ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-181 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Prove the identity

$$\sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{u^m v^n}{m!n!} = \frac{1}{(1 - ax)(1 - dy) - bcxy}$$

where

$$u = xe^{-(ax+by)}, \qquad v = ye^{-(cx+dy)} .$$

H-182 Proposed by S. Krishnar, Berthampur, India.

Prove or disprove

(i) $$\sum_{k=1}^{m} \frac{1}{k^2} \equiv 0 \pmod{2m + 1} ,$$

and

(ii) $$\sum_{k=1}^{m} \frac{1}{(2k-1)^2} \equiv 0 \pmod{2m+1},$$

when $2m + 1$ is prime and larger than 3.
[See Special Problem on page 216.]

SOLUTIONS

GONE BUT NOT FORGOTTEN

H-102 Proposed by J. Arkin, Suffern, New York. (For convenience, the problem is restated, using $B_n = A_m$.)

Find a closed expression for B_n in the following recurrence relation.

(H) $$\left[\frac{n}{2}\right] + 1 = B_n - B_{n-3} - B_{n-4} - B_{n-5} + B_{n-7} + B_{n-8} + B_{n-9} - B_{n-12},$$

where $n = 0, 1, 2, \cdots$ and the first thirteen values of B_0 through B_{12} are 1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 30, 37, and 47, and $[x]$ is the greatest integer contained in x.

Solution by the Proposer.

In a recent paper* this author introduced a new notation, and because of the new method in the paper, we are, for the first time, able to find explicit formulas in such recurrence relations as H-102.

We denote by $p_m(n)$ the number of partitions of n into parts not exceeding m, where

(1) $$F_m(x) = 1/(1-x)(1-x^2)\cdots(1-x^m) = \sum_{n=0}^{\infty} p_m(n)x^n,$$

and $p_m(0) = 1$.

The new notation we mentioned above is defined as follows:

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*Joseph Arkin, "Researches on Partitions," Duke Mathematical Journal, Vol. 38, No. 3 (1970), pp. 304-409.

$$A(m,n) = 1 \quad \text{if} \quad m \text{ divides } n \tag{2}$$

$$A(m,n) = 0 \quad \text{if} \quad m \text{ does not divide } n ,$$

where

$$m = 1, 2, 3, \cdots, \qquad n = 0, 1, 2, \cdots,$$

and

$$A(m,0) = 1 .$$

Now, in (1), it is plain that

$$F_2(x)/(1 - x^3)(1 - x^4)(1 - x^5) = \sum_{n=0}^{\infty} p_5(n)x^n ,$$

and we have

$$F_2(x) = (1 - x^3)(1 - x^4)(1 - x^5) \sum_{n=0}^{\infty} p_5(n)x^n . \tag{3}$$

Then, combining the coefficients in (3) leads to

$$\begin{aligned} p_2(n) = {} & p_5(n) - p_5(n-3) - p_5(n-4) - p_5(n-5) + p_5(n-7) \\ & + p_5(n-8) + p_5(n-9) - p_5(n-12) , \end{aligned} \tag{4}$$

and it is evident that the right side of (4) is identical to the right side of (H).

Now* it was shown that

*Joseph Arkin, "Researches on Partitions," Duke Mathematical Journal, Vol. 38, No. 3 (1970), Eq. (6), p. 404.

$$p_2(2u) = u + 1$$

and

$$p_2(2u + 1) = u + 1 \qquad (u = 0, 1, 2, \cdots),$$

so that

$$p_2(n) = [n/2], \tag{5}$$

where $n = 0, 1, 2, \cdots$, and $[x]$ is the greatest integer contained in x.

Then, combining (5) with the left side of (4) and since

$$B_n = p_5(n) \qquad (n = 0, 1, 2, \cdots),$$

it remains to find an explicit formula for the $p_5(n)$.

To this end*, we see that

$$p_5(n) = \frac{1}{17280}\begin{bmatrix} 6n^4 + 180n^3 + 1860n^2 + 7650n + 7719 \\ (270n + 2025)(-1)^n \\ 1920A(3,n) \\ 2160(A(4,n) + A(4,n+3)) \\ 3456A(5,n) \end{bmatrix}$$

A LARGE ORDER

H-161 Proposed by David Klarner, University of Alberta, Edmonton, Alberta, Canada.

