N. Gauthier and Paul S. Bruckman
Sums of the even integral powers of the cosecant and secant,
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Abstract

Special finite sums of the even powers of the cosecant and of the secant are studied, $\sum_k \csc^{2m}(k\pi/N)$ and $\sum_k \sec^{2m}(k\pi/N)$, with positive integers $N \geq 3, m$ and $1 \leq k < N/2$. The main result of this article is that these power sums are even polynomials in N, of order $2m$, whose coefficients are rational. The approach is based on new differential identities for the functions $\csc^2 z$ and $\sec^2 z$. The Mittag-Leffler expansions for these functions are invoked and the corresponding infinite series are summed to give closed form expressions for the desired sums. Specific polynomial coefficients are obtained, for $1 \leq m \leq 6$ and for all $N \geq 3$, to illustrate the method. Similar sums involving the cotangent and the tangent are also examined.