ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS

$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n;\ L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$

PROBLEMS PROPOSED IN THIS ISSUE

B-214 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let n be a random positive integer. What is the probability that L_n has a remainder of 11 on division by 13? [Hint: Look at the remainders for $n = 1, 2, 3, 4, 5, 6, \cdots$.]

B-215 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that for all positive integers n the quadratic $q(x) = x^2 - x - 1$ is an exact divisor of the polynomial

$$p_n(x) = x^{2n} - L_n x^n + (-1)^n$$

and establish the nature of $p_n(x)/q(x)$. [Hint: Evaluate $p_n(x)/q(x)$ for $n = 1, 2, 3, 4, 5.$]

B-216 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Solve the recurrence $D_{n+1} = D_n + L_{2n} - 1$ for D_n, subject to the initial condition $D_1 = 1$.

B-217 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array of numbers $A(n,k)$ $(n = 0, 1, 2, \cdots;\ 0 \le k \le n)$ is defined by the recurrence

$$A(n+1, k) = A(n, k-1) + (n+k+1)A(n,k) \qquad (1 \le k \le n)$$

together with the boundary conditions

$$A(n,0) = n!\ , \qquad A(n,n) = 1\ .$$

Find an explicit formula for $A(n,k)$.

B-218 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $a = (1 + \sqrt{5})/2$ and show that

$$\text{Arctan} \sum_{n=1}^{\infty} [1/(aF_{n+1} + F_n)] = \sum_{n=1}^{\infty} \text{Arctan}\,(1/F_{2n+1})\ .$$

B-219 Proposed by Tomas Djerverzon, Albrook College, Tigertown, New Mexico.

Let k be a fixed positive integer and let $a_0, a_1, \cdots, a_k$ be fixed real numbers such that, for all positive integers n,

$$\frac{a_0}{n} + \frac{a_1}{n+1} + \cdots + \frac{a_k}{n+k} = 0\ .$$

Prove that $a_0 = a_1 = \cdots = a_k = 0$.

SOLUTIONS

INVERTING A CONVOLUTION

B-196 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots,$ and $b_0, b_1, b_2, \cdots$ be two sequences such that

$$b_n = \binom{n}{0}a_n + \binom{n}{1}a_{n-1} + \binom{n}{2}a_{n-2} + \cdots + \binom{n}{n}a_0 \qquad a = 0, 1, 2, \cdots .$$

Give the formula for a_n in terms of $b_n, \cdots, b_0$.

Solution by A. C. Shannon, New South Wales, I. T., N.S.W., Australia.

We are given

$$b_n = \sum_{r=0}^{n} \binom{n}{r} a_r ,$$

and so

$$\sum_{n=0}^{\infty} b_n x^n/n! = \sum_{n=0}^{\infty}\sum_{r=0}^{n} a_r x^n/r!\,(n-r)!$$

$$= e^x \sum_{n=0}^{\infty} a_n x^n/n! ,$$

Thus

$$\sum_{n=0}^{\infty} a_n x^n/n! = e^{-x}\sum_{n=0}^{\infty} b_n x^n/n!$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} b_r(-x)^{n-r} x^r/r!\,(n-r)!$$

which gives

$$a_n = \sum_{r=0}^{n} (-1)^{n-r} {n \atop r} b_r$$

$$= \binom{n}{0} b_n - \binom{n}{1} b_{n-1} + \binom{n}{2} b_{n-2} + \cdots + (-1)^n \binom{n}{n} b_0 \ .$$

Also solved by J. L. Brown, Jr., T. J. Cullen, Herta T. Freitag, M. S. Klamkin, and the Proposer.

AN IM-PELL-ING FORMULA

B-197 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the Pell Sequence be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that there is a sequence Q_n such that

$$P_{n+2k} = Q_k P_{n+k} - (-1)^k P_n \ ,$$

and give initial conditions and the recursion formula for Q_n.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

We have

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \ ,$$

where α, β are the roots of

$$\alpha^2 = 2\alpha + 1 \ .$$

Since $\alpha\beta = -1$,

$$P_{n+2k} + (-1)^k P_n = \frac{\alpha^{n+2k} - \beta^{n+2k}}{\alpha - \beta} + \alpha^k\beta^k \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$= \frac{(\alpha^k + \beta^k)(\alpha^{n+k} - \beta^{n+k})}{\alpha - \beta} .$$

Thus if we put

$$Q_k = \alpha^k + \beta^k ,$$

we have

$$P_{n+2k} = Q_k P_{n+k} - (-1)^k P_n .$$

Clearly,

$$Q_{k+2} = 2Q_{k+1} + Q_k, \qquad Q_0 = Q_1 = 2 .$$

Also,

$$\sum_{k=0}^{\infty} Q_k x^k = \frac{1}{1 - \alpha x} + \frac{1}{1 - \beta x} = \frac{2 - 2x}{1 - 2x - x^2} .$$

Also solved by Clyde A. Bridger, T. J. Cullen, Herta T. Freitag, M. S. Klamkin, and the Proposer.