Let

$$b(n) = \sum_{a_1+a_2+\cdots+a_i=n} \binom{a_i + a_2}{a_2}\binom{a_2 + a_3}{a_3}\cdots\binom{a_{i-1} + a_i}{a_i},$$

*Joseph Arkin, "Researches on Partitions," Duke Mathematical Journal, Vol. 38, No. 3 (1970), Eq. (19), p. 406.

where the sum is extended over all compositions of n and the contribution to the sum is 1 when there is only one part in the composition. Find an asymptotic estimate for $b(n)$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$b_k(n) = \sum_{a_1+\cdots+a_k=n} \binom{a_1+a_2}{a_2}\binom{a_2+a_3}{a_3}\cdots\binom{a_{k-1}+a_k}{a_k} ,$$

$$\frac{1}{f_k(x)} = \sum_{n=0}^{\infty} b_k(n)\, x^n .$$

It is known (see "A Binomial Identity Arising from a Sorting Problem," SIAM Review, Vol. 6 (1964), pp. 20-30), that $f_k(x)$ is equal to the following determinant of order $k+1$:

$$\begin{vmatrix} 1 & x & & & & \\ 1 & 1 & x & & & \\ & 1 & 1 & x & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & 1 & x \\ & & & & 1 & 1 \end{vmatrix}$$

It follows that

$$f_{n+1}(x) = f_n(x) - x f_{n-1}(x) .$$

Since $f_0(x) = 1$, $f_1(x) = 1 - x$, we find that

$$F(z) = \sum_{k=0}^{\infty} f_k(x)\, z^k = \frac{1 - xz}{1 - z + xz^2} ,$$

In the next place,

$$\frac{1 - xz}{1 - z + xz^2} = \frac{1}{\alpha - \beta}\left(\frac{\alpha^2}{1 - \alpha z} - \frac{\beta^2}{1 - \beta z}\right) ,$$

where

$$\alpha + \beta = 1, \qquad \alpha\beta = x .$$

It follows that

$$f_k(x) = \frac{\alpha^{k+2} - \beta^{k+2}}{\alpha - \beta} ,$$

so that

$$\sum_{n=0}^{\infty} b_k(n)\, x^n = \frac{\alpha - \beta}{\alpha^{k+2} - \beta^{k+2}} . \tag{1}$$

Now, if $k = 2r + 1$,

$$\begin{aligned}
\frac{\alpha^k - \beta^k}{\alpha - \beta} &= \prod_{s=1}^{k-1} (\alpha - \beta e^{2\pi i s/k}) \\
&= \prod_{s=1}^{r} (\alpha - \beta e^{2\pi i s/k})(\alpha - \beta e^{-2\pi i s/k}) \\
&= \prod_{s=1}^{r} \left(\alpha^2 - 2\alpha\beta \cos \frac{2\pi s}{k} + \beta^2\right) \\
&= \prod_{s=1}^{r} \left(1 - 4x \cos^2 \frac{\pi s}{k}\right) .
\end{aligned}$$

If we put

$$(3) \qquad \prod_{s=1}^{r}\left(1 - 4x\cos^2\frac{\pi s}{k}\right)^{-1} = \sum_{s=1}^{r}\frac{A_s}{1 - 4x\cos^2\frac{\pi s}{k}} \quad ,$$

we find that

$$A_s = \frac{\cos^{2(r-1)}\frac{\pi s}{k}}{\prod_{\substack{t=1\\t\neq s}}^{r}\left(\cos^2\frac{\pi s}{k} - \cos^2\frac{\pi t}{k}\right)} = \frac{2^{r-1}\cos^{2(r-1)}\frac{\pi s}{k}}{\prod_{\substack{t=1\\t\neq s}}^{r}\left(\cos\frac{2\pi s}{k} - \cos\frac{2\pi t}{k}\right)}$$

$$= \frac{\cos^{2(r-1)}\frac{\pi s}{k}}{\prod_{\substack{t=1\\t\neq s}}^{r}\sin\frac{\pi(t+s)}{k}\sin\frac{\pi(t-s)}{k}} \quad .$$

