ON A CLASS OF DIFFERENCE EQUATIONS

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The purpose of this article is to examine sequences generated by a certain class of difference equations and to encourage further investigations into their properties. We shall be interested in sequences satisfying the recurrence relation,

(1) $$v_{n+2} = v_{n+1} + v_n + kv_nv_{n+1}; \qquad v_1 = v_2 = 1 \qquad (n \geq 1),$$

where k is a positive integer.

It may be shown by a simple inductive argument that

(2) $$v_n = \frac{(k+1)^{F_n} - 1}{k} \qquad (n \geq 1),$$

where F_n denotes the n^{th} Fibonacci number.

When we wish to emphasize the dependence on the parameter, k, we shall write $v_n \equiv v_n(k)$.

A MODEL FOR $\{v_n\}_{n=1}^{\infty}$

Let b denote an integer $(b \geq 2)$. Consider the sequence defined as follows:

(3) $$\theta_n = \overbrace{11\cdots1}^{F_n}\ (b) \qquad (n \geq 1).$$

where (b) denotes base b. Obviously,

(4) $$\theta_n = \sum_{i=0}^{F_n-1} b^i = \frac{b^{F_n} - 1}{b - 1} \qquad (n \geq 1)$$

As above, we shall write $\theta_n \equiv \theta_n(b)$. From Eqs. (2) and (4), we see that

$$v_n(k+1) = \theta_n(b) .$$

$b^n - 1$ has been called the n^{th} Fermatian function of b and

$$B_n \equiv \frac{b^n - 1}{b - 1}$$

has been called a reduced Fermatian of index b. (See [1].) We note that $B_{F_n} = \theta_n$.

If we are willing to abuse the language, we may extend the allowed values of b. Formally, if $k = 0$, Eq. (1) becomes the usual Fibonacci recurrence relation. Then $b = k + 1 = 1$, and if we interpret the 1's in (3) as tally marks,

$$\begin{aligned}\theta_n &= 1(1)^{F_n - 1} + \cdots + 1(1)^0 \\ &= \overbrace{11\cdots 1}^{F_n}(1) .\end{aligned}$$

Similarly, if $k = -1$, then $b = 0$. With the agreement that $0^0 = 1$,

$$\begin{aligned}\theta_n &= 1(0)^{F_n - 1} + \cdots + 1(0)^0 \\ &= \overbrace{11\cdots 1}^{F_n}(0)\end{aligned}$$

Thus $\theta_n \equiv 1$. But the solution of (1) in this case is

$$v_n(-1) \equiv 1 \qquad (n \geq 1) .$$

Using similar interpretations for negative bases, we can extend (1) and (3) to negative integers.

DIVISIBILITY PROPERTIES OF $\{v_n\}_{n=1}^{\infty}$

It is interesting to note that if

$$\{v_n(1)\}_{n=1}^{\infty}$$

contains an infinite number of primes, then there would be an infinite number of Fibonacci and Mersenne primes.

In this section, we shall assume $k = 9$ $(b = 10)$ unless otherwise specified.

<u>Theorem 1.</u>

(a) $(\theta_n, {}_{n+1}) = 1 \quad (n \geq 1)$;

(b) $(\theta_n, {}_{n+2}) = 1 \quad (n \geq 1)$.

<u>Proof.</u> a) Deny! Then there is a pair such that $(\theta_m, \theta_{m+1}) = d > 1$. But $d | v_{n+2}$, $d | v_{n+1}$ implies $d | v_n$. Thus, after repeated use of the above, we would have $(\theta_1, \theta_2) \geq d \geq 1$. Contradiction.

b) Similar to part a).

<u>Theorem 2.</u> None of the θ_n are perfect.

<u>Proof.</u> Any odd perfect number is congruent to 1 modulo 4 (see [2]). But

$$\theta_n \equiv 3 \pmod 4 \quad \text{for} \quad n \geq 3.$$

<u>Theorem 3.</u> $3 | \theta_n$ if and only if $4 | n$.

<u>Proof.</u> Clearly, $3 | \theta_n$ if and only if $3 | F_n$. Thus $F_4 | F_n$ and the result follows.

<u>Theorem 4.</u> $11 | \theta_n$ if and only if $3 | n$.

<u>Proof.</u> $11 | \theta_n$ if and only if $2 = F_3 | F_n$ and the result follows.

