SOME PROPERTIES OF CERTAIN GENERALIZED FIBONACCI MATRICES

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INTRODUCTION

1. In this paper, we will derive a number of identities for the generalized Fibonacci sequence $\{H_n\}$ of Horadam [4] defined by the second-order recurrence relation

$$(1.1)\qquad H_{n+2} = H_{n+1} + H_n \qquad (n \text{ an integer, unrestricted}),$$

with initial values

$$(1.2)\qquad H_0 = q \quad \text{and} \quad H_1 = p,$$

by the use of generalized (square) Fibonacci matrices.

2. A generalized Fibonacci matrix is a matrix whose elements are generalized Fibonacci numbers.

3. The technique adopted is basically paralleling that due to Hoggatt and Bicknell [1], [2], and [3], where we establish numerous identities by examining the lambda functions or the characteristic equations of certain generalized Fibonacci matrices.

4. If we were to proceed as in Hoggatt and Bicknell [1] by selecting the 2-by-2 matrix defined by

$$(4.1)\qquad A = \begin{bmatrix} p+q & p \\ p & q \end{bmatrix},$$

which becomes the Q matrix of [1] when $q = 0$ and $p = 1$, and where

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$|A| = -d$ where $d = p^2 - pq - q^2$ (which is the e of [4]), we would find that we would be unable to obtain a compact expression for the matrix A^n.

5. Instead, we commence our investigations by starting with the generalized Fibonacci matrix defined by

$$(5.1) \qquad A_n = \begin{bmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{bmatrix}$$

where

$$(5.2) \qquad \begin{aligned} |A_n| &= H_{n+1}H_{n-1} - H_n^2 \\ &= (-1)^n d \end{aligned}$$

Then the matrix A defined by (4.1) is a special case of A_n when $n = 1$. The matrix A_n becomes the matrix Q^n of [1] when $q = 0$ and $p = 1$. This approach is used throughout this paper where, by changing the powers of various characteristic equations to suffixes, we are able to develop numerous easily verified identities.

THE LAMBDA FUNCTION

6. We adopt the definition of the lambda function $\lambda(M)$ of the matrix M used by Hoggatt and Bicknell [1] where, if a_{ij} is the $i - j^{th}$ element in M, then

$$(6.1) \qquad \lambda(M) = |a_{ij} + 1| - |a_{ij}|$$

7. Thus, for the Fibonacci matrix A_n defined by (5.1), we have

$$(7.1) \qquad \begin{aligned} \lambda(A_n) &= \begin{vmatrix} H_{n+1} + 1 & H_n + 1 \\ H_n + 1 & H_{n-1} + 1 \end{vmatrix} - |A_n| \\ &= H_{n-3} \end{aligned}$$

on simplification.

Hence, from (7.1) and the easily verified identity (1) of [1], viz:

$$|a_{ij} + k| = |a_{ij}| + k\lambda(M) \tag{7.2}$$

we have

$$\begin{vmatrix} H_{n+1} + k & H_n + k \\ H_n + k & H_{n-1} + k \end{vmatrix} = (H_{n+1}H_{n-1} - H_n^2) + k(H_{n-1} + H_{n+1} - 2H_n) \tag{7.3}$$

$$= |A_n| + kH_{n-3}$$

8. For a 3-by-3 matrix, the associated lambda function may be found more conveniently by the application of a theorem of [1], where, for the matrix

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

$$\lambda(M) = \begin{vmatrix} 1 & b & c \\ 1 & e & f \\ 1 & h & j \end{vmatrix} + \begin{vmatrix} a & 1 & c \\ d & 1 & f \\ g & 1 & j \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{vmatrix} \tag{8.1}$$

or

$$\lambda(M) = \begin{vmatrix} a + e - (b + d) & b + f - (c + e) \\ d + h - (g + e) & e + j - (h + f) \end{vmatrix} . \tag{8.2}$$

For example, consider the generalized Fibonacci matrix E, where

$$E = \begin{bmatrix} H_{2p} & H_{2p+1} & H_m \\ H_{2p+1} & H_{2p+2} & H_m \\ H_{2p+2} & H_{2p+3} & H_m \end{bmatrix} \tag{8.3}$$

so that

$$|E| = H_m[H_{2p+1}H_{2p+3} - H^2_{2p+2} - H_{2p}H_{2p+3} + H_{2p+1}H_{2p+2} + H_{2p}H_{2p+2} - H^2_{2p+1}]$$

$$\begin{aligned} &= H_m[H_{2p+1}H_{2p+2} - H_{2p}H_{2p+3}] \qquad (8.4)\\ &= (-1)^{2(p+1)}\,dH_m \\ &= d\,H_m \end{aligned}$$

on using (12) of Horadam [4] where $n = 2p+1$, $r = 0$, and $s = 1$.

