ON DETERMINANTS WHOSE ELEMENTS ARE PRODUCTS OF RECURSIVE SEQUENCES

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1. INTRODUCTION

Let W_0, W_1, $p \neq 0$, and $q \neq 0$ be arbitrary real numbers, and define

$$(1.1)\qquad W_{n+2} = pW_{n+1} - qW_n, \quad p^2 - 4q \neq 0, \qquad (n = 0, 1, \cdots),$$

$$(1.2)\qquad U_n = (A^n - B^n)/(A - B) \qquad (n = 0, 1, \cdots),$$

$$(1.3)\qquad V_n = A^n + B^n, \quad V_{-n} = V_n/q^n, \qquad (n = 0, 1, \cdots),$$

where $A \neq B$ are roots of $y^2 - py + q = 0$. Carlitz [1, p. 132 (6)], using a well-known result for linear transformations of a quadratic form, has given a closed form for the class of determinants

$$(1.4)\qquad \left| W^k_{n+r+s} \right| \qquad (r, s = 0, 1, \cdots, k).$$

As a first generalization of (1.4), we will show that for $m = 1, 2, \cdots$, and $n_0 = 0, 1, \cdots$,

$$(1.5)\qquad \left| W^k_{m(n+r+s)+n_0} \right| \qquad (r, s = 0, 1, \cdots, k)$$

$$= (-1)^{(k+1)(k/2)} \cdot q^{(mn+n_0)(k+1)(k/2)+(mk/3)(k^2-1)} \cdot \prod_{j=0}^{k} \binom{k}{j}$$

$$\cdot (W_1^2 - pW_0W_1 + qW_0^2)^{(k+1)k/2} \cdot \prod_{i=1}^{k} U_{mi}^{2(k+1-i)}.$$

For $m = 1$ and $n_0 = 0$, Eq. (1.5) gives the main result (1.4) of [1]. As in [1], our proof of (1.5) will require the following known result for quadratic forms (e.g., see [2, pp. 127-128]):

Lemma 1. Let a quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}x_i x_j \qquad (\alpha_{ij} = \alpha_{ji})$$

be transformed by a linear transformation

$$x_i = \sum_{k=1}^{n} \beta_{ik} Y_k \qquad (i = 1, 2, \cdots, n)$$

to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}Y_i Y_j \qquad (c_{ij} = c_{ji}) .$$

Then

$$(1.6) \qquad \left|c_{ij}\right| = \left|\alpha_{ij}\right| \cdot \left|\beta_{ij}\right| \qquad (i,j = 1, 2, \cdots, n) .$$

2. STATEMENT OF THEOREM 1

We note that (1.5) is a special case of Theorem 1.

Theorem 1. Let W_n, $n = 0, 1, \cdots$, satisfy (1.1), where $A \neq B \neq 0$ are the roots of $y^2 - py + q = 0$. Let $m,k = 1, 2, \cdots$, and define

$$(2.1) \qquad P_n = \prod_{i=1}^{k} W_{mn+n_i} \qquad (n = 0, 1, \cdots) ,$$

where n_i, $i = 1, 2, \cdots, k$, are arbitrary integers or zero. Let $N_k = n_1 + n_2 + \cdots + n_k$. Then, with $u + 1$ as the row index and $v + 1$ as the column index, we have

$$(2.2)\quad \left| \prod_{i=1}^{k} W_{m(n+u+v)+n_i} \right| \qquad (u, v = 0, 1, \cdots, k)$$

$$= (-1)^{(k+1)k/2} \cdot q^{mn(k+1)(k/2)+(mk/3)(k^2-1)} \cdot \prod_{r=0}^{k} C_r$$

$$\cdot (W_1^2 - pW_0W_1 + qW_0^2)^{(k+1)k/2} \prod_{i=1}^{k} U_{mi}^{2(k+1-i)} ,$$

with $C_0 = A^{N_k}$,

$$(2.3)\quad C_r = \sum_{j=1}^{\binom{k}{r}} A^{N_k - S(j,r)} B^{S(j,r)} \qquad (r = 1, 2, \cdots, k) ,$$

$$(2.4)\quad S(j,r) = n_1^{(j)} + n_2^{(j)} + n_3^{(j)} + \cdots + n_r^{(j)} \qquad \left(j = 1, 2, \cdots, \binom{k}{r}\right),$$

where, for each j, $S(j,r)$, as the sum of r integers, $n_i^{(j)}$, $i = 1, 2, \cdots, r$, represents one of the $\binom{k}{r}$ combinations obtained by choosing r numbers from the k numbers, $n_1, n_2, n_3, \cdots, n_k$.

