ON AN INITIAL-VALUE PROBLEM FOR LINEAR PARTIAL DIFFERENCE EQUATIONS *

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SUMMARY

Sufficient conditions are given for the existence and unity of the solution of an initial-value problem with linear partial difference equations. From this, in particular, assertions about the existence of compatibility conditions between initial values can be derived in case, by the formulation of a problem (perhaps a discretization of a partial differential equation) or by the method of solution, more than the required initial values goes into the calculation. With the aid of a two-dimensional operational calculus, certain applications are investigated.

INTRODUCTION

In the classical work [1] of A. A. Markoff, there is an existence and uniqueness theorem for partial difference equations of the form

$$x_{m+1,n+1} - a_{mn}x_{m,n+1} = b_{mn}x_{m+k,n} \,, \tag{1}$$
$$(m,n \geq 0, \ \text{integral}, \ k \ \text{fixed natural number}) \,,$$

for a desired complex-valued function $x = x_{mn}$ with given initial values x_{mo} $(m \geq k)$ and x_{on} $(n \geq 1)$. The proof is conducted by investigation of a system of infinitely many ordinary difference equations equivalent to (1). Here, in Theorem 1, an essentially more general initial-value problem for linear partial difference equations of arbitrary order will be treated by which the ideas of Ch. Jordan [2] on the subject are made precise.

The applications in the second part of the work show that the two-dimensional discrete operational calculus developed in [3] is appropriate to give in certain cases the solution, determined uniquely according to Theorem 1, in closed form and the possibly necessary compatibility conditions between the initial values explicitly.

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EXISTENCE AND UNIQUENESS THEOREMS

We consider the linear partial difference equation

$$D(x) = \sum_{k,j=1,0}^{k,\ell} a_{ij} x_{m+i,n+j} = b_{mn} \qquad (m,n \geq 0, \text{ integral}), \tag{2}$$

of order (k,ℓ) with given complex-valued functions

$$a_{ij} = a_{ij}(m,n),\ b_{mn}\,.$$

Let $k \geq 1$, $\ell \geq 1$, and for at least one i or j the coefficients a_{io}, a_{oj}, a_{i1}, a_{kj} should not vanish.

The question arises which of the initial values

$$\begin{array}{ll} x_{mj} & (j = 0, 1, \cdots, \ell - 1) \\ x_{in} & (i = 0, 1, \cdots, k - 1) \end{array}$$

should be prescribed so that the function x_{mn} is uniquely determined by (2) for all remaining $m,n \geq 0$. An answer to this is given by the following:

Theorem 1. The difference equation (2) of order (k,ℓ) possesses exactly one solution if, for all $m,n \geq 0$,

(a) $a_{kj} \neq 0$ for $j = \ell_k \leq 1$ and for $j = \ell_o \leq \ell_k$, $\quad a_{kj} = 0$ for $j > \ell_k$

holds, and the initial values

$$\begin{array}{ll} x_{mj} = \alpha^j_m & (j = 0, 1, \cdots, \ell_k;\ j \neq \ell_0;\ m \geq 0)\,, \\ x_{in} = \beta^i_n & (i = 0, 1, \cdots, k - 1;\ n \geq 0) \\ & \alpha^j_i = \beta^i_j \end{array} \tag{3}$$

are prescribed, or if

$$\begin{array}{l} a_{i\ell} \neq 0 \text{ for } i = k_1 \leq k \quad \text{and for } i = k_0 \leq k_1 \\ a_{i\ell} = 0 \quad \text{for} \quad i > k_1 \end{array} \tag{b}$$

holds and the initial values

$$
\begin{aligned}
x_{mj} &= \alpha_m^j \quad (j = 0, 1, \cdots, \ell - 1;\ m \geq 0)\,, \\
x_{in} &= \beta_n^i \quad (i = 0, 1, \cdots, k_\ell;\ i \neq k_0;\ n \geq 0) \text{ with } \alpha_i^j = \beta_j^i
\end{aligned} \tag{4}
$$

are prescribed. For $\ell_k = 0$ (in the case (a)) or $k_\ell = 0$ (case (b)) the first equation of (3) or the second of (4), respectively, drops out.

