REDUCTION FORMULAS FOR FIBONACCI SUMMATIONS

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1. INTRODUCTION

In a recent paper [1], Brother Alfred Brousseau has obtained a chain of formulas of the following kind.

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}} \quad ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}} = \frac{5}{12} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \quad ,$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} = \frac{97}{2640} + \frac{40}{11} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} \cdots F_{n+6}} \quad .$$

As an application he has computed the value of the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{F_n}$$

to twenty-five decimal places. It does not seem to be known whether the sum S is a rational number.

If we define

$$S_k = \sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{F_n F_{n+1} \cdots F_{n+2k}} \qquad (k = 0, 1, 2, \cdots) ,$$

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the above special results suggest that generally

$$S_{k+1} = a_k + b_k S_k , \tag{1.1}$$

where a_k, b_k are rational numbers. We shall show below that this is indeed true and moreover we shall obtain explicit formulas for a_k, b_k. Also we obtain explicit formulas for the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{F_n F_{n+1} \cdots F_{n+2k-1}} \qquad (k = 1, 2, 3, \cdots) .$$

Indeed we shall prove these results in a somewhat more general setting. In place of the Fibonacci numbers F_n we take the numbers u_n defined by

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = (\alpha + \beta)u_n - \alpha\beta u_{n-1} \qquad (n = 1, 2, 3, \cdots),$$

where α, β are distinct, and consider the sums

$$U_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}}$$

and

$$T_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k-1}} .$$

We show that

$$U_{k+1} = c_k + d_k U_k , \tag{1.2}$$

where c_k, d_k are rational functions of α, β that are determined explicitly. As for T_k, we show that

$$T_k = c_k' + \frac{d_k'}{\alpha} ,$$

where c_k', d_k' are rational functions of α, β that are determined explicitly. Also it is assumed, in order to assure convergence, that

$$|\alpha| > |\beta|, \qquad |\alpha| > 1 .$$

2. SOME PRELIMINARY RESULTS

To begin with, let α, β denote indeterminates and put

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n . \tag{2.1}$$

Then, of course,

$$\begin{cases} u_{n+1} = (\alpha + \beta)u_n - \alpha\beta u_{n-1} \\ v_{n+1} = (\alpha + \beta)v_n - \alpha\beta v_n \end{cases} . \tag{2.2}$$

Next define

$$(u)_0 = 1, \qquad (u)_n = u_1 u_2 \cdots u_n \tag{2.3}$$

and

$$\left\{ {n \atop k} \right\} = \frac{(u)_n}{(u)_k (u)_{n-k}} = \left\{ {n \atop n - k} \right\} . \tag{2.4}$$

It follows from the definition that

$$\begin{aligned} \left\{ {n + 1 \atop k} \right\} &= \alpha^k \left\{ {n \atop k} \right\} + \beta^{n-k+1} \left\{ {n \atop k - 1} \right\} \\ &= \beta^k \left\{ {n \atop k} \right\} + \alpha^{n-k+1} \left\{ {n \atop k - 1} \right\} . \end{aligned} \tag{2.5}$$

Clearly $u_n v_n$, $(u)_n$, $\left\{ {n \atop k} \right\}$ are symmetric polynomials in α, β; the last assertion is a consequence of (2.5).

Let

$$R_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^j \ , \tag{2.6}$$

$$R_{2k-1}(x) = \sum_{j=0}^{2k-1} (-1)^j \left\{ \begin{matrix} 2k-1 \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \alpha^{j+1} x^j \ . \tag{2.7}$$

