CONTINUED FRACTIONS OF QUADRATIC FIBONACCI RATIOS

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In a previous article [1] the author investigated the continued fraction representation of linear Fibonacci ratios. As a sequence of this work a study has been made of certain quadratic ratios and their representation in continued fractions. The program as carried out was twofold: (1) Ascertaining the pattern or patterns; (2) Proving that these patterns hold in general. We shall begin with a couple of elementary examples and then report more fully a case of greater difficulty. Other patterns discovered and proved will then be listed.

THE RATIO F_{n+1}^2/F_n^2

The pattern in this case can be devined readily from a few examples.

$$F_5^2/F_4^2 = 25/9 = (2, 1, 3, 1, 1)$$
$$F_6^2/F_5^2 = 64/25 = (2, 1, 1, 3, 1, 1, 1)$$
$$F_7^2/F_6^2 = 169/64 = (2, 1, 1, 1, 3, 1, 1, 1, 1)$$

It appears that in general

$$F_{n+1}^2/F_n^2 = (2, 1_{n-3}, 3, 1_{n-2})$$

where the subscripts of the 1's indicate the number of times the quotient 1 occurs at the point in question.

We first examine the initial portion of the expansion represented by $2, 1_{n-3}$. Forming a table of convergents:

		2	1	1	1	1	1	1
0	1	2	3	5	8	13	21	34
1	0	1	1	2	3	5	8	13

we can conclude that

$$(2, 1_{n-3}) = F_n / F_{n-2} \quad .$$

If we now adjoin the 3 to the above table we have

$$\begin{array}{lll} 1 & 1 & 3 \\ F_{n-1} & F_n & 3F_n + F_{n-1} \\ F_{n-3} & F_{n-2} & 3F_{n-2} + F_{n-3} \end{array} \quad .$$

Additional 1's simply mean that the last two convergents are being treated as the first two terms of a Fibonacci sequence. Now if we start a sequence with a and b, the n^{th} term is

$$T_n = F_{n-2}a + F_{n-1}b \ .$$

In the present instance, therefore, we have for the numerator

$$F_{n-2}F_n + F_{n-1}(3F_n + F_{n-1}) \ ,$$

which can be shown to be equal to F^2_{n+1}. Similarly the denominator comes out F^2_n.

THE RATIO L^2_n / F^2_n

In this case, the pattern can be derived directly from two formulas, namely:

$$\begin{aligned} L^2_{2n} &= 3F^2_{2n} + 4 \\ L^2_{2n+1} &= 3F^2_{2n+1} - 4 \ . \end{aligned}$$

From the first relation it follows that

$$L^2_{2n} / F^2_{2n} = 5 + 4/F^2_{2n} \ .$$

Then if $F_{2n} \equiv 0 \pmod 2$,

$$L_{2n}^2/F_{2n}^2 = (5,\ F_{2n}^2/4)\ .$$

If $F_{2n} \equiv 1 \pmod 2$,

$$L_{2n}^2/F_{2n}^2 = (5,\ [F_{2n}^2/4],\ 4)$$

where the square brackets indicate the greatest integer function.

From the second relation,

$$L_{2n+1}^2/F_{2n+1}^2 = 4 + (F_{2n+1}^2 - 4)/F_{2n+1}^2\ .$$

Then

$$F_{2n+1}^2/(F_{2n+1}^2 - 4) = 1 + 4/(F_{2n+1}^2 - 4)\ .$$

If $F_{2n+1} \equiv 0 \pmod 2$, the final outcome is

$$L_{2n+1}^2/F_{2n+1}^2 = (4,\ 1,\ (F_{2n+1}^2 - 4)/4)\ .$$

If $F_{2n+1} \equiv 1 \pmod 2$,

$$L_{2n+1}^2/F_{2n+1}^2 = (4,\ 1,\ [(F_{2n+1}^2 - 4)/4],\ 4).$$

THE RATIO F_n^2/F_{n-3}^2

The case we shall consider in some detail is the ratio F_n^2/F_{n-3}^2 as it is sufficiently complex to bring out the techniques required in finding and priving the patterns. We list first the continued fraction expansions for $n = 4$ to $n = 35$. (See Table 1.)