Proof. We consider case (a) and solve equation (2) for $x_{m+k, n+\ell_0}$ which is possible because $a_{k,\ell_0} \neq 0$. For $m = n = 0$, there results, after inserting the initial values (3),

$$
x_{k,\ell_0} = 1/a_{k,\ell_0}\left(b_{00} - \sum_{i,j=0,0}^{k-1,\ell} a_{ij}\beta_j^i - \sum_{\substack{j=0\\ j\neq\ell_0}}^{\ell_k} a_{kj}\alpha_k^j\right).
$$

For $\ell_k = 0$ and thus $\ell_0 = 0$, the sum $\sum a_{kj}\alpha_k^j$ drops out in agreement with the concluding remark of the theorem. Since $a_{k,\ell_k} \neq 0$, the equation

$$
x_{m+k, n+\ell_k} = 1/a_{k,\ell_k}\left(b_{mm} - \sum_{i,j=0,0}^{k-1,\ell} a_{ij}x_{m+j,n+j} \sum_{j=0}^{\ell_k - 1} a_{kj}x_{m+k,n+j}\right) \tag{5}
$$

follows from (2), and there results with (3) the function values x_{kn} $(n > \ell_k)$. If the function values up to x_{k,ℓ_k+p-1} $(p \geq 2)$ are determined, then it follows for $2 \leq p \leq \ell_k$ that

$$
x_{k,\ell_k+p} = 1/a_{k,\ell_k}\left(b_{op} - \sum_{i,j=0,0}^{k-1,\ell} a_{ij}\beta_{j+p}^i - \sum_{i=\ell_k-p+1}^{\ell_k-1} a_{kj}x_{k,j+p} - \sum_{j=0}^{\ell_k-p} a_{kj}\alpha_k^{j+p}\right)
$$

For $p > \ell_k$, the last sum drops out and the lower limit of the second sum is set to zero. The elements x_{mn} result analogously for the rows $m > k$ (the function x_{mn} being regarded as an infinite matrix) by use of the elements standing at hand in the immediate upper k rows, which are given either by (3) or are determined by (3) and (5).

For the case (b), one notes that (2) can be solved respectively for $x_{m+k_0, n+\ell}$ or $x_{m+k_1, n+\ell}$ because $a_{k_0,\ell} \neq 0$ or $a_{k_1,\ell} \neq 0$. In an analogous way as with (a) the function values x_{mn} ($m = k_0$, $m > k_1$, $n \geq 1$) are determined column-wise.

The proof of uniqueness of solution is trivial. If there were two solutions $x \neq y$ in the case (a) and if $x_{m_0,n_0} \neq y_{m_0,n_0}$ for $m_0 \geq k$ and $n_0 \geq \ell_k$, while $x_{mn} = y_{mn}$ for $m < m_0$ and $m = m_0$, $n < n_0$, then there immediately results from (2), for $m = m_0 - k$, $n = n_0 - \ell_k$ because $a_{k,\ell_k} \neq 0$, a contradiction. In case (b), the same holds for $n_0 = \ell_0 < \ell_k$.

In applications, the case when $\ell_0 = \ell_k = 1$ and $k_0 = k_1$ often occurs; then it follows that $k_i = k$ and the distinction between cases is cancelled. The solution x_{mn} of (2) is then uniquely determined by the specification of $k + \ell$ initial functions, namely by the first k rows and the first columns. (See example 1^0, 2^0.) Also, if $\ell_k < \ell$ or $k_1 < k$, occasionally $k + \ell$ initial values x_{mn} ($j = 0, \cdots, \ell - 1$), x_{in} ($i = 0, \cdots, k - 1$) are considered as prescribed. Compatibility conditions between these must then exist so that in the case (a) the $\ell - \ell_k$ functions x_{mj} ($j = \ell_0, \ell_k + 1, \cdots, \ell - 1$) and in case (b) the $k - k_1$ functions x_{in} ($i = k_0, k_1 + 1, \cdots, k - 1$) are already respectively determined by the remaining $k + \ell_k$ or $\ell + k_1$ functions. (See example 3^0, 4^0, 5^0.)

