A GENERALIZED PYTHAGOREAN THEOREM

A. G. SHANNON
University of Papua and New Guinea, Boroko, T. P. N. G.
and
A. F. HORADAM
University of New England, Armidale, Australia

1. INTRODUCTION

The results in this paper arose from the efforts of the first-named author to adapt Horadam's Fibonacci number triples [3] to generate direction of numbers in solid geometry for a multivariable calculus course. This effort was unsuccessful in that the equation obtained was true for quadratic diophantine equations in general, but it did not use any properties of the Fibonacci sequence. However, it did give rise to some results for higher order sequences.

Horadam [2], [3], [4] has studied the properties of a generalized Fibonacci sequence defined by

$$H_{n+2} = H_{n+1} + H_n, \qquad (n \geq 1)\ , \tag{1}$$

with $H_1 = p$, $H_2 = p + q$. One of the properties he found was that

$$(H_n H_{n+3}) + (2H_{n+1}H_{n+2})^2 = (2H_{n+1}H_{n+2} + H_n^2)^2\ , \tag{2}$$

Which connects generalized Fibonacci numbers with Pythagorean triples.

In the next section of this paper, an analogous result is obtained for generalized "Tribonacci" numbers. The theorem is then extended to general linear difference equations of order r with unit coefficients.

2. TRIBONACCI NUMBER TRIPLES

The general Tribonacci series (see Feinberg [1]) is defined by

$$U_{n+3} = U_{n+2} + U_{n+1} + U_n, \qquad (n \geq 1) \tag{3}$$

with initial values U_1, U_2, U_3.

Theorem 1.

$$(U_n U_{n+4})^2 + (2(U_{n+1} + U_{n+2})U_{n+3})^2 = (U_n^2 + 2(U_{n+1} + U_{n+2})U_{n+3})^2 . \tag{4}$$

Proof.

$$\begin{aligned} U_n^2 &= (U_{n+3} - (U_{n+1} + U_{n+2}))^2 \\ &= U_{n+3}^2 + (U_{n+1} + U_{n+2})^2 - 2(U_{n+1} + U_{n+2})U_{n+3} \end{aligned}$$

and so

$$U_n^2 + 2(U_{n+1} + U_{n+2})U_{n+3} = U_{n+3}^2 + (U_{n+1} + U_{n+2})^2 .$$

This gives

$$\begin{aligned} &(U_n^2 + 2(U_{n+1} + U_{n+2})U_{n+3})^2 \\ &\quad = U_{n+3}^4 + (U_{n+1} + U_{n+2})^4 + 2(U_{n+1} + U_{n+2})U_{n+3})^2 \\ &\quad = \left\{U_{n+3}^2 - (U_{n+1} + U_{n+2})^2\right\}^2 + (2(U_{n+1} + U_{n+2})U_{n+3})^2 . \end{aligned} \tag{5}$$

Now

$$\begin{aligned} &\left\{U_{n+3}^2 - (U_{n+1} + U_{n+2})^2\right\}^2 \\ &\quad = (U_{n+3} - (U_{n+1} + U_{n+2}))^2 + (U_{n+3} + (U_{n+1} + U_{n+2}))^2 \\ &\quad = U_n^2\, U_{n+4}^2 . \end{aligned} \tag{6}$$

Substitution of (6) in (5) gives the result (4).

Theorem 2. All Pythagorean triples are Fibonacci triples.

Proof. Put $U_1 = x - y, \quad U_2 = y, \quad U_3 = 0.$ Then

$$U_4 = x \quad \text{and} \quad U_5 = x + y .$$

For $n = 1$, Eq. (4) becomes

$$(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 .$$

For example, when $x = 5$ and $y = 2$, we get the triple 20, 21, 29.

3. GENERALIZED PYTHAGOREAN THEOREM

Comparison of (4) with (2) suggests that for a general recurring sequence $\{V_n\}$ of order r where

$$V_{n+r} = \sum_{i=0}^{r-1} V_{n+i} , \qquad (n \geq 1) , \tag{7}$$

with initial values $V_1, V_2, \cdots, V_r$, there is a Pythagorean theorem of the form

Theorem 3.

$$\begin{aligned}(V_n V_{n+r+1})^2 &+ (2V_{n+r}(V_{n+r} - V_n))^2 \\ &= (V_n^2 + 2V_{n+r}(V_{n+r} - V_n))^2 .\end{aligned} \tag{8}$$

For example, when $r = 2$, we get

$$(V_n V_{n+3})^2 + (2V_{n+2}V_{n+1})^2 = (V_n^2 + 2V_{n+2}V_{n+1})^2 ,$$

which agrees with (2). When $r = 3$, we get

$$\begin{aligned}(V_n V_{n+4})^2 &+ (2V_{n+3}(V_{n+2} + V_{n+1}))^2 \\ &= (V_n^2 + 2(V_{n+1} + V_{n+2})V_{n+3})^2 ,\end{aligned}$$

which agrees with (4).