Then, by (2.5),

$$R_{2k}(x) = \sum_{j=0}^{2k} (-1)^j (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^j \left[\alpha^j \left\{ \begin{matrix} 2k-1 \\ j \end{matrix} \right\} + \beta^{2k-j} \left\{ \begin{matrix} 2k-1 \\ j-1 \end{matrix} \right\} \right]$$

$$= \sum_{j=0}^{2k-1} (-1)^j \left\{ \begin{matrix} 2k-1 \\ j \end{matrix} \right\} x^j \Big[(\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \alpha^j$$

$$- (\alpha\beta)^{\frac{1}{2}j(j+1)(j+2)-(j+1)k} \beta^{2k-j+1} x \Big]$$

$$= \sum_{j=0}^{2k-1} (-1)^j \left\{ \begin{matrix} 2k-1 \\ j \end{matrix} \right\} x^j \Big[\alpha^{\frac{1}{2}j(j+1)(j+2)-jk-1} \beta^{\frac{1}{2}j(j+1)-jk}$$

$$- \alpha^{\frac{1}{2}(j+1)(j+2)-jk-j} \beta^{\frac{1}{2}j(j+1)-jk+k} x \Big]$$

$$= (\alpha^{-1} - \alpha^k \beta^k x) \sum_{j=0}^{2k-1} (-1)^j \left\{ \begin{matrix} 2k-1 \\ j. \end{matrix} \right\} \alpha^{\frac{1}{2}(j+1)(j+2)-jk} \beta^{\frac{1}{2}j(j+1)-jk} x^j \ ,$$

so that

$$R_{2k}(x) = (\alpha^{-1} - \alpha^{-k}\beta^k x) R_{2k-1}(x) \ . \tag{2.8}$$

Similarly,

$$R_{2k+1}(x) = \sum_{j=0}^{2k+1} (-1)^j \left\{ {2k+1 \atop j} \right\} \alpha^{\frac{1}{2}(j+1)(j+2)-j(k+1)} \beta^{\frac{1}{2}j(j+1)-j(k+1)} x^j$$

$$= \sum_{j=0}^{2k+1} (-1)^j \alpha^{\frac{1}{2}(j+1)(j+2)-j(k+1)} \beta^{\frac{1}{2}j(j+1)-j(k+1)} x^j$$

$$\cdot \left[\beta^j \left\{ {2k \atop j} \right\} + \alpha^{2k-j+1} \left\{ {2k \atop j-1} \right\} \right]$$

$$= \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} x^j \left[\alpha^{\frac{1}{2}(j+1)(j+2)-jk-j} \beta^{\frac{1}{2}j(j+1)-jk} \right.$$

$$- \alpha^{\frac{1}{2}(j+2)(j+3)-(j+1)(k+1)+2k-j}$$

$$\left. \cdot \beta^{\frac{1}{2}(j+1)(j+2)-(j+1)(k+1)} x \right]$$

$$= \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} x^j \left[\alpha^{\frac{1}{2}j(j+1)-jk+1} \beta^{\frac{1}{2}j(j+1)-jk} \right.$$

$$\left. - \alpha^{\frac{1}{2}j(j+1)-jk+k+2} \beta^{\frac{1}{2}j(j+1)-jk-k} x \right]$$

$$= (\alpha - \alpha^{k+1}\beta^{-k}x) \sum_{s=0}^{2k} (-1)^k \left\{ {2k \atop j} \right\} \alpha^{\frac{1}{2}j(j+1)-jk} \beta^{\frac{1}{2}j(j+1)-jk} x^j$$

and so

$$R_{2k+1}(x) = (\alpha - \alpha^{k+1}\beta^{-k}x) R_{2k}(x) . \tag{2.9}$$

Combining (2.8) and (2.9) we get

$$R_{2k}(x) = (\alpha\beta)^{-k+1}(\alpha^{k-1} - \beta^k x)(\beta^{k-1} - \alpha^k x) R_{2k-2}(x) \qquad (k \geq 1) \tag{2.10}$$

and therefore

$$R_{2k}(x) = (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} (\alpha^{j-1} - \beta^j x)(\beta^{j-1} - \alpha^j x) \tag{2.11}$$

$$= (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j-1}x + (\alpha\beta)^j x^2 \right] ,$$

with v_{2j-1} defined by (2.1).