From this table, it appears that for $n > 12$, the patterns arrange themselves modulo 6 as follows:

$n = 6k$.	$(17,\ (1,\ 16)_{k-2},\ 1,\ 15,\ 17,\ (1,\ 16)_{k-3},\ 1,\ 17)$
$n = 6k + 1.$	$(17,\ (1,\ 16)_{k-2},\ 1,\ 17,\ 1,\ 3,\ (1,\ 2,\ 1,\ 3)_{k-1},\ 2)$
$n = 6k + 2.$	$(17,\ (1,\ 16)_{k-1},\ 1_3,\ 3,\ (1,\ 2,\ 1,\ 3)_{k-1},\ 2)$
$n = 6k + 4.$	$(17,\ (1,\ 16)_{k-1},\ 1,\ 8,\ (1,\ 16)_k)$
$n = 6k + 5.$	$(17,\ (1,\ 16)_{k-1},\ 1,\ 24,\ (1,\ 16)_k)$

Table 1
CONTINUED FRACTION EXPANSION OF F_n^2/F_{n-3}^2

n	
4	(9)
5	(25)
6	(16)
7	(18,1,3,2)
8	(17,1,1,1,3,2)
9	(18,16)
10	(17,1,8,1,16)
11	(17,1,24,1,16)
12	(17,1,15,18)
13	(17,1,17,1,3,1,2,1,3,2)
14	(17,1,16,1,1,1,3,1,2,1,3,2)
15	(17,1,17,15,1,17)
16	(17,1,16,1,8,1,16,1,16)
17	(17,1,16,1,24,1,16,1,16)
18	(17,1,16,1,15,17,1,17)
19	(17,1,16,1,17,1,3,1,2,1,3,1,2,1,3,2)
20	(17,1,16,1,16,1,1,1,3,1,2,1,3,1,2,1,3,2)
21	(17,1,16,1,17,15,1,16,1,17)
22	(17,1,16,1,16,1,8,1,16,1,16,1,16)
23	(17,1,16,1,16,1,24,1,16,1,16,1,16)
24	(17,1,16,1,16,1,15,17,1,16,1,17)
25	(17,1,16,1,16,1,17,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
26	(17,1,16,1,16,1,16,1,1,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
27	(17,1,16,1,16,1,17,15,1,16,1,16,1,17)
28	(17,1,16,1,16,1,16,1,8,1,16,1,16,1,16,1,16)
29	(17,1,16,1,16,1,16,1,24,1,16,1,16,1,16,1,16)
30	(17,1,16,1,16,1,16,1,15,17,1,16,1,16,1,17)
31	(17,1,16,1,16,1,16,1,17,1,3,1,2,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
32	(17,1,16,1,16,1,16,1,16,1,1,1,3,1,2,1,3,1,2,1,3,1,2,1,3,1,2,1,3,2)
33	(17,1,16,1,16,1,16,1,17,15,1,16,1,16,1,16,1,17)
34	(17,1,16,1,16,1,16,1,16,1,8,1,16,1,16,1,16,1,16,1,16)
35	(17,1,16,1,16,1,16,1,16,1,24,1,16,1,16,1,16,1,16,1,16)

To establish these patterns, it is first necessary to examine various portions of the expansion and prove that their forms continue to hold for all values of k. For the first portion, we have Table 2.

Table 2

FIRST PORTION OF THE EXPANSION

Quotient	Numerator	Denominator	Numerator	Denominator
17	17	1	$F_9/2$	$F_3/2$
1	18	1	$F_{12}/8$	$F_6/8$
16	305	17	$F_{15}/2$	$F_9/2$
1	323	18	$F_{18}/8$	$F_{12}/8$
16	5473	305	$F_{21}/2$	$F_{15}/2$
1	5796	323	$F_{24}/8$	$F_{18}/8$
16	98209	5473	$F_{27}/2$	$F_{21}/2$
1	104005	5796	$F_{30}/8$	$F_{24}/8$

Let p_n/q_n be the partial quotient for the n^{th} step in Table 2. Then assuming that for n odd

$$\begin{aligned} p_n &= (F_{3n+6})/2 , \\ q_n &= F_{3n}/2 , \\ p_{n+1} &= (F_{3n+9})/8 , \\ q_{n+1} &= (F_{3n+3})/8 , \end{aligned}$$

it would follow that

$$\begin{aligned} p_{n+2} &= 16(F_{3n+9})/8 + (F_{3n+6})/2 \\ &= (4F_{3n+9} + F_{3n+6})/2 = (F_{3n+12})/2 . \end{aligned}$$

Similarly, for q_{n+2},

$$p_{n+3} = (F_{3n+12})/2 + (F_{3n+9})/8 = (F_{3n+15})/8 .$$

Similarly for q_{n+3}.