Remarks. If $n_i \equiv n_0$, $i = 1, 2, \cdots, k$, then $N_k = kn_0$, $S(j,r) = rn_0$, and

$$C_r = \binom{k}{r} A^{(k-r)n_0} B^{rn_0} .$$

Since $AB = q$, we have

$$\prod_{r=0}^{k} C_r = q^{n_0(k+1)k/2} \cdot \prod_{j=0}^{k} \binom{k}{j},$$

and thus (2.2) gives (1.5) as a special case.

For the case $n_1 = n_2 = \cdots = n_{k-1} = d$ and $n_k \neq d$, it is readily seen that

$$C_r = \binom{k-1}{r-1} A^{(k-r)d} B^{(r-1)d+n_k} + \binom{k-1}{r} A^{(k-r-1)d+n_k} B^{rd}.$$

As a footnote to Theorem 1, we have

<u>Lemma 2.</u> For $r < k - r$, $r = 0, 1, \cdots,$ we have

$$(2.5)\quad C_r C_{k-r} = \binom{k}{r} q^{N_k} + \sum_{j=2}^{\binom{k}{r}} \sum_{i=1}^{j-1} q^{S(i,r)-S(j,r)+N_k} \cdot V_{2S(j,r)-2S(i,r)}.$$

Thus,

$$(2.6)\quad \prod_{r=0}^{k} C_r = \prod_{r=0}^{(k-1)/2} C_r C_{k-r} \qquad (k = 1, 3, 5, \cdots),$$

$$(2.7)\quad \prod_{r=0}^{k} C_r = C_{k/2} \cdot \prod_{r=0}^{(k-2)/2} C_r C_{k-r} \qquad (k = 2, 4, 6, \cdots),$$

where

$$(2.8)\quad C_{k/2} = \sum_{j=1}^{\binom{k-1}{k/2}} q^{S(j,k/2)} \cdot V_{N_k - 2S(j,k/2)} \qquad (k = 2, 4, 6, \cdots).$$

Proof of Lemma 2. Since $AB = q$, we obtain from (2.3),

$$C_r = \sum_{j=1}^{\binom{k}{r}} q^{S(j,r)} \cdot A^{N_k - 2S(j,r)} .$$

Noting that a choice of r numbers from k numbers leaves a complement choice of $k - r$ numbers, we have from (2.3)

$$(2.9)\quad C_{k-r} = \sum_{j=1}^{\binom{k}{r}} A^{N_k - S(j,k-r)} B^{S(j,k-r)} = \sum_{j=1}^{\binom{k}{r}} A^{S(j,r)} B^{N_k - S(j,r)}$$

$$= \sum_{i=1}^{\binom{k}{r}} q^{S(i,r)} \cdot B^{N_k - 2S(i,r)} .$$

In forming the product $C_r C_{k-r}$, we note that $\binom{k}{r}$ product pairs have equal i and j indices and the same value q^{N_k}. For the cross products with $i \neq j$, we combine those pairs having the same values of i and j, noting that

$$q^{S(j,r)} A^{N_k - 2S(j,r)} \cdot q^{S(i,r)} B^{N_k - 2S(i,r)} + q^{S(i,r)} A^{N_k - 2S(i,r)} \cdot q^{S(j,r)} B^{N_k - 2S(j,r)}$$
$$= q^{S(i,r) - S(j,r) + N_k} V_{2S(j,r) - 2S(i,r)} .$$

Set $k = 2r$ in (2.3). Since a choice of r numbers from a set of 2r numbers leaves another set of r numbers, we may again pair off related terms of the sum in (2.3). Since

$$A^{N_{2r} - S(j,r)} B^{S(j,r)} + A^{S(j,r)} B^{N_{2r} - S(j,r)} = q^{S(j,r)} V_{N_{2r} - 2S(j,r)}$$

and

$$\binom{2r}{r} = 2\binom{2r-1}{r} ,$$

we obtain (2.8) from (2.3) with $r = k/2$.