APPLICATIONS

In the treatment of the following applications, we make use of the operational calculus developed in [3]. It is shown there that the set of complex-valued functions $x = x_{mn}$ of integral variables m, n with vanishing function values for $m < M$ and all n for $n < N(m)$, $m \geq M$ (for each function an integer M exists and a function $N(m)$) forms a field by means of ordinary addition and of two-dimensional Cauchy product as multiplication. The subset D of functions with $M = 0$ and $N(m) = 0$ is an integral domain. For functions $x \in D$, the difference theorem

$$x_{m+k,n+1} = p^k q^\ell x_{mn} - q^\ell \sum_{i=0}^{k-1} p^{k-i} x_{in} - p^k \sum_{j=0}^{\ell-1} q^{\ell-j} x_{mj} + \sum_{i,j=0,0}^{k-1,\ell-1} p^{k-i} q^{\ell-j} x_{ij} , \tag{6}$$

holds, where x_{mj}, x_{in} and x_{ij} can be understood as functions from D which at least for $n = 0$ or $m = 0$ and $m = n = 0$ possess nonvanishing function values; p, q are displacement functions from Q, with k, ℓ being natural numbers.

1^0. The equation

$$x_{m+2,n+2} - x_{m+1,n+2} - x_{m+2,n+1} - x_{m,n+2} + 3x_{m+1,n+1} - x_{m+2,n} = 0 \quad (m,n \geq 0)$$

related to Fibonacci numbers was treated in [4] and [5]. Its solution according to Theorem 1 is uniquely determined because

$$\ell_k = \ell = k_1 = k = 2 ,$$

if the $k + 1 = 4$ initial values $x_{m0}, x_{m1}, x_{0n}, x_{1n}$ (so far as $k_0 = \ell_0 = 2$ is chosen) are prescribed independently of one another. This solution was represented in [5] in closed form.

2^0. The equation

$$x_{m+1,n+1} = x_{m+1,n} + \frac{2m + n + 3}{2m + 2} x_{m,n+1} \qquad (m,n \geq 0)$$

possesses the solutions[1]

$$x = x_{mn} = \sum_{i=0}^{\infty} \binom{m+i}{m} \binom{2m+n+1}{2m+2i+1} \quad \text{and} \quad y = y_{mn} = 2^n \binom{m+n}{m} .$$

Here,

$$\ell_k = \ell = k_1 = k = 1 .$$

Thus if one chooses $\ell_0 = k_0 = 1$, then it follows from Theorem 1 that the equation is uniquely solvable if the initial functions x_{mo}, x_{on} are prescribed. Since $x_{mo} = y_{mo} = 1$ $(m \geq 0)$ and $x_{on} = y_{on} = 2^n$ $(n \geq 0)$, it immediately follows that $x \equiv y$, and thus

[1]According to a written communication from A. Kotzauer (treated there by complete induction).

$$\sum_{i=0}^{\infty} \binom{m+i}{m}\binom{2m+n+1}{2m+2i+1} = 2^n \binom{m+n}{m} \qquad m,n = 0,1,\cdots .$$

3^0. The equation

$$x_{m+3,n} + x_{m,n+2} = 0 \qquad (m,n \geq 0) \tag{7}$$

of order (3.2) possesses, on account of $\ell_k = 0 < 2 = \ell$, $k_\ell = 0 < 3 = k$, exactly one solution from D if either in the case (a) the three initial functions $x_{in} = \beta_n^i$ ($i = 0, 1, 2$) according to (3), or in the case (b) the two functions $x_{mj} = \alpha_m^j$ ($j = 0,1$) are prescribed according to (4). With application of the difference theorem (6) there appears, however, $k + \ell = 5$ initial functions in the operational representation of equation (7):

$$x = x_{mn} = \frac{y}{p^3}(p^3\beta_n^0 + p^2\beta_n^1 + p\beta_n^2 + q^2\alpha_m^0 + q\alpha_m^1) . \tag{8}$$

With it,

$$y = \frac{p^3}{p^3+q^3} = \begin{cases} (-1)^{m/3} & \text{for } n = 2m/3, \ n = 0,3,\cdots \\ 0 & \text{otherwise} \end{cases}$$

The required compatibility conditions between the initial functions are, as result from (8) for $n = 0$ or $n = 1$ after easy calculation in the field Q,

$$\alpha_m^j = (-1)^{[m/3]}\beta_{j+2[m/3]}^{\epsilon_m} \quad (j = 0,1;\ m \geq 0) \text{ with } \epsilon_m = \begin{cases} 0 \text{ for } m \equiv 0(3) \\ 1 \text{ for } m \equiv 1(3) \\ 2 \text{ for } m \equiv 2(3) \end{cases} \tag{9}$$

or, after β_n^i is solved,

$$\beta_n^i = (-1)^{[n/2]}\alpha_{i+3\,n/2}^{\delta_n} \quad (i = 0,1,2;\ n \geq 0) \text{ with } \delta_n = \begin{cases} 0 \text{ for } n \equiv 0\ (2) \\ 1 \text{ for } n \equiv 1\ (2) \end{cases} . \tag{10}$$