PERMUTATIONS, DERANGEMENTS, AND THESE THINGS

B-198 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let c_n be the coefficient of $x_1x_2 \cdots x_n$ in the expansion of

$$(-x_1 + x_2 + x_3 + \cdots + x_n)(x_1 - x_2 + x_3 + \cdots + x_n)(x_1 + x_2 - x_3 + \cdots + x_n)$$
$$\cdots (x_1 + x_2 + x_3 + \cdots + x_{n-1} - x_n) .$$

For example,

$$c_1 = -1, \quad c_2 = 2, \quad c_3 = -2, \quad c_4 = 8 \quad \text{and} \quad c_5 = 8 .$$

Show that

$$c_{n+2} = nc_{n+1} + 2(n+1)c_n, \qquad c_n = nc_{n-1} + (-2)^n ,$$

and

$$\lim_{n\to\infty} (c_n/n!) = e^{-2} .$$

Solution by M. S. Klamkin, Ford Motor Company, Dearborn, Michigan.

Letting $x = \Sigma x_1$, the given produce is a special case of (i.e., for $a = 2$)

$$(x - ax_1)(x - ax_2) \cdots (x - ax_n) =$$
$$x^n - ax^{n-1}\Sigma x_i + a^2 x^{n-2}\Sigma x_i x_j - a^3 x^{n-3}\Sigma x_i x_j x_k + \cdots .$$

Then by the multinomial theorem, the coefficient of $x_1x_2 \cdots x_n$ is given by the sum

$$n! - a\binom{n}{1}[(n-1)!] + a^2\binom{n}{2}[(n-2)!] - \cdots$$

or

$$c_n(a) = n!\left\{1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots + \frac{(-a)^n}{n!}\right\} .$$

It now immediately follows that

$$c_n - nc_{n-1} = (-a)^n ,$$
$$c_{n+2} - (n+2)c_{n+1} = -a[c_{n+1} - (n+1)c_n] .$$

Also, since

$$e^{-a} = 1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots ,$$

$$\lim_{n \to \infty} c_n/n! = e^{-a} .$$

Now let $a = 2$ to give the desired special case.

Also solved by L. Carlitz, Herta T. Freitag, Graham Lord, David Zeitlin, and the Proposer.

A FIBONACCI-PELL INEQUALITY

B-199 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

Define the Fibonacci and Pell numbers by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad n \geq 1;$$
$$P_1 = 1, \quad P_2 = 2, \quad P_{n+2} = 2P_{n+1} + P_n \quad n \geq 1.$$

Prove or disprove that $P_{6k} < F_{11k}$ for $k \geq 1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Let α, β be the roots of $x^2 = x + 1$, and A, B the roots of $x^2 = 2x + 1$. Now,

$$Y_k \equiv F_{11k} = (\alpha^{11k} - \beta^{11k})/(\alpha - \beta)$$

satisfies

$$(E - \alpha^{11})(E - \beta^{11})Y_k = 0, \qquad \text{or} \qquad Y_{k+2} - 199\,Y_{k+1} - Y_k = 0,$$

and

$$Z_k \equiv P_{6k} = (A^{6k} - B^{6k})/(A - B)$$

satisfies

$$(E - A^6)(E - B^6)Z_k = 0 ,$$

or

$$Z_{k+2} - 198 Z_{k+1} + Z_k = 0,$$

where $E^n R_k = R_{k+n}$.

Let

$$W_k \equiv Z_k - Y_k \equiv P_{6k} - F_{11k} .$$

Then

$$\text{(1)} \quad W_{k+2} = Z_{k+2} - Y_{k+2} = 198(Z_{k+1} - Y_{k+1}) - Y_{k+1} - Y_k - Z_k ,$$

or

$$W_{k+2} - 198 W_{k+1} \quad 0, \qquad k = 0, 1, \cdots,$$

and thus

$$W_k < (198)^{k-1} W_1 \qquad k = 2, 3, \cdots .$$

Since

$$W_1 = P_6 - F_{11} = 70 - 89 = -19 < 0 ,$$

$W_k \equiv P_{6k} - F_{11k} < 0$ for $k = 1, 2, \cdots$. Thus, the stated inequality is true.

Also solved by Wayne Vucenic and the Proposer.