But

$$\prod_{\substack{t=1\\t\neq s}}^{r}\sin\frac{\pi(t+s)}{k}\sin\frac{\pi(t-s)}{k} = (-1)^{s-1}\frac{\prod_{t=1}^{2r}\sin\frac{\pi t}{k}}{\sin\frac{\pi s}{k}\sin\frac{2\pi s}{k}}$$

$$= \frac{(-1)^{s-1}k}{2^k\sin^2\frac{\pi s}{k}\cos\frac{\pi s}{k}} \quad ,$$

so that

$$(4) \qquad A_s = (-1)^{s-1}\frac{2^k\sin^2\frac{\pi s}{k}\cos^{2r-1}\frac{\pi s}{k}}{k} \quad .$$

Then, by (2), and (3) and (4),

$$\frac{\alpha - \beta}{\alpha^{k+2} - \beta^{k+2}} = \frac{2^{k+2}}{k+2} \cdot \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \frac{\sin^2 \frac{\pi s}{k+2} \cos^k \frac{\pi s}{k+2}}{1 - 4x \cos^2 \frac{\pi s}{k+2}}$$

$$= \frac{2^{k+2}}{k+2} \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^k \frac{\pi s}{k+2} \sum_{n=0}^{\infty} (4x)^n \cos^{2n} \frac{\pi s}{k+2}$$

$$= \frac{2^{k+2}}{k+2} \sum_{n=0}^{\infty} (4x)^n \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2} .$$

Therefore, by (1),

$$(5) \qquad b_k(n) = \frac{2^{k+2n+2}}{k+2} \sum_{s=1}^{\frac{1}{2}(k+1)} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2} \qquad (k \text{ odd}) .$$

This implies the asymptotic formula

$$(6) \qquad b_k(n) \sim \frac{2^{k+2n+2}}{k+2} \sin^2 \frac{\pi}{k+2} \cos^{k+2n} \frac{\pi}{k+2} \qquad (k \text{ odd}) .$$

Next, if $k = 2r$,

$$\frac{\alpha^k - \beta^k}{\alpha - \beta} = \frac{\alpha^k - \beta^k}{\alpha^2 - \beta^2} = \prod_{s=1}^{r-1} (\alpha - \beta e^{2\pi i s/k})(\alpha - \beta e^{-2\pi i s/k})$$

$$= \prod_{s=1}^{r-1} \left(1 - 4x \cos^2 \frac{\pi s}{k}\right) .$$

If we put

$$\prod_{s=1}^{r-1} \left(1 - 4x \cos^2 \frac{\pi s}{k}\right) = \sum_{s=1}^{r-1} \frac{A_s}{1 - 4x \cos^2 \frac{\pi s}{k}} ,$$

we get

$$A_s = \frac{\cos^{2(r-2)} \frac{\pi s}{k}}{\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \left(\cos \frac{2\pi s}{k} - \cos \frac{2\pi t}{k} \right)}$$

$$= \frac{2^{r-2} \cos^{2(r-2)} \frac{\pi s}{k}}{\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \left(\cos \frac{2\pi s}{k} - \cos \frac{2\pi t}{k} \right)} = \frac{\cos^{2(r-2)} \frac{\pi s}{k}}{\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \sin \frac{\pi(t + s)}{k} \sin \frac{\pi(t - s)}{k}} .$$

Since

$$\prod_{\substack{t=1 \\ t \neq s}}^{r-1} \sin \frac{\pi(t - s)}{k} \sin \frac{\pi(t - s)}{k} = (-1)^{s-1} \frac{\prod_{t=1}^{2r-1} \sin \frac{\pi t}{k}}{\sin \frac{\pi s}{k} \sin \frac{2\pi s}{k} \sin \frac{\pi(r + s)}{k}}$$

$$= (-1)^{s-1} \frac{k}{2^k \sin^2 \frac{\pi s}{k} \cos^2 \frac{\pi s}{k}} ,$$

it follows that

$$A_s = (-1)^{s-1} \frac{2^k \sin^2 \frac{\pi s}{k} \cos^{k-2} \frac{\pi s}{k}}{k} .$$