If one combines the conditions (9) with the representation (8), there results the solution of equation (7) determined according to case (a) of Theorem 1 in D, namely,

$$x_{mn} = (-1)^{[m/3]} \beta^{\epsilon_m}_{2[m/3]+n} \qquad (m,n \geq 0), \tag{11}$$

while in case (b), the solution can be represented with the aid of (10) in dependence of initial functions $x_{mj} = \alpha^j_m$ $(j = 0, 1)$, in the form

$$x_{mn} = (-1)^{[n/2]} \alpha^{\delta_n}_{m+3[n/2]} \qquad (m,n \geq 0) \tag{12}$$

4⁰. As an example of a discretized partial differential equation, let us consider the difference equation

$$z_{m+2,n+1} - z_{m+1,n+2} - z_{m+1,n} + z_{m,n+1} = 0 \qquad (m,n \geq 0) \tag{13}$$

of order (2.2) appropriate for the wave equation $z_{xx} = z_{tt}$. Because $\ell_k = k_1 = 1$, the solution of (13) according to Theorem 1 is uniquely secured if three initial functions are prescribed, in the case (a) z_{on}, z_{1n}, z_{mo}, and in the case (b), z_{mo}, z_{ml}, z_{on}. For k_0, ℓ_0, only the possibility $k_0 = \ell_0 = 1$ exists. A compatibility condition between the four initial functions z_{mj} $(j = 0, 1)$, z_{in} $(i = 0, 1)$ is thus necessary. One obtains in [6] further evidence and the proof of existence of a solution from D only after application of an operational calculus to equation (13) where the initial functions are specially selected. We again use the difference law (6) with which, for arbitrary initial values $z_{mj} = \alpha^j_m$ $(j = 0, 1)$, $z_{in} = \beta^i_n$ $(i = 0, 1)$, $\alpha^j_i = \beta^i_j$ $(i,j = 0,1)$, there results the operational representation

$$z = pq/(pq - 1)(\beta^{0'}_n + \alpha^0_m) + uy(\beta^{1'}_n - \alpha^{1'}_m - v\beta^0_n + u\alpha^0_m) \tag{14}$$

(u,v in Q inverse to p,q) with

$$\beta^{i'}_n = \begin{cases} 0_i & \text{for } n = 0 \\ \beta^i_n & \text{for } n > 0 \end{cases}, \quad (i = 0,1), \qquad \alpha^{1'}_m = \begin{cases} 0 & \text{for } m = 0 \\ \alpha^1_m & \text{for } m > 0 \end{cases},$$

and

$$y = \frac{p^2q}{(p - q)(pq - 1)} = \begin{cases} m + n + 1 & \text{for } |n| \leq m, \; m \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \notin D$$

From this, there follows, after easy calculation in Q, upon use of

$$pq/(pq - 1) = \delta_{mn} \in D$$

(δ_{mn} Kronecker delta) for $n = 0$, the required compatibility condition

$$\alpha^1_{m+1} - \beta^1_{m+1} = \alpha^0_m - \beta^0_m, \qquad m \geq 0 . \tag{15}$$

If one specializes the initial functions according to [6], namely,

$$z_{m0} = z_{m1} = \alpha^0_m, \qquad z_{on} = 0, \qquad z_{in} = \beta^1_n \quad , \tag{16}$$

then (15) transforms to the condition

$$\alpha^0_m - \alpha^0_{m-1} = \beta^1_m \qquad (m \geq 1) , \tag{17}$$

which is equivalent to the equation

$$\alpha^0_n = \sum_1^n \beta^1_i \qquad (n \geq 1) \quad ,$$

given in [6].

