MAGIC SQUARES CONSISTING OF PRIMES IN A. P.

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Let a be the first prime in A. P. (not necessarily positive), d the common difference, s the last prime in A. P., n the number of primes in A. P., and let the residue r be the smallest positive integer such that $a \equiv r \pmod{d}$; if we keep a constant and increase d, we may speak of a search limit on d, designated by a_d; if we keep d constant and increase a and s, we may speak of a search limit on s, designated by s_n.

The standard magic square of order 3 with elements 1, 3, ⋯, 9 and center element c = 5, may be defined as

8	1	6
3	5	7
4	9	2

and the standard magic square of order 4 with elements 1, 2, ⋯, 16 and center square

8	5
9	12

whose sum is the magic constant C = 34, may be defined as

1	15	14	4
10	8	5	11
7	9	12	6
16	2	3	13

A magic square consisting of primes in A. P. is formed by letting an increasing sequence of 9 or 16 primes in A. P. occupy the locations corresponding to the elements 1, 2, ⋯, 9 or 1, 2, ⋯, 16, respectively, in the standard magic squares. To avoid ambiguity, let $|a| < s$.

The disposition of the prime factors and their powers in d to achieve maximum n was the topic for about 140 years and is reflected in four classical papers by Edward Waring (1734-1798) [15, p. 379], Peter Barlow (1776-1862) [1, p. 67], Moritz Cantor (1829-1920) [3], and Artemas Martin (1835-1918) [8]. In 1910, E. B. Escott (1868-1946) [5, p. 426 and 2, p. 221] found a string of 11 primes in A. P. yielding one case of almost 16 primes in A. P. (except for two composite elements)

-1061	1879	1669	-431
829	409	-13.17	1039
199	619	1249	-11
2089	-23.37	-641	1459

with C = 2056 .

This sequence is treated with loving care in [9, pp. 152-54] and [14]. The concept of magic squares consisting of primes in A. P. may be extended to more or less magic squares of reversible primes [4].

The modern period of this interesting subject starts in 1944 with a paper by Victor Thebault (1882-1960) [13]. In 1958, V. A. Golubev (Kouvshinovo, USSR) [10, p. 348 and 6, p. 120] found a string of 12 primes in A. P. yielding two cases of almost 16 primes in A. P. (except for two composite elements

23143	443563	413533	113233
293413	233353	143263	323443
203323	263383	353473	173293
23.59.349	53173	83203	17^2.1327

53173	23.59.349	443563	143263
323443	263383	173293	353473
233353	293413	17^2.1327	203323
503623	83203	133233	413533

with C = 993472 and C = 1113592 .

In 1963, V. N. Seredinskij (Moscow, USSR) [11, p. 121 and 12, p. 48] found a string of 14 primes in A.P., yielding four cases of almost 16 primes in A.P. (except for two composite elements).

-19.23.401	665603	605543	4943
365303	245183	65003	425363
185123	305243	485423	125063
725663	-149.773	-55117	545483

-149.773	725663	665603	65003
425363	305243	125063	485423
245183	365303	545483	185123
17.46219	-55117	4943	605543

with C = 1100852 and with C = 1341092,

-55117	17.46219	725663	125063
485423	365303	185123	545483
305243	425363	605543	245183
37.22859	4943	65003	665603

4943	37.22859	17.46219	185123
545483	425363	245183	605543
365303	485423	665603	305243
905843	65003	125063	725663

with C = 1581332 and C = 1821572 .

The magnitude of the last six C could be lowered essentially, when in 1966, the author found a string of 10 primes in A. P. yielding one case of almost 16 primes in A. P. (except for two composite elements) with C = 30824, and in 1967 [7], found a string of 12 primes in A. P. yielding the only known case (so far) of almost 16 primes in A. P. (except for one composite element) with C = 857548.

-9619	22721	20411	-2689
11171	6551	-379	13.17.61
4241	8861	15791	1931
25031	-7309	-4999	23.787

110437	304477	290617	152017
235177	207457	165877	249037
193597	221317	262897	179737
318337	124297	138157	13.61.349

with C = 30824 and with C = 857548 .