A CLOSE CALL

B-200 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

With the notation of B-199, prove or disprove that

$$F_{11k} < P_{6k+1} \quad \text{for} \quad k \geq 1 .$$

Solution by Phil Mana and Wayne Vucenic, University of New Mexico, Albuquerque, New Mexico.

Let

$$a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2, \quad c = 1 + \sqrt{2}, \quad \text{and} \quad d = 1 - \sqrt{2} .$$

Then

$$F_{11k} = (a^{11k} - b^{11k})/\sqrt{5}, \qquad P_{6k+1} = (c^{6k+1} - d^{6k+1})/2\sqrt{2} .$$

Since $|a| > |b|$ and $|c| > |d|$, it can easily be seen that as $k \to \infty$ the limit of F_{11k}/P_{6k+1} is a positive constant times the limit of $(a^{11}/c^6)^k$. Since

$$a^{11} = (L_{11} + F_{11}\sqrt{5})/2 = (199 + 89\sqrt{5})/2 \qquad 99 + 70\sqrt{2} = c^6 ,$$

$(a^{11}/c^6)^k \to +\infty$ as $k \to \infty$ and so ultimately $F_{11k} > P_{6k+1}$. Computer calculation shows that when $k = 128$, $F_{11k} > 8 \times 10^{293} > P_{6k+1}$.

Also solved by the Proposer.

PARITY OF n

B-201 Proposed by Mel Most, Ridgefield Park, New Jersey.

Given that a very large positive integer k is a term F_n in the Fibonacci Sequence, describe an operation on k that will indicate whether n is even or odd.

I. Solution by F. D. Parker, St. Lawrence University, Canton, New York.

Undoubtedly, there are many possible tests. Here is only one test. From the identities

$$F_n^2 = F_{n-1}F_{n+1} - (-1)^n$$
$$F_{n+1} = F_n + F_{n-1} \quad ,$$

we get

$$F_n + 2F_{n-1} = \sqrt{5F_n^2 + 4(-1)^n}$$

Therefore n is even if $5F_n^2 + 4$ is a perfect square; otherwise, n is odd, with the single exception of $n = 1$ or $n = 2$. In this case, no test prevails since $F_1 = F_2 = 1$.

II. Solution by Wayne Vucenic, Student, University of New Mexico, Albuquerque, New Mexico.

The ratio between consecutive terms of the Fibonacci sequence, F_{n+1}/F_n, approaches α by oscillation as n approaches infinity, where α is the Greek golden ratio, or $\frac{1}{2}(1 + \sqrt{5})$, which is $1.61803\cdots$. Thus, if n is even,

(1) $$\frac{F_{n+1}}{F_n} = \alpha + A_n, \qquad \text{and } A_n \text{ decreases as } n \text{ increases;}$$

if n is odd,

(2) $$\frac{F_{n+1}}{F_n} = \alpha - B_n, \qquad \text{and } B_n \text{ decreases as } n \text{ increases.}$$

If n is even, from Equation (1),

$$F_{n+1} = F_n(\alpha + A_n)$$

$$F_{n+1} = \alpha F_n + A_n F_n$$
$$\alpha F_n = F_{n+1} - A_n F_n ;$$

if n is odd, from Equation (2),

$$\alpha F_n = F_{n+1} + B_n F_n .$$

These equations show that αF_n will be less than F_{n+1} if n is even, and will be greater than F_{n+1} if n is odd.

As n increases, A_n and B_n decrease fast enough that, if $n \geq 2$, $A_n F_n < 0.5$ and $B_n F_n < 0.5$.

Thus, if $n \geq 2$, it is possible to determine whether n is even or odd by multiplying F_n by α, then seeing if the product is greater than or less than the nearest integer which will be F_{n+1}. For example, given that $F_n = 21$, $21 \times 1.618 = 33.978$. This is less than the nearest integer, 34, thus a is even.

Also solved by the Proposer.

[Continued from page 437.] When X and Y are -ve integers,

$$X = (2 - L_{4k})/5, \quad Y = (X - F_{4k})/2, \quad k = 1, 2, 3, \ldots.$$

And the general solution in +ve integers is:

$$X = (2 + L_{4k-2})/5 = F_{2k-1}^2; \quad Y = (X + F_{4k-2})/2 = F_{2k-1}F_{2k}$$

The author found the first set of integral solutions while others were found by Guy Guillotte

REFERENCES

1. J. A. H. Hunter, "Fibonacci to the Rescue," Fibonacci Quarterly, Oct. 1970, p. 406.
2. David Ferguson, "Letters to the Editor," Fibonacci Quarterly, Feb. 1970, p. 88.