Lemma 1.

$$2V_{n+r} - V_{n+r+1} = V_n . \tag{9}$$

Proof.

$$
\begin{aligned}
2V_{n+r} &- V_{n+r+1} \\
&= V_{n+r} + V_{n+r} - V_{n+r+1} \\
&= (V_{n+r-1} + V_{n+r-2} + \cdots + V_{n+1} + V_n) + V_{n+r} \\
&\quad - (V_{n+r} + V_{n+r-1} + V_{n+r-2} + \cdots + V_{n+1}) \\
&= V_n \;.
\end{aligned}
$$

Lemma 2.

$$2V_{n+r} + V_{n+r+1} = 4V_{n+r} - V_n \;. \tag{10}$$

Proof. This reduces immediately to

$$2V_{n+r} - V_{n+r+1} = V_n \;,$$

which has just been proved.

Proof of Theorem 3. From Lemmas 1 and 2, we have

$$(2V_{n+r} - V_{n+r+1})(2V_{n+r} + V_{n+r+1}) = V_n(4V_{n+r} - V_n) \;,$$

which becomes

$$4V^2_{n+r} - V^2_{n+r+1} = V_n(4V_{n+r} - V_n) \;.$$

This can be rearranged as

$$V^2_{n+r+1} = V^2_n + 4V_{n+r}(V_{n+r} - V_n) \;. \tag{11}$$

On multiplication by V^2_n and addition of $(2V_{n+r}(V_{n+r} - V_n))^2$ to each side of (11), the result in (8) follows.

For example, when $r = 4$, we get a "tetranacci" series [1] and (7) becomes

$$V_{n+r} = \sum_{i=0}^{3} V_{n+i} \;,$$

with $V_1 = V_2 = V_3 = V_4 = 1$, say. Then $V_5 = 4$, $V_6 = 7$. At the same time, (8) becomes

$$(12)\qquad \begin{aligned}&(V_n^2 + 2(V_{n+1} + V_{n+2} + V_{n+3})V_{n+4})^2 \\ &\quad = (V_n V_{n+5})^2 + (2(V_{n+1} + V_{n+2} + V_{n+3})V_{n+4})^2 .\end{aligned}$$

$n = 1$ gives the Pythagorean triple 7, 24, 25.

If we call the type of triangle in (8) a recurrence triple, we get

Theorem 4. All Pythagorean triples are recurrence triples.

Proof. Put

$$V_1 = x - y, \quad V_2 = y, \quad V_3 = V_4 = 0$$

in (7). Then $V_5 = x$, $V_6 = x + y$, and for $n = 1$, Eq. (8) becomes

$$(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 .$$

4. CONCLUDING COMMENTS

The results in Theorem 3 can be used to produce various properties for recurrence relations of different orders. For instance, when $r = 2$ and $n = m - 1$, we get

$$(13)\qquad 4(H_{m+1}H_{m-1} - H_{m+1}^2) = H_{m-1}^2 - H_{m+2}^2 ,$$

which, in conjunction with Eq. (11) of [2], gives

$$(14)\qquad 4(H_m^2 - H_{m+1}^2 + (-1)^m e) = H_{m-1}^2 - H_{m+2}^2 ,$$

where $e = p^2 - pq - q^2$.

For a third-order relation, the property analogous to (13) is

$$(15)\qquad 4(U_{m+2}U_{m-1} - U_{m+2}^2) = U_{m-1}^2 - U_{m+3}^2 .$$

This may provide a convenient method for the development of properties of third and higher order recurrence relations, which have been studied in a number of papers in the Fibonacci Quarterly in recent years. For earlier studies, Morgan Ward [5] provides a useful reference.

REFERENCES

1. M. Feinberg, "Fibonacci-Tribonacci," Fibonacci Quarterly, Vol. 1, No. 3, 1963, pp. 71-74.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," American Mathematical Monthly, Vol. 68, 1961, pp. 455-459.
3. A. F. Horadam, "Fibonacci Number Triples," American Mathematical Monthly, Vol. 68, 1961, pp. 751-753.
4. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," American Mathematical Monthly, Vol. 70, 1963, pp. 289-291.
5. M. Ward, "The Algebra of Recurring Series," Annals of Mathematics, Vol. 32, 1931, pp. 1-9.

[Continued from page 306.]

3. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," Fibonacci Quarterly, Vol. 1, No. 2, 1963, pp. 1-16.
4. H. W. Gould, "Up-Down Permutations," SIAM Review, Vol. 10, 1968, pp. 225-226.
5. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1965, Ch. XVII.
6. A. F. Horadam, "A Generalized Fibonacci Sequence," The American Mathematical Monthly, Vol. 68, 1961, pp. 445-459.
7. E. Landau, "Sur la sérle des inverses des nombres de Fibonacci, Bulletin de la Société Mathématique de France, Vol. 27, 1899, pp. 298-300.