With the compatibility condition (15), the solution of (13) can be represented in dependence on three initial functions. In the case of (a), these are α^0_m, β^i_n $(i = 0, 1)$ and there results

$$z = \delta_{mn}(\beta^{0'}_n + \alpha^0_m) + uy(\beta^{1'}_n - \beta^1_m + \beta^0_{m-1} - \beta^0_{n-1}) .$$

If one carrys out the multiplication (in Q), one obtains finally the solution $z \in D$ in the form

$$\left(\sum_1^0 a_i = 0 \quad \text{set} \right)$$

$$(18)\qquad z_{mn} = \sum_{i=1}^{\min(m,n)} (\beta^1_{m+n+1-2i} - \beta^0_{m+n-2i}) + \begin{cases} \beta^0_{m-n} & \text{for } 0 \le m \le n, \\ \alpha^0_{mon} & \text{for } 0 \le n \le m . \end{cases}$$

For the special initial functions (16), equation (18) yields

$$z_{mn} = \sum_{i=1}^{\mathrm{Min}(m,n)} \beta^1_{m+n+1-2i} + \begin{cases} 0 & \text{for } 0 \le m \le n , \\ \alpha^0_{m-n} & \text{for } 0 \le n \le m \end{cases}$$

and one easily recognizes with the aid of the special compatibility condition (17) that this function is in agreement with that given in [6].

5^0. The linear difference equation of order (1,1) with constant coefficients

$$(19)\qquad ax_{m+1,n} - bx_{m,n+1} - cx_{mn} = 0 \qquad (m,n \ge 0;\ a,b \ne 0)$$

leads to the operator representation

$$(20)\qquad x = \frac{ap}{ap - bq - c}\left(\beta - \frac{b}{a}\,uq\alpha\right)$$

with initial functions $x_{m0} = \alpha^0_m = a$, $x_{on} = \beta^0_n = \beta$. On account of the vanishing of the coefficients $x_{m+1;n+1}$, there exists, according to Theorem 1, a compatibility condition between and B. This results from (20), since, for $n = 0$,

$$y = \frac{ap}{ap - bq - c} = \begin{cases} \binom{m}{-n}\left(\frac{c}{a}\right)^m\left(\frac{c}{b}\right)^n & \text{for } -m \le n \le 0 , \\ 0 & \text{otherwise} \end{cases}$$

and $(qy\alpha)_{m0} = 0$, in the form

$$(21)\qquad \alpha_m = \left(\frac{c}{a}\right)^m \sum_{i=0}^{m} \binom{m}{i}\left(\frac{b}{c}\right)^i \beta_i \qquad (m \ge 0) .$$

In the case (a) of Theorem 1 (x_{on} prescribed), the solution of (19) can be represented, with the aid of the compatibility condition (21), as a function of β alone, namely

$$x_{mn} = \left(\frac{c}{a}\right)^m \sum_{i=0}^{m} \binom{m}{i}\left(\frac{b}{c}\right)^i \beta_{n+1} \in D \,, \tag{22}$$

which results, after easy calculation[1].

If one eliminates x and α in (20) with the aid of (21) and (22), there results the operator relation

$$a^{-m}\sum_{i=0}^{m}\binom{m}{i} b^i c^{m-i}\beta_{n+i} = \frac{1}{ap-bq-c}\left(ap\beta_n - bqa^{-m}\sum_{i=0}^{m}\binom{m}{i} b^{-i}c^{m-i}\beta_i\right)$$
$$(m, n \geq 0)$$

which for $\beta^n = d^n$ (d = constant) changes to

$$\left(\frac{c+bd}{a}\right)^m d^n = \frac{1}{ap - bq - c}\left(apd^n - bq\left(\frac{c+bd}{a}\right)^m\right) \qquad (m, n \geq 0) \,,$$

[1]For $-m \leq n \leq -1$

$$x_{mn} = \left(\frac{c}{a}\right)^m\left(\frac{c}{b}\right)^n \sum_{i=0}^{m+n} c^{-i}\left(\binom{m}{-n+i}b^i\beta_i - \binom{m-1-i}{-n-1}a^i\alpha_i\right) \,,$$

and from (21) and

$$\sum_{i=0}^{p}\binom{p+q-i}{p-i}\binom{r+i}{i} = \binom{p+q+r+1}{r} \qquad (p, q, r \geq 0)$$

it follows that $x_{mn} = 0$.

and, for $a = b$, $c = 0$, to

$$(23)\quad \beta_{mn} = \frac{1}{p - q}(p\beta_n - q\beta_m) \qquad (m,n \geq 0;\ \beta_{mn} = \beta_{m+n} \in D).$$

A formula analogous to (23) is known in the operational calculus for functions of two continuous variables (see perhaps [7]; p,q difference operators) in the theory of two-dimensional Laplace transformation (see [8]).

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