One may evaluate $\lambda(E)$ by the use of (8.1) and a few simple column operations, whence

$$\lambda(E) = d\ . \qquad (8.5)$$

The matrix E defined by (8.3) reduces to the matrix U of [1].

9. If we let $k = H_{m-1}$ in (7.2), we have

$$\begin{aligned} |E + H_{m-1}| &= |E| + H_{m-1}\cdot d \\ &= dH_m + dH_{m-1} \qquad (9.1)\\ &= dH_{m+1}\ . \end{aligned}$$

Similarly, if we put $k = H_n$ in (7.2), then we have

$$\begin{aligned} |A_n + H_n| &= \begin{vmatrix} H_{n+1} + H_n & 2H_n \\ 2H_n & H_{n-1} + H_n \end{vmatrix} \qquad (9.2)\\ &= |A_n| + H_n\lambda(A_n) \end{aligned}$$

so that, by (5.2) and (7.1),

$$\begin{vmatrix} H_{n+2} & 2H_n \\ 2H_n & H_{n+1} \end{vmatrix} = (-1)^n d + H_nH_{n-3} \qquad (9.3)$$

from which we have

(9.4) $$4H_n^2 = H_{n+2}H_{n+1} - H_nH_{n-3} + (-1)^{n+1}d\,.$$

10. From Paragraphs 6 to 9, we can see that it is possible to derive many identities for the generalized Fibonacci sequence $\{H_n\}$ by the use of generalized Fibonacci matrices and the lambda function.

CHARACTERISTIC EQUATIONS

11. As a special case of the generalized Fibonacci matrix

(11.1) $$W_n = \begin{bmatrix} H_{n-1}^2 & H_{n-1}H_n & H_n^2 \\ 2H_{n-1}H_n & H_{n+1}^2 - H_{n-1}H_n & 2H_nH_{n+1} \\ H_n^2 & H_nH_{n+1} & H_{n+1}^2 \end{bmatrix},$$

when $n = 1$, we have the matrix W (say) where, on calculation, we have

$$W = W_1 = \begin{bmatrix} q^2 & pq & p^2 \\ 2pq & (p+q)^2 - pq & 2p(p+q) \\ p^2 & p(p+q) & (p+q)^2 \end{bmatrix}$$

whence

(11.2) $$|W| = -d^3$$

Since the Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, namely,

$$|W - \lambda I| = \lambda^3 - h\lambda^2 - dh\lambda + d^3 = 0\,,$$

W satisfies the equation

(11.3) $$W^3 - hW^2 - dhW + d^3I = 0$$

where $h = 2p^2 + 3pq + 3q^2$.

Hence, from (11.3), we have, on multiplying throughout by W^n,

$$W^{n+3} - hW^{n+2} - dhW^{n+1} + d^3W^n = 0 . \tag{11.4}$$

Now, from the relations

$$\left\{\begin{array}{l} H_{n+3}^2 - 2H_{n+2}^2 - 2H_{n+1}^2 + H_n^2 = 0 \\ H_{n+3}H_{n+4} - 2H_{n+2}H_{n+3} - 2H_{n+1}H_{n+2} + H_nH_{n+1} = 0 \\ H_{n+4}^2 - H_{n+2}H_{n+3} - 2H_{n+3}^2 + 2H_{n+1}H_{n+2} - 2H_{n+2}^2 + 2H_nH_{n+1} \\ \qquad + H_{n+1}^2 - H_{n-1}H_n = 0 \end{array}\right. \tag{11.5}$$

and so on, we can form the matrices W_{n+3}, W_{n+2}, and W_{n+1}, which will satisfy the recurrence relation

$$W_{n+3} - 2W_{n+2} - 2W_{n+1} + W_n = 0 \tag{11.6}$$

adapted from Eq. (11.4) by analogy with the special case for the ordinary Fibonacci sequence $\{F_n\}$ for which $p = 1$, $q = 0$, $h = 2$, $d = 1$.

