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*Pairs of Reciprocal Quadratic Congruences Involving Primes*,
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**Abstract**

Using Pell equations and known solutions that involve Lucas sequences, we find all solutions of the reciprocal pair of quadratic congruences $p^2 \equiv \pm 1 \pmod{q}$, $q^2 \equiv \pm 1 \pmod{p}$ for odd primes $p, q$. In particular, we show that there is exactly one solution $(p, q) = (3, 5)$ when the right-hand sides are $-1$ and $1$. When the right-hand sides are both $-1$, there are four known solutions, all of them pairs of Fibonacci primes, and when the right-hand sides are both $1$, there are no solutions. By partly different methods we completely characterize the solutions of $p^2 \equiv \pm N \pmod{q}$, $q^2 \equiv \pm N \pmod{p}$ for $N = 2$ and $4$, and give partial results for $N = 3$ and $5$. In the process we indicate how the general case can be treated.