3. PROOF OF THEOREM 1

Since $A \neq B$, the general solution to (1.1) is

$$W_n = aA^n + bB^n, \qquad n = 0, 1, \cdots,$$

where a and b are arbitrary constants whose values satisfy $W_0 = a + b$ and $W_1 = aA + bB$. We readily find that $(B - A)a = W_0 B - W_1$ and $(B - A)b = W_1 - AW_0$. Since $A + B = p$ and $AB = q$, we have that

$$(3.1) \qquad (A - B)^2 ab = -(W_1^2 - pW_0W_1 + qW_0^2) .$$

We observe that

$$(3.2) \qquad P_n = \prod_{i=1}^{k} W_{mn+n_i} = \sum_{j=0}^{k} K_j (B^{m(k-j)}A^{mj})^n \qquad (n = 0, 1, \cdots) ,$$

where K_j, $j = 0, 1, \cdots, k$, denote arbitrary constants independent of n. The quadratic form

$$Q = \sum_{r,s=0}^{k} P_{n+r+s} Y_r Y_s = \sum_{r,s=0}^{k} Y_r Y_s \cdot \sum_{j=0}^{k} K_j (B^{m(k-j)}A^{mj})^{n+r+s}$$

$$(3.3) \qquad = \sum_{j=0}^{k} K_j (B^{m(k-j)}A^{mj})^n \sum_{r,s=0}^{k} A^{mj(r+s)} B^{m(k-j)(r+s)} Y_r Y_s$$

$$= \sum_{j=0}^{k} K_j (B^{m(k-j)}A^{mj})^n x_j^2 ,$$

where

$$(3.4)\qquad x_j = \sum_{r=0}^{k} (A^{mj}B^{m(k-j)})^r Y_r \qquad (j = 0, 1, \cdots, k) .$$

Thus, by means of the linear transformation (3.4), we have reduced Q to a diagonal form. If M denotes the determinant of the linear transformation (3.4), it follows from Lemma 1 (see (1.6)), that

$$(3.5)\qquad \left|P_{n+r+s}\right| = M^2 \cdot \prod_{j=0}^{k} K_j(B^{m(k-j)}A^{mj})^n = M^2 \cdot q^{mn(k+1)k/2} \cdot \prod_{j=0}^{k} K_j ,$$

where

$$(3.6)\qquad M = \left|(A^{mj}B^{m(k-j)})^r\right| \qquad (j, r = 0, 1, \cdots, k) ,$$

is a Vandermonde determinant.

We find now that

$$M = \prod_{0\le j<r\le k} (A^{mr}B^{m(k-r)} - A^{mj}B^{m(k-j)}) = \prod_{j=0}^{k-1}\prod_{r=j+1}^{k} A^{mj}B^{m(k-r)}(A-B)U_{m(r-j)}$$

$$(3.7)\qquad \begin{aligned} &= (A - B)^{k(k+1)/2} \cdot \prod_{j=0}^{k-1}\prod_{s=1}^{k-j} A^{mj}B^{m(k-j-s)}U_{ms} \\ &= (A - B)^{k(k+1)/2} \cdot \prod_{j=0}^{k-1} A^{mj(k-j)}B^{m(k-j)(k-j-1)/2} \cdot \prod_{i=0}^{k-1}\prod_{s=1}^{k-i} U_{ms} \\ &= (A - B)^{k(k+1)/2} \cdot q^{mk(k^2-1)/6} \cdot \prod_{i=1}^{k} U_{mi}^{k+1-i} .\end{aligned}$$