As a special case of (11.6) for $n = 0$, we may re-write

$$W_3 - 2W_2 - 2W_1 + W_0 = 0 \tag{11.7}$$

in the equivalent form

$$W_3 + 3W_2 + 3W_1 + W_0 = 5W_2 + 5W_1 = 5(W_2 + W_1) , \tag{11.8}$$

from which, in general, it can be shown that

$$\binom{2n+1}{0}W_{2n+1} + \binom{2n+1}{1}W_{2n} + \cdots + \binom{2n+1}{2n+1}W_0 = 5^n(W_{n-1} - W_n). \tag{11.9}$$

On equating those elements in the first row and third column, and after using (9) of Horadam [4], we can deduce the result

(11.10)
$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} = 5^n (H_{n+1}^2 + H_n^2)$$
$$= 5^n [(2p - q)H_{2n+1} - dF_{2n+1}] .$$

12. We can find a number of identities for the generalized Fibonacci sequence $\{H_n\}$ by proceeding as in Hoggatt and Bicknell [3] as follows.

Consider the generalized Fibonacci matrix defined by

(12.1)
$$J_n = \begin{bmatrix} H_{2n+2} & H_{2n} \\ -H_{2n} & -H_{2n-2} \end{bmatrix}$$

where, as a special case of (12.1), we have the matrix

$$J = J_1 = \begin{bmatrix} 3p + 2q & p + q \\ -p - q & -q \end{bmatrix}$$

for $n = 1$. Since J satisfies its own characteristic equation

(12.2)
$$J^2 - (3p + q)J + dI = 0 ,$$

we can show that

(12.3)
$$(J + H_2 I)^2 = H_5 J + H_0 H_4 I .$$

This leads to the equations

(12.4)
$$J^m (J + H_2 I)^{2n} = J^m (H_5 J + H_0 H_4 I)^n$$

and

(12.5)
$$\sum_{k=0}^{2n} \binom{2n}{k} H_2^{2n-k} J^{k+m} = J^m (H_5 J + H_0 H_4 I)^n .$$

From the easily verified matrix equation

$$J_2 \; = \; 3J_1 + J_0 \; = \; 0 \tag{12.6}$$

obtained from observation of Eq. (12.2), we have the rearranged equation

$$J_2 + 2J_1 + J_0 = 5\,J_1 \quad . \tag{12.7}$$

In general, it can be shown that the J-matrices satisfy the equation

$$\binom{2n}{0}J_{2n} + \binom{2n}{1}J_{2n-1} + \cdots + \binom{2n}{2m}J_0 = 5^n J_n \;, \tag{12.8}$$

whence

$$\sum_{k=0}^{2n} \binom{2n}{k} J_k = 5^n J_n \;. \tag{12.9}$$

Hence, on equating those elements in the first row and second column, we have

$$\sum_{k=0}^{2n} \binom{2n}{k} H_{2k} = 5^n H_{2n} \;. \tag{12.10}$$

13. If we now consider the same auxiliary matrix S as in [3], viz:

$$S = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} , \tag{13.1}$$

we have, on calculation:

$$J_n S = \begin{bmatrix} H_{2n+3} & H_{2n+1} \\ -H_{2n+1} & -H_{2n-1} \end{bmatrix} \tag{13.2}$$

By proceeding as in Paragraph 12, we can similarly establish the summation

$$(13.3)\qquad \sum_{k=0}^{2n} \binom{2n}{k} J_k S = 5^n J_n S ,$$

from which we deduce the result

$$(13.4)\qquad \sum_{k=0}^{2n} \binom{2n}{k} H_{2k+1} = 5^n H_{2n+1} .$$

Similarly, we can generalize the equation

$$(13.5)\qquad J_3 + 3J_2 + 3J_1 + J_0 = 5(J_2 + J_1)$$

so that

$$(13.6)\qquad \sum_{k=0}^{2n+1} \binom{2n+1}{k} J_k = 5^n [J_{n+1} + J_n] ,$$

from which we deduce that

$$(13.7)\qquad \sum_{k=0}^{2n+1} \binom{2n+1}{k} H_{2k} = 5^n [H_{2n+2} + H_{2n}]$$