Then

$$\frac{\alpha - \beta}{\alpha^{k+2} - \beta^{k+2}} = \frac{2^{k+2}}{k + 2} \sum_{s=1}^{\frac{1}{2}k} (-1)^{s-1} \frac{\sin^2 \frac{\pi s}{k + 2} \cos^k \frac{\pi s}{k + 2}}{1 - 4x \cos^2 \frac{\pi s}{k + 2}}$$

$$= \frac{2^{k+2}}{k + 2} \sum_{n=0}^{\infty} (4x)^n \sum_{s=1}^{\frac{1}{2}k} (-1)^{s-1} \sin^2 \frac{\pi s}{k + 2} \cos^{k+2n} \frac{\pi s}{k + 2} ,$$

so that

$$(7)\qquad b_k(n) = \frac{2^{k+2n+2}}{k+2}\sum_{s=1}^{\frac{1}{2}k} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2} \qquad (k \text{ even}) .$$

This implies the asymptotic result

$$(8)\qquad b_k(n) \sim \frac{2^{k+2n+2}}{k+2} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi}{k+2} \qquad (k \text{ even}) .$$

We may combine (5) and (7) in the single formula

$$(9)\qquad b_k(n) = \frac{2^{k+2n+2}}{k+2} \sum_{s=1}^{[\frac{1}{2}(k+1)]} (-1)^{s-1} \sin^2 \frac{\pi s}{k+2} \cos^{k+2n} \frac{\pi s}{k+2}$$

and (6) and (8) in

$$(10)\qquad b_k(n) \sim \frac{2^{k+2n+2}}{k+2} \sin^2 \frac{\pi}{k+2} \cos^{k+2n} \frac{\pi}{k+2} .$$

LUCA-NACCI

H-163 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the following identities:

$$(1)\qquad \sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} = 2^{2n} F_{n+1}^2 - 1$$

$$(2)\qquad 5\sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} = 2^{2n} L_{n+1}^2 - 1 ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

Solution by A. G. Shannon, Mathematics Department, University of Papua and New Guinea, Boroko, T.P.N.G.

1. <u>n = 1</u>; $$\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} = L_1 F_4 = 3 ,$$

and

$$2^n F_{n+1}^2 - 1 = 2^2 F_2^2 - 1 = 3 .$$

Assume identity true for n. Then,

$$\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} + 2^{2n} L_{n+1} F_{n+4} = \sum_{k=1}^{n+1} 2^{2k-2} L_k F_{k+3}$$

$$\begin{aligned} 2^{2n} F_{n+1}^2 - 1 + 2^{2n} L_{n+1} F_{n+4} & \\ &= 2^{2n} (F_{n+1}^2 + (F_n + F_{n+2})(F_{n+3} + F_{n+2})) - 1 \\ &= 2^{2n} (F_{n+1}^2 + 2F_{n+2}^2 + 2F_n F_{n+2} + F_n F_{n+1} + F_{n+1} F_{n+2}) - 1 \\ &= 2^{2n} (2F_{n+2}^2 + F_{n+2}(2F_n + 2F_{n+1})) - 1 \\ &= 2^{2n+2} F_{n+2}^2 - 1 , \end{aligned}$$

which proves the result.

2. It can be readily shown that

$$L_k F_{k+3} = F_k L_{k+3} + 4(-1)^k , \tag{3}$$

by using

$$L_k = \alpha^k + \beta^k$$

and

$$F_k = (\alpha^k - \beta^k)(\alpha - \beta)^{-1} .$$

From (1) above, it follows that

$$(4) \qquad 5\sum_{k=1}^{n} 2^{k-2} L_k F_{k+3} = 2^{2n}(\alpha^{n+1} - \beta^{n+1})^2 - 5 .$$

With (3), the left-hand side of (4) becomes

$$5\sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} + 20\sum_{k=1}^{n} 2^{2k-2}(-1)^k$$

$$= 5\sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} + (2^{2n+2}(-1)^n - 4) .$$

The right-hand side of (4) reduces to

$$2^{2n}(\alpha^{2n+2} + \beta^{2n+2} + 2(-1)^n) - 5$$

$$= (2^{2n}L_{n+1} - 1) + (2^{2n+2}(-1)^n - 4) ,$$

and result (2) follows.

Also solved by M. Yoder, C. B. A. Peck, J. Milsom, M. Ratchford, D. V. Jaiswal, and the Proposer.