The recurrence (2.10) can be generalized in the following way. Let

$$\xi = (x_0, x_1, x_2, \cdots)$$

denote an arbitrary sequence and define

$$R_{2k}(\xi) = \sum_{j=0}^{2k} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x_j .$$

Then, exactly as above, we have

$$(2.12) \qquad R_{2k}(\xi) = (\alpha\beta)^{-k+1} \sum_{j=0}^{2k-2} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} (\alpha\beta)^{\frac{1}{2}j(j+1)-j(k-1)} \cdot \left[(\alpha\beta)^{k-1} x_j - v_{2k-1}x_{j+1} + (\alpha\beta)^k x_{j+2}\right] .$$

It follows from (2.12) that

$$(2.13) \qquad R_{2k+2}(\xi) - R_{2k}(\xi) = \sum_{j=0}^{2k} (-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} \alpha\beta^{\frac{1}{2}j(j+1)-jk}\left[-(\alpha\beta)^{-k} v_{2k+1}x_{j+1} + \alpha\beta x_{j+2}\right]$$

3. A SECOND PROOF OF EQ. (2.11)

It may be of interest to show that (2.11) can be obtained from a known result. We recall that

(3.1) $$\prod_{j=0}^{k-1}(1 - q^j x) = \sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)} x^j \quad ,$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q^{k-j+1})}{(1 - q)(1 - q^2) \cdots (1 - q^j)} \quad .$$

Replacing q by α/β ,

$$\begin{bmatrix} k \\ j \end{bmatrix} \rightarrow \frac{(\beta^k - \alpha^k)(\beta^{k-1} - \alpha^{k-1}) \cdots (\beta^{k-j+1} - \alpha^{k-j+1})}{(\beta - \alpha)(\beta^2 - \alpha^2) \cdots (\beta^j - \alpha^j)} \beta^{j^2-jk}$$

$$= \begin{Bmatrix} k \\ j \end{Bmatrix} \beta^{j^2-jk}$$

Thus (2.1) becomes

$$\beta^{-\frac{1}{2}k(k-1)} \prod_{j=0}^{k-1}(\beta^j - \alpha^j x) = \sum_{j=0}^{k}(-1) \begin{Bmatrix} k \\ j \end{Bmatrix} \alpha^{\frac{1}{2}j(j-1)} \beta^{\frac{1}{2}j(j+1)-jk} x^j \quad .$$

In particular, if k is replaced by 2k, we get

(3.2) $$\beta^{-k(2k-1)} \prod_{j=0}^{2k-1}(\beta^j - \alpha^j x) = \sum_{j=0}^{2k}(-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} \alpha^{\frac{1}{2}j(j-1)} \beta^{j(j+1)-2jk} x_j \; .$$

Now replace x by $\alpha^{1-k}\beta^k x$ and (3.2) becomes

$$\beta^{-k(2k-1)} \prod_{j=0}^{2k-1}(\beta^j - \alpha^{j-k+1}\beta^k x)$$

$$= \sum_{j=0}^{2k}(-1)^j \begin{Bmatrix} 2k \\ j \end{Bmatrix} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^j \quad ,$$

so that

$$\tag{3.3} \begin{aligned} R_{2k}(x) &= \beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^{j-k+1}\beta^k x) \\ &= \prod_{j=0}^{2k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) = \prod_{j=1}^{2k} (1 - \alpha^{k-j+1}\beta^{j-k}x) \ ; \end{aligned}$$

at the last step we have replaced j by $2k - j$.

Now on the other hand,

$$\begin{aligned} &\prod_{j=1}^{k} (\alpha^{j-1} - \beta^j x)(\beta^{j-1} - \alpha^j x) \\ &= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{k=1}^{k} (1 - \alpha^{-j+1}\beta^j x)(1 - \alpha^j\beta^{-j+1}x) \\ &= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x)(1 - \alpha^{k-j}\beta^{j-k+1}x) \\ &= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) \cdot \prod_{j=k}^{2k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) \\ &= (\alpha\beta)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{2k-1} (1 - \alpha^{j-k+1}\beta^{k-j}x) \ . \end{aligned}$$

Substitution in (3.3) gives

$$R_{2k}(x) = (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} (\alpha^{j-1} - \beta^j x)(\beta^{j-1} - \alpha^j x) \ ,$$

which is the first of (2.11).