Thus, the pattern is seen to hold by mathematical induction. Consider next the portion $(1,16)_k$.

Table 3

PORTION $(1,16)_k$

Quotient	Numerator	Denominator
1	1	1
16	17	1
1	18	17
16	305	288
1	323	305
16	5473	5168
1	5796	5473
16	98209	92736

It appears that for n odd,

$$p_n = (F_{3n+3})/8 \ ,$$

$$q_n = F_{3n}/2 \ ,$$

$$p_{n+1} = (F_{3n+6})/2 \ ,$$

$$q_{n+1} = (F_{3n+6})/2 - F_{3n}/2 \ .$$

Again, this pattern can be shown to hold by mathematical induction. Another part of some of the patterns is $(1, 2, 1, 3)_k$. (See Table 4).

This pattern continues leading to assumed values for $n \equiv 0 \pmod 4$ as follows.

$$p_n = (L_{3+6k})/4 - F_{6k}/2 \ ,$$

$$p_{n+1} = (L_{3+6k})/4$$

From this assumption,

Table 4

PORTION $(1, 2, 1, 3)_k$

Quotient	Numerator	Denominator	Numerator	Denominator
1	1	1	$L_3/4$	$F_1 + F_0/8$
2	3	2	$F_6/2 - L_3/4$	$(3L_3 + 4F_2)/8$
1	4	3	$F_6/2$	$F_6/2 - L_3/4$
3	15	11	$L_9/4 - F_6/2$	$11 F_6/8$
1	19	14	$L_9/4$	$F_7 + F_6/8$
2	53	39	$F_{12}/2 - L_9/4$	$(3L_9 + 4F_8)/8$
1	72	53	$F_{12}/2$	$F_{12}/2 - L_9/4$
3	269	198	$L_{15}/4 - F_{12}/2$	$11 F_{12}/8$

$$\begin{aligned} p_{n+2} &= 2(L_{3+6k})/4 + (L_{3+6k})/4 - F_{6k}/2 \\ &= L_{3+6k} - F_{6k}/2 - (L_{3+6k})/4 \\ &= F_{4+6k} + F_{2+6k} - F_{6k}/2 - (L_{3+6k})/4 \\ &= F_{4+6k} + (F_{3+6k})/2 - (L_{3+6k})/4 \\ &= (F_{6+6k})/2 - (L_{3+6k})/4 , \end{aligned}$$

which agrees with the observed pattern.

From this,

$$p_{n+3} = (F_{6+6k})/2 - (L_{3+6k})/4 + (L_{3+6k})/4 = (F_{6+6k})/2 .$$

Then

$$\begin{aligned} p_{n+4} &= 3(F_{6+6k})/2 + (F_{6+6k})/2 - (L_{3+6k})/4 \\ &= 10\,(F_{6+6k})/4 - (L_{3+6k})/4 - (F_{6+6k})/2 \\ &= (F_{9+6k})/4 + 2\,(F_{8+6k})/4 - (F_{6+6k})/2 \\ &= (L_{9+6k})/4 - (F_{6+6k})/2 . \end{aligned}$$

Finally,

$$p_{n+5} = (L_{9+6k})/4 - (F_{6+6k})/2 + F_{6+6k})/2 = (L_{9+6k})/4 \ .$$

Similar considerations show that the q's follow the observed pattern.

The next step is to put the pieces together for the six given cases. For $n = 6k$, the first part is given by the partial quotients $(17, (1,16)_{k-2}, 1)$. The last part can be remodeled to this same form by changing the final 17 to 16, 1. Between these two sets of quantities is 15. Thus, the numerator and denominator can be evaluated from Table 5.