<u>Theorem 5.</u> a) $7 | \theta_n$ if and only if $12 | n$;

b) $13 | \theta_n$ if and only if $12 | n$.

<u>Proof.</u> a) Consider the congruences,

$$1 \equiv 1 \pmod 7, \qquad 10 \equiv 3 \pmod 7, \qquad 100 \equiv 2 \pmod 7,$$
$$1{,}000 \equiv -1 \pmod 7, \quad 10{,}000 \equiv -3 \pmod 7, \quad 100{,}000 \equiv -2 \pmod 7.$$

Clearly $7 | \theta_n$ if and only if $6 | F_n$. But $6 | F_n$ is equivalent to $2 | F_n$ and $3 | F_n$ of $3 | n$ and $4 | n$ and the result follows.

b) Similar to a), considering the congruences modulo 13.

In light of the above, we have the unusual result that $3 | \theta_n$ and $11 | \theta_n$ implies $7 | \theta_n$ and $13 | \theta_n$.

We mention some other results which the reader might like to establish.

Assertion 1: $18 | F_n$ implies $19 | \theta_n$.

Assertion 2: $41 | \theta_n$ if and only if $5 | n$.

Assertion 3: $271 | \theta_n$ if and only if $5 | n$.

Assertion 4: $73 | \theta_n$, $101 | \theta_n$, $137 | \theta_n$ if and only if $6 | n$.

GENERATING FUNCTIONS FOR $\{v_n(k)\}_{n=1}^{\infty}$

One area which might be worth investigating is that of obtaining generating functions for the sequences. Of course, since

$$\frac{1}{1 - x - x^2} = \sum_{i=1}^{\infty} F_i x^{i-1} , \tag{6}$$

we have

$$\frac{1}{1 - x - x^2} = \sum_{i=1}^{\infty} \frac{\log [1 + kv_i(k)]}{\log (k + 1)} x^{i-1} , \tag{7}$$

but one should be able to do better than this.

ALTERNATE RELATIONSHIPS

We present two results along these lines.

Theorem 6.

$$\theta_{n+2}(2) = 2 \prod_{i=1}^{n} [1 + \theta_i(2)] - 1 \quad (n \geq 1) .$$

Proof. Since

$$2^{F_n} = 1 + \theta_n(2)$$

and

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \qquad (n \geq 1) ,$$

the result easily follows.

Theorem 7.

$$1 + \theta_{2n}(2) = \prod_{i=1}^{n} [1 + \theta_{2i-1}(2)] \qquad (n \geq 1).$$

Proof. The result is readily obtained from

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n} .$$

GENERALIZATION TO OTHER RECURSIVELY DEFINED SEQUENCES

We conclude our discussion with one result in this area.

Theorem 8. If

$$\{u_n\}_{n=1}^{\infty}$$

is a recursively defined positive integer sequence satisfying the linear difference equation

$$(8) \qquad \sum_{i=0}^{m} \alpha_i u_{n+i} = \beta \qquad (n \geq 1) \qquad (\text{order } m) ,$$

and boundary conditions $\{u_1, u_2, \cdots, u_{m-1}\}$, where β and α_i for $i \in \{0, 1, \cdots, m\}$ are constants, and if

$$\beta_n = \overbrace{11\cdots1}^{u_n}\,(b) \qquad (n \geq 1)\,;$$

then

$$\prod_{i=0}^{m} \left[1 + (b-1)\beta_{n+i}\right]^{\alpha_i} = b^{\beta} \qquad (n \geq 1)\,. \tag{9}$$

Proof. Since

$$\beta_n = \frac{b^{u_n} - 1}{b - 1} \qquad (n \geq 1)\,,$$

we have

$$b^{u_k} = 1 + (b-1)\beta_k$$

for $k \geq 1$ and the result readily follows.

REFERENCES

1. L. E. Dickson, History of the Theory of Numbers, Vol. 1, p. 385.
2. Ibid, p. 19.

ON A CONJECTURE OF DMITRI THORO*

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Denoting the n^{th} term of the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$, by F_n, where $F_{n+2} = F_{n+1} + F_n$, it is well known that

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}\,.$$

If odd prime p divides F_{n-1}, then

$$F_n^2 \equiv (-1)^{n+1} \pmod{p}\,,$$

so that $(-1)^{n+1}$ is a quadratic residue modulo p. Clearly, for $n = 2k$, this implies -1 is a quadratic residue modulo p, and accordingly, $p \equiv 1$ (mod

[Continued on page 537.]