4. THE MAIN RESULTS

We consider next the expansion into partial fractions of

$$\frac{x^k}{(1 - x)(\alpha - \beta x)(\alpha^2 - \beta^2 x) \cdots (\alpha^{2k} - \beta^{2k} x)} = \sum_{j=0}^{2k} \frac{A_j}{\alpha^j - \beta^j x} , \tag{4.1}$$

where A_j is independent of x. We find that

$$(\alpha - \beta)^{2k} A_j = (-1)^j \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{(u)_j (u)_{2k-j}} , \tag{4.2}$$

where, as above,

$$(u)_n = \prod_{j=1}^{n} \frac{\alpha^j - \beta^j}{\alpha - \beta} .$$

Thus we have the identity

$$\frac{(\alpha - \beta)^{2k} x^k}{(1 - x)(\alpha - \beta x) \cdots (\alpha^{2k} - \beta^{2k} x)} \tag{4.3}$$

$$= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{\alpha^j - \beta^j x} .$$

For $x = \alpha^{-n}\beta^n$, the left member of (4.3) becomes

$$\frac{}{\alpha - \beta} \frac{\alpha^{n(k+1)} \beta^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} ,$$

while the right member becomes

$$\frac{\alpha^n}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{\alpha^{n+j} - \beta^{n+j}} .$$

We have therefore the identity

$$\frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2j \\ j \end{matrix} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}}{\alpha^{n+j} - \beta^{n+j}} \quad . \tag{4.4}$$

Now put

$$U_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} \qquad (k = 0, 1, 2, \cdots) \tag{4.5}$$

and in particular, for $k = 0$,

$$U = U_0 = \sum_{n=1}^{\infty} \frac{1}{u_n} \quad . \tag{4.6}$$

To assure convergence, we assume that

$$|\alpha| > |\beta|, \qquad |\alpha| > 1 \ .$$

Then, by (4.4) and (4.5),

$$U = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{\infty} \frac{1}{u_{n+j}}$$

$$= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \cdot U$$

$$- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{j} \frac{1}{u_n} \quad .$$

The coefficient on the right is equal to

$$\frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$= \frac{1}{(u)_{2k}} S_{2k}(1)$$

$$= \frac{(\alpha\beta)^{-\frac{1}{2}k(k-1)}}{(u)_{2k}} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j+1} + (\alpha\beta)^j \right] .$$

We have therefore

$$U_k = \frac{(\alpha\beta)^{-\frac{1}{2}k(k-1)}}{(u)_{2k}} U \cdot \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j+1} + (\alpha\beta)^j \right]$$

$$(4.7) \qquad - \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{j} \frac{1}{u_n} .$$

More generally, if we put

$$(4.8) \qquad U_k(x) = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk} x^{n+2k}}{u_n u_{n+1} \cdots u_{n+2k}}$$

and in particular, for $k = 0$,

$$(4.9) \qquad U(x) = U_0(x) = \sum_{n=1}^{\infty} \frac{x^n}{u_n} ,$$

then as above,

$$U_k(x) = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} \sum_{n=1}^{\infty} \frac{x^{n+2k}}{u_{n+j}}$$

$$= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^{2k-j} U(x)$$

$$- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^{2k-j} \sum_{n=1}^{j} \frac{x^n}{u_n} .$$

Since

$$\sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk} x^{2k-j} = x^{2k} S_{2k}(x^{-1})$$

$$= (\alpha\beta)^{-\frac{1}{2}k(k-1)} x^{2k} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} - v_{2j-1} x^{-1} + (\alpha\beta)^j x^{-2} \right]$$

$$= (\alpha\beta)^{-\frac{1}{2}k(k-1)} \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} x^2 - v_{2j-1} x + (\alpha\beta)^j \right] ,$$

it is clear that

$$U_k(x) = \frac{(\alpha\beta)^{-\frac{1}{2}k(k-1)}}{(u)_{2k}} U(x) \prod_{j=1}^{k} \left[(\alpha\beta)^{j-1} x^2 - v_{2j-1} x + (\alpha\beta)^j \right]$$

(4.10)

$$- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{j(j+1)-jk} x^{2k-j} \sum_{n=1}^{j} \frac{x^n}{u_n} .$$

