FIBONACCI NUMBERS AS PATHS OF A ROOK ON A CHESSBOARD

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The purpose of this article is to show that Fibonacci numbers can be derived by enumerating the number of different routes of a rook from one corner of a chessboard to the opposite corner when the moves of the rook are limited by restrictive fences.

Consider the chessboard array of binomial coefficients or figurate numbers in Fig. 1. It is well known that the number in any square or cell represents the number of different routes of a rook from the upper left corner to that cell, provided that the rook moves are either horizontal to the right or vertically downward.*

1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8
1	3	6	10	15	21	28	36
1	4	10	20	35	56	84	120
1	5	15	35	70	126	210	330
1	6	21	56	126	252	462	792
1	7	28	84	210	462	924	1716
1	8	36	120	330	792	1716	3432

Fig. 1. Number of Rook Paths from Corner of Chessboard

Figure 2 shows the same chessboard array in standard combinatorial notation:

$$\binom{h}{k} = h!/k!\,(h-k)! = \binom{h}{h-k} .$$

*Edouard Lucas, Théorie des Nombres, Paris, 1891, page 83.

$\binom{0}{0}$	$\binom{1}{0}$	$\binom{2}{0}$	$\binom{3}{0}$	$\binom{4}{0}$	$\binom{5}{0}$	$\binom{6}{0}$	$\binom{7}{0}$
$\binom{1}{1}$	$\binom{2}{1}$	$\binom{3}{1}$	$\binom{4}{1}$	$\binom{5}{1}$	$\binom{6}{1}$	$\binom{7}{1}$	$\binom{8}{1}$
$\binom{2}{2}$	$\binom{3}{2}$	$\binom{4}{2}$	$\binom{5}{2}$	$\binom{6}{2}$	$\binom{7}{2}$	$\binom{8}{2}$	$\binom{9}{2}$
$\binom{3}{3}$	$\binom{4}{3}$	$\binom{5}{3}$	$\binom{6}{3}$	$\binom{7}{3}$	$\binom{8}{3}$	$\binom{9}{3}$	$\binom{10}{3}$
$\binom{4}{4}$	$\binom{5}{4}$	$\binom{6}{4}$	$\binom{7}{4}$	$\binom{8}{4}$	$\binom{9}{4}$	$\binom{10}{4}$	$\binom{11}{4}$
$\binom{5}{5}$	$\binom{6}{5}$	$\binom{7}{5}$	$\binom{8}{5}$	$\binom{9}{5}$	$\binom{10}{5}$	$\binom{11}{5}$	$\binom{12}{5}$
$\binom{6}{6}$	$\binom{7}{6}$	$\binom{8}{6}$	$\binom{9}{6}$	$\binom{10}{6}$	$\binom{11}{6}$	$\binom{12}{6}$	$\binom{13}{7}$
$\binom{7}{7}$	$\binom{8}{7}$	$\binom{9}{7}$	$\binom{10}{7}$	$\binom{11}{7}$	$\binom{12}{7}$	$\binom{13}{7}$	$\binom{14}{7}$

Fig. 2. Rook Paths in Combinatorial Notation

Figure 3 shows a chessboard array where the moves of a rook are limited by the indicated pattern of horizontal and vertical restrictive fences.

1	1	1					
1	2	3	3				
	2	5	8	8			
		5	13	21	21		
			13	34	55	55	
				34	89	144	144
					89	233	377
						233	610

Fig. 3 Rook Paths Limited by Restrictive Fences

The array begins with number one in the top left corner. Inasmuch as the number in any cell is the sum of the numbers immediately above it and to the left of it, the pattern of restrictive fences results in the entire array being composed of Fibonacci numbers.

Figure 4 shows the chessboard with the same pattern of fences as in Fig. 3, but with the numbers in the Fibonacci notation where $F_0 = F_1 = 1$; $F_2 = 2$; $F_3 = 3$; and, in general, $F_{n+2} = F_{n+1} + F_n$.

F_0	F_1	F_1					
F_0	F_2	F_3	F_3				
	F_2	F_4	F_5	F_5			
		F_4	F_6	F_7	F_7		
			F_6	F_8	F_9	F_9	
				F_8	F_{10}	F_{11}	F_{11}
					F_{10}	F_{12}	F_{13}
						F_{12}	F_{14}

Fig. 4. Limited Rook Paths in Fibonacci Notation

Comparing Fig. 4 with Fig. 2, it is noted that F_0 corresponds to $\binom{0}{0}$ and F_{14} corresponds to $\binom{14}{7}$. Comparing Fig. 3 with Fig. 1, we see that the number of Fibonacci rook paths from corner to corner is 610, whereas the number of unrestricted paths is 3432. The difference of 2822 must be the number of routes which are eliminated because of the restrictive fences. This can be verified by tabulating the effect of each restrictive fence as in the analysis on the following page.

To generalize, in a chessboard of $(n + 1)^2$ cells, the number of unrestricted rook paths from corner to corner is $\binom{2n}{n}$, the number of Fibonacci rook paths is F_{2n}, and the number of paths that are eliminated by the pattern of horizontal and vertical fences is

$$\begin{aligned}\binom{2n}{n} - F_{2n} = {} & F_0\binom{2n-2}{n} + F_1\binom{2n-3}{n} \\ & + F_2\binom{2n-4}{n-1} + F_3\binom{2n-5}{n-1} \\ & + \cdots \qquad\qquad + \cdots \\ & + F_{2n-6}\binom{4}{3} \quad + F_{2n-5}\binom{3}{3} \\ & + F_{2n-4}\binom{2}{2}.\end{aligned}$$

ANALYSIS OF ELIMINATED ROOK PATHS

A	B		A x B
Number of Fibonacci paths from origin to cells with fences	Number of unrestricted rook paths from fence to lower right corner		Number of paths eliminated by fences
Cells with horizontal fences			
$F_0 = 1$	$\binom{12}{7} = 792$		792
$F_2 = 2$	$\binom{10}{6} = 210$		420
$F_4 = 5$	$\binom{8}{5} = 56$		280
$F_6 = 13$	$\binom{6}{4} = 15$		195
$F_8 = 34$	$\binom{4}{3} = 4$		136
$F_{10} = 89$	$\binom{2}{2} = 1$		89
		Subtotal	1912
Cells with vertical fences			
$F_1 = 1$	$\binom{11}{7} = 792$		330
$F_3 = 3$	$\binom{9}{6} = 84$		262
$F_5 = 8$	$\binom{7}{5} = 21$		168
$F_7 = 21$	$\binom{5}{4} = 5$		105
$F_9 = 55$	$\binom{3}{3} = 1$		55
		Subtotal	910
Total number of eliminated paths			2822
Total number of Fibonacci paths, F_{14}			610
Total number of unrestricted paths, $\binom{14}{7}$			3432