha$ and the next row is all zeros, in each case, these corresponding to the

$$\binom{n + s - 1 - (k - 1)(m - p) + \binom{k-1}{2}\alpha + q - p}{n - (k - 1)m - 1} \tag{3.2}$$

paths from $((k - 1)m, 0)$ to

$$\left(n, q + (k - 2)p + \binom{k-1}{2}\alpha + s\right)$$

with an initial horizontal step.

Thus the total number of arrays is

$$(3.3)\quad 1 + \sum_{s=1}^{q}\binom{n+s-1}{s} + \sum_{k=2}^{\left[\frac{n+m-1}{m}\right]} \sum_{s=1}^{p+(k-1)\alpha} \binom{n+s-1(k-1)(m-p)+\binom{k-1}{2}\alpha+q-p}{n-(k-1)m-1}$$

We note that (3.3) can be simplified by replacing the first two terms by the right member of (3.1) and the inner sum by applying (2.8).

We note some special cases of (3.3). First the case $\alpha = 0$ is, with obvious notational changes, Roselle's $N_j(m,k)$ [2, §3]. If, in addition, we take $p = q$, Eq. (3.3) reduces to (2.5), which in turn reduces to (2.2) for $p = 1$.

As we remarked at the beginning, it is now quite clear that the description of a very general case of these types of arrays would be quite complicated. However, it is clear that in any given instance, the method used above is easy to apply.

REFERENCES

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TAKE-AWAY GAMES

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