It follows from (4.10) that

$$(4.11)\quad U_{k+1}(x) - \frac{x^2 - (\alpha\beta)^k v_{2k+1}x + \alpha\beta}{u_{2k+1}u_{2k+2}} U_k(x)$$

$$= -\frac{x^{2k+2}}{(u)_{2k+2}}\left\{\sigma_{2k+2}(x) - \left[1 - (\alpha\beta)^{-k}v_{2k+1}x^{-1} + \alpha\beta x^{-2}\right]\sigma_{2k}(x)\right\},$$

where

$$\sigma_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{j(j+1)-jk} x^{-j} \sum_{n=1}^{j} \frac{x^n}{u_n} .$$

If we now apply (2.13) to $\sigma_{2k}(x)$ with

$$(4.12)\qquad x_j = x^{-j}\sum_{n=1}^{j}\frac{x^n}{u_n} ,$$

we get

$$\sigma_{2k+2}(x) - \sigma_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$\cdot \left[-(\alpha\beta)^{-k} v_{2k+1}x_{j+1} + \alpha\beta x_{j+2}\right] .$$

Thus, by (4.12), (4.11) reduces to

$$U_{k+1}(x) - \frac{x^2 - (\alpha\beta)^{-k} v_{2k+1}x + \alpha\beta}{u_{2k+1}u_{2k+2}} U_k(x)$$

$$(4.13)\qquad = \frac{x^{2k+2}}{(u)_{2k+2}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$\cdot \left[(\alpha\beta)^{-k}\frac{v_{2k+1}}{u_{j+1}} - \alpha\beta\left(\frac{x^{-1}}{u_{j+1}} + \frac{1}{u_{j+2}}\right)\right] .$$

In particular, for $x = 1$, (4.13) becomes

$$U_{k+1} - \frac{1 - (\alpha\beta)^{-k} v_{2k+1} x + \alpha\beta}{u_{2k+1} u_{2k+2}} U_k$$

(4.14)
$$= \frac{1}{(u)_{2k+2}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-jk}$$

$$\cdot \left[(\alpha\beta)^{-k} \frac{v_{2k+1}}{u_{j+1}} - \alpha\beta \left(\frac{1}{u_{j+1}} + \frac{1}{u_{j+2}} \right) \right] .$$

5. APPLICATION TO FIBONACCI SUMMATIONS

We now consider the special case

(5.1)
$$\alpha + \beta = 1, \qquad \alpha\beta = -1 .$$

Then

(5.2)
$$u_n = F_n, \qquad v = L_n ,$$

the Fibonacci and Lucas numbers, respectively. Also $U_k(x)$ becomes

(5.3)
$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} x^{n+2k}}{F_n F_{n+1} \cdots F_{n+2k}}$$

and in particular U_k becomes

(5.4)
$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_n F_{n+1} \cdots F_{n+2k}} = (-1)^k S_k .$$

Formula (4.10) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk} x^{n+2k}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{1}{(F)_{2k}} \prod_{j=1}^{k} (x^2 + (-1)^j L_{2j-1} x - 1) \cdot \sum_{n=1}^{\infty} \frac{x^n}{F_n}$$
$$\tag{5.5} - \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j-1)-jk} \begin{Bmatrix} 2k \\ j \end{Bmatrix} x^{2k-j} \sum_{n=1}^{j} \frac{x^n}{F_n} ,$$

where now

$$\begin{Bmatrix} n \\ j \end{Bmatrix} = \frac{F_n F_{n-1} \cdots F_{n-j+1}}{F_1 F_2 \cdots F_j}$$

and

$$(F)_{2k} = F_1 F_2 \cdots F_{2k} .$$

In particular, for $x = 1, -1$, (5.5) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{(-1)^{\frac{1}{2}k(k+1)}}{(F)_{2k}} \prod_{j=1}^{k} L_{2j-1} \cdot \sum_{n=1}^{\infty} \frac{1}{F_n}$$
$$\tag{5.6} - \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j-1)-jk} \begin{Bmatrix} 2k \\ j \end{Bmatrix} \sum_{n=1}^{j} \frac{1}{F_n} ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(k+1)}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{(-1)^{\frac{1}{2}k(k-1)}}{(F)_{2k}} \prod_{j=1}^{k} L_{2j-1} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}$$
$$\tag{5.7} - \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j+1)-jk} \begin{Bmatrix} 2k \\ j \end{Bmatrix} \sum_{n=1}^{j} \frac{(-1)^n}{F_n} .$$