We proceed now to evaluate

$$\prod_{j=0}^{k} K_j$$

of (3.5). From (3.2) we have

$$(3.8) \qquad \prod_{i=1}^{k} W_{mn+n_i} = B^{mkn} \cdot \sum_{j=0}^{k} K_j \left((A/B)^{mn}\right)^j ,$$

which is a polynomial in the variable $(A/B)^{mn}$. Since $W_n = aA^n + bB^n$, we have

$$W_{mn+n_i} = B^{mn}\left(aA^{n_i}(A/B)^{mn} + bB^{n_i}\right) ,$$

and thus

$$(3.9) \qquad \prod_{i=1}^{k} W_{mn+n_i} = B^{mkn} \cdot \prod_{i=1}^{k} \left(aA^{n_i}(A/B)^{mn} + bB^{n_i}\right)$$

$$= B^{mkn}a^k \cdot A^{N_k} \cdot \prod_{i=1}^{k} \left((A/B)^{mn} + (b/a)(B/A)^{n_i}\right) .$$

Recalling the definition of the elementary symmetric functions of the roots of a polynomial, we conclude, after comparing (3.8) and (3.9), that (see (2.3))

$$(3.10) \quad K_r = a^k \cdot A^{N_k} \cdot (-1)^r \cdot \sum_{j=1}^{\binom{k}{r}} (-b/a)^r \prod_{s=1}^{r} (B/A)^{n_{i_s}^{(j)}} = a^{k-r}b^r C_r \qquad (r = 0, 1, \cdots, k) .$$

Using (3.1), we obtain from (3.10)

$$\prod_{r=0}^{k} K_r = (ab)^{k(k+1)/2} \prod_{r=0}^{k} C_r$$

(3.11)

$$= (-1)^{k(k+1)/2}(A - B)^{-k(k+1)}(W_1^2 - pW_0W_1 + qW_0^2)^{k(k+1)/2} \prod_{r=0}^{k} C_r .$$

Thus, (3.5), with the use of (3.7) and (3.11), gives the desired result, (2.2).

4. THE CASE $p^2 - 4q = 0$

In [1], Carlitz gave an alternate proof of (1.4) for the case $p^2 - 4q = 0$. Although (1.4) was proved for the case $p^2 - 4q \neq 0$, the two results are shown to be the same for the case $p^2 - 4q = 0$.

In the derivation of (2.2), we assumed that $p^2 - 4q \neq 0$. It can be shown (by a repetition of the argument in [1]) that (2.2) is also valid for the case $p^2 - 4q = 0$, where now $U_n = n(p/2)^{n-1}$, and $W_n = (a + bn)(p/2)^n$, with $a = W_0$ and $pb = 2W_1 = pW_0$. Since $A = B = p/2$, we obtain from (2.3) that

$$C_r = \binom{k}{r}(p/2)^{N_k} .$$

Moreover, we have

$$\prod_{i=1}^{k} U_{mi}^{2(k+1-i)} = \prod_{i=1}^{k} (mi)^{2(k+1-i)} \cdot (p/2)^{2(mi-1)(k+1-i)}$$

$$= m^{k(k+1)} \cdot (p/2)^{k(k+1)(mk+2m-3)/3} \cdot \prod_{r=0}^{k} (r!)^2 ,$$

and

$$\prod_{r=0}^{k} \binom{k}{r} (r!)^2 = (k!)^{k+1} .$$

Thus, from Theorem 1, we obtain the simplified result

Theorem 2

$$\left| \prod_{i=1}^{k} \left(a + bn_i + bm(n + r + s)\right) (p/2)^{m(n+r+s)+n_i} \right| \quad (r,s = 0,1,\cdots,k)$$

$$(4.1) \quad = (-1)^{(k+1)k/2} \cdot (p/2)^{(k+1)(k(mn+1)+(2/3)mk(k-1)+(k/3)(mk+2m-3)+N_k)} \cdot (bm)^{k(k+1)} \cdot (k!)^{k+1} .$$

Remarks. If $m = 1$ and $n_i \equiv 0$, $i = 1, 2, \cdots, k$, then $N_k = 0$, and thus (4.1) contains, as a special case, the second (and the last) principal result, (7), of [1].

Additional simplifications of (4.1) are readily obtained.

REFERENCES

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