Again, we have the summation

$$(13.8)\qquad \sum_{k=0}^{2n+1} \binom{2n+1}{k} J_k S = 5^n [J_{n+1} S + J_n S]$$

from which we have

(13.9) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} H_{2k+1} = 5^n [H_{2n+3} + H_{2n+1}] \quad .$$

Finally, since we may re-write (12.6) in the form

(13.10) $$J_2 - 2J_1 + J_0 = J_1 \quad ,$$

we have, in general, that

(13.11) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} J_k = J_n \quad ,$$

so that, as before, we have the summation

(13.12) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_{2k} = H_{2n} \quad .$$

Similarly, from the summation

(13.13) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} J_k S = J_n S$$

we deduce the result

(13.14) $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_{2k+1} = H_{2n+1} \quad .$$

FURTHER SUMMATION IDENTITIES

14. As in [3], we can continue to establish further identities for the generalized Fibonacci sequence $\{H_n\}$ by letting

(14.1)
$$G_nS_0 = \begin{bmatrix} H_{4n+4} & H_{4n} \\ -H_{4n} & -H_{4n-4} \end{bmatrix} \quad \text{where } S_0 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$G_nS_1 = \begin{bmatrix} H_{4n+5} & H_{4n+1} \\ -H_{4n+1} & -H_{4n-3} \end{bmatrix} \qquad S_1 = \begin{bmatrix} 5 & 1 \\ -1 & -2 \end{bmatrix}$$

$$G_nS_2 = \begin{bmatrix} H_{4n+6} & H_{4n+2} \\ -H_{4n+2} & -H_{4n-2} \end{bmatrix} \qquad S_2 = \begin{bmatrix} 8 & 1 \\ -1 & 1 \end{bmatrix}$$

$$G_nS_3 = \begin{bmatrix} H_{4n+7} & H_{4n+3} \\ -H_{4n+3} & -H_{4n-1} \end{bmatrix} \qquad S_3 = \begin{bmatrix} 13 & 2 \\ -2 & -1 \end{bmatrix}$$

so that we have

(14.2)
$$G_n = \frac{1}{3}\begin{bmatrix} H_{4n+4} & H_{4n} \\ -H_{4n} & -H_{4n-4} \end{bmatrix}$$

As a special case of (14.2) we have, for $n = 1$, the matrix G which satisfies its characteristic equation $|G - \lambda I| = 0$, so that

(14.3)
$$G^2 - (7p + 4q)G + dI = 0 .$$

We can easily verify the matrix equation

(14.4)
$$G_2 - 7G_1 + G_0 = 0 ,$$

so that, in general, we have

(14.5) $$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} G_j = 5^n G_n .$$

Multiplying on the right by the auxiliary matrix S_s (s = 0, 1, 2, 3) and equating the elements in the first row and second column gives

(14.6) $$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} H_{4j+s} = 5^n H_{4n+s} .$$

Further, the matrix equation

(14.7) $$G_3 - 3G_2 + 3G_1 - G_0 = 5(G_2 - G_1) ,$$

may be generalized so that we have

(14.8) $$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} G_j = 5^n [G_{n+1} - G_n] .$$

On postmultiplying by S_s, we have, therefore:

(14.9) $$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} H_{4j+s} = 5^n [H_{4(n+1)+s} - H_{4n+s}] .$$

Again, Eq. (14.4) is equivalent to

(14.10) $$G_2 + 2G_1 + G_0 = 3^2 G_1 ,$$

which may be generalized to give

(14.11) $$\sum_{j=0}^{2n} \binom{2n}{j} G_j = 3^{2n} G_n .$$

Postmultiplying by S_s leads to the identity

$$\sum_{j=0}^{2n} \binom{2n}{j} H_{4j+s} = 3^{2n} H_{4n+s} \quad . \tag{14.12}$$

Similarly, the matrix equation

$$G_3 + 3G_2 + 3G_1 + G_0 = 3^2(G_2 + G_1) \quad , \tag{14.13}$$

can be generalized, so that we have

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} G_j = 3^{2n}[G_{n+1} + G_n] \quad , \tag{14.14}$$

from which, on postmultiplying by S_s, we have the final identity

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} H_{4n+s} = 3^{2n}[H_{4(n+1)+s} + H_{4n+s}] \quad . \tag{14.15}$$

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