For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = -S + 3 ,$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = -\frac{2}{3} S + \frac{41}{18} ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+6}} = \frac{11}{60} S + \frac{17749}{28800} ,$$

where

$$S = \sum_{n=1}^{\infty} \frac{1}{F_n} .$$

We note also that (4.14) yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(k+1)}}{F_n F_{n+1} \cdots F_{n+2k+2}} + \frac{(-1)^k L_{2k+1}}{F_{2k+1} F_{2k+2}} \sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_n F_{n+1} \cdots F_{n+2k}}$$

(5.8)
$$= \frac{1}{(F)_{2k+2}} \sum_{j=0}^{2k} (-1)^{\frac{1}{2}j(j-1)-jk} \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} \left[(-1)^k \frac{L_{2k+1}}{F_{j+1}} + \frac{1}{F_{j+1}} + \frac{1}{F_{j+2}} \right] .$$

For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} + \sum_{n=1}^{\infty} \frac{1}{F_n} = 3 ,$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} - \frac{2}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = \frac{5}{18} ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+6}} + \frac{11}{40} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 8} \, \frac{97}{40} = \frac{97}{9600} ,$$

in agreement with the special results obtained in [1].

It should be observed that the formulas of this section depend essentially on $\alpha\beta = -1$. Very similar results can be stated for $\alpha\beta = 1$. Thus, in particular we can obtain results like the above for such sums as

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+2} \cdots F_{2n+4k}}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_{3n} F_{3n+3} \cdots F_{3n+6k}} .$$

6. SOME ADDITIONAL RESULTS

Returning to the general case, we shall now evaluate the sum

$$T_k = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k-1}} \qquad (k = 1, 2, 3, \cdots) . \tag{6.1}$$

Multiplying (4.4) by $(\alpha\beta)^n / u_{n+2k+1}$, we get

$$\frac{(\alpha\beta)^{n(k+1)}}{u_n u_{n+1} \cdots u_{n+2k+1}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \frac{(\alpha\beta)^{\frac{1}{2}j(j+1)-jk+n}}{u_{n+j} u_{n+2k+1}}$$

so that

$$\sum_{n=1}^{\infty} \frac{(\alpha\beta)^{n(k+1)}}{u_n u_{n+1} \cdots u_{n+2k+1}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} (\alpha\beta)^{\frac{1}{2}j(j+1)-j(k+1)}$$

(6.2)

$$\cdot \sum_{n=1}^{\infty} \frac{(\alpha\beta)^{n+j}}{u_{n+j} u_{n+2k+1}} .$$

Now consider the sum

(6.3)
$$A_r = \sum_{n=1}^{\infty} \frac{(\alpha\beta)^n}{u_n u_{n+r}} .$$

Since

$$u_{n+r} u_{n-1} - u_n u_{n+r-1} = (\alpha\beta)^n u_r ,$$

we have

$$\frac{u_{n-1}}{u_n} - \frac{u_{n+r-1}}{u_{n+r}} = \frac{(\alpha\beta)^n u_r}{u_n u_{n+r}} .$$

In this identity, take $n = 1, 2, \cdots, N$ and sum. Then

$$u_r \sum_{n=1}^{N} \frac{(\alpha\beta)^n}{u_n u_{n+r}} = \sum_{n=1}^{N} \frac{u_{n-1}}{u_n} - \sum_{n=1}^{N} \frac{u_{n+r-1}}{u_{n+r}}$$

(6.4)

$$= \sum_{n=1}^{r} \frac{u_{n-1}}{u_n} - \sum_{n=1}^{r} \frac{u_{N+n-1}}{u_{N+n}} .$$

Since we have assumed that

[Continued on page 510.]