Time may be near to find the first entire sequence of 16 primes in A. P.

Even more fascinating is the magic square of order 3. One is tempted to ask: Given any c, is there always a magic square of 9 primes in A. P. belonging to this c? This question may once be answered in the positive. For c = 5 and 7, Golubev found

41	-43	17
-19	5	29
-7	53	-31

and

97	-113	37
-53	7	67
-23	127	-83

Nevertheless, the center element may not be unique, since we have also the magic square

2703607	-3604793	901207
-1802393	7	1802407
-901193	3604807	-2703593

discovered recently by Seredinskij. The author found three further magic squares with low c, namely

2089	-27701	6949
-13841	19	13879
-6911	27739	-20771

127	-83	67
-23	37	97
7	157	-53

1327	-1613	487
-773	67	907
-353	1747	-1193

With the increasing scarcity of primes, one may wonder if there exists a magic square of order 3 and of primes in A. P. whose smallest element is greater than, say, 8 million. In 1968, the author found 10 primes in A. P. starting with $a = 8081737$, and yielding the magic squares

8291947	8081737	8231887
8141797	8201857	8261917
8171827	8321977	8111767

and

8321977	8111767	8261917
8171827	8231887	8291947
8201857	8352007	8141797

Or one could ask for a magic square of order 3 and of primes in A. P. whose greatest element is greater than, say, 40 million. The author found recently 9 primes in A. P. starting with 2657, ending with 49011617, and yielding the magic square

42885497	2657	30633257
12254897	24507137	36759377
18381017	49011617	6128777

All we need for further research is a list of known results of at least 9 primes in A. P. with headings d, r, a, c, s, n, and s (in million). Such a list is published in the Appendix for the first time. Compiled from the newest discoveries around the globe, the author will be pleased to keep them up to date. Two additional tables are available from the author.*

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APPENDIX

MULTIPLES OF d = 210, YIELDING AT LEAST 9 PRIMES IN A. P.
(s_z for 8^{th} term)

d	r	a	c	z	n	s_z
210	47	-373	467	1307	9	0.5
	139	3499	4339	5179	9	
	149	10859	11699	12539	9	
	179	-241	599	1439	9	
	199	-11	829	2089	11	
			1039			
			1249			
420	67	-1613	67	1747	9	0.5
	193	-647	1033	2713	9	
	317	61637	63317	64997	9	
	379	52879	54559	56659	10	
			54979			
	11.37	56267	57947	59627	9	
630	137	279857	282377	284897	9	0.5
840	97	-2423	937	4297	9	0.5
	163	6043	9403	12763	9	
	181	201781	205141	208501	9	
	11.47	103837	107197	110557	9	
	11.71	10861	14221	17581	9	

1050 on next page.

d	r	a	c	z	n	s_z
1050	443	-2707	1493	5693	9	0.5
1260	11.73	2063	7103	12143	9	0.5
1470	859	363949	369829	375709	9	0.5
	11.97	101027	106907	112787	9	
1680	227	216947	223667	230387	9	0.5
	11.71	316621	323341	330061	9	
	1093	31333	38053	44773	9	
	1487	258527	265247	271967	9	
1890	11.37	45767	53327	60887	9	0.5
	487	15607	23167	30727	9	
	31.43	194113	201673	209233	9	
	1543	-4127	3433	10993	9	
2100	13.101	34913	43313	53813	10	0.5
			45413			
	1787	176087	184487	192887	9	
	19.109	102871	111271	119671	9	

[Continued from page 280.]

his works. Also, Fibonacci numbers with prime subscripts need not necessarily be primes (p. 83).

In conclusion, we have in this publication a very readable work that fills a much needed place in the literature. We now have an answer to the many requests for information on Leonard of Pisa which come to the Fibonacci Association.

Specific information regarding the book is as follows:

Publisher: Thomas Y. Crowell Company

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Authors: Joseph and Frances Gies

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