Table 5

Quotients	16	1	$15 + (F_{6k-6})/F_{6k}$
Numerator	$(F_{6k-3})/2$	$F_{6k}/8$	
Denominator	F_{6k-9}	$(F_{6k-6})/8$	

The numerator would therefore be

$$\begin{aligned} 15\,F_{6k}/8 &+ (F_{6k-6})F_{6k}/8F_{6k} + (F_{6k-3})/2 \\ &= 16\,F_{6k}^2/8F_{6k} \end{aligned} \ .$$

The denominator evaluates to

$$15\,(F_{6k-6})/8 + F_{6k-6}^2 + (F_{6k-9})/2 \ ,$$

which after some calculation gives $16\,F_{6k-3}^2/8F_{6k}$. Thus, the ratio represented is F_{6k}^2/F_{6k-3}^2. Similar considerations apply to the other five cases.

SUMMARY OF RESULTS

1. F_{n+1}^2/F_n^2. Pattern already given.
2. Patterns of F_n^2 / F_{n-2}^2.

$$F_{4k+1}^2/F_{4k-1}^2 = (6,\ (1,5)_{k-1},\ 3,\ (1,5)_{k-2},\ 1,\ 6)$$

$$F_{4k+2}^2/F_{4k}^2 = (6,\ (1,5)_{k-2},\ 1,\ 6,\ 8,\ (1,5)_{k-2}, 1, 6)$$

$$F_{4k+3}^2/F_{4k+1}^2 = (6, (1,5)_{k-1},\ 1, 3, 5, (1,5)_{k-2},\ 1, 6)$$

$$F_{4k+4}^2/F_{4k+2}^2 = (6,\ (1,5)_{k-1},\ 1, 8, 6,\ (1,5)_{k-2},\ 1, 6)\ .$$

All these results hold down to $k = 2$.

3. Patterns of F_n^2/F_{n-3}^2. Already given.
4. Patterns of L_n^2/L_{n-1}^2.

$$L_n^2/L_{n-1}^2 = (2, L_{n-5}, 2, 9 \text{ expansion of } L_{n-3}/L_{n-8}) \text{ for } n \quad 9.$$

5. Patterns of L_n^2/L_{n-2}^2.

$$L_{4k}^2/L_{4k-2}^2 = \left(6, (1,5)_{k-2}, 1, 4, 2, 33, \text{ expansion of } \frac{(L_{4k-6})/3}{L_{4k-1}}\right)$$

$$L_{4k+1}^2/L_{4k-1}^2 = (6, (1,5)_{k-2}, 1, 6, 1, 1, 2, 1, 3, 33, \text{ expansion of } (L_{4k-10}/3)/L_{4k-15})$$

$$L_{4k+2}^2/L_{4k}^2 = (6, (1,5)_{k-1}, 1, 1, 1, 1, 2, 3, 2, 1, 3, 33, \text{ expansion of } (L_{4k-14}/3)/L_{4k-19})$$

$$L_{4k+3}^2/L_{4k+1}^2 = (6, (1,5)_{k-1}, 1, 19, \text{ expansion of } (L_{4k+2}/3)/L_{4k-3})$$

6. Pattern of L_n^2/F_n. Previously given.
7. Patterns of L_n^2/F_{n-1}^2.

$$L_{5k+5}^2/F_{5k+4}^2 = (13, 11_k, 2, 4, 11_k) \text{ down to } k = 0.$$

$$L_{5k+6}^2/F_{5k+5}^2 = (13, 11_{k-1}, 10, 1, 24, 11_k) \text{ down to } k = 1.$$

$$L_{5k+7}^2/L_{5k+6}^2 = (13, 11_k, 7, 9, 11_k) \text{ down to } k = 0.$$

$$L_{5k+8}^2/F_{5k+7}^2 = (13, 11_k, 14, 12, 11_k) \text{ down to } k = 0.$$

$$L_{5k+9}^2/F_{5k+8}^2 = (13, 11_k, 10, 3, 1, 10, 11_k) \text{ down to } k = 0.$$

REFERENCE

1. Brother Alfred Brousseau, "Continued Fractions of Fibonacci and Lucas Ratios," Fibonacci Quarterly, Dec. 1964, pp. 269-276.