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Sommerville’s Symmetrical Cyclic Compositions of a Positive Integer with Parts Avoiding Multiples of an Integer, Fibonacci Quart. 55 (2017), no. 1, 54–73.

Abstract

A linear composition of a positive integer \( N \) is an ordered list of positive integers (called parts) whose sum equals \( N \). A linear composition of \( N \) is called palindromic of type I if it stays the same when it is read in reverse order, while it is called palindromic of type II if it becomes a palindromic composition of type I (of an integer smaller than \( N \)) when we remove the first part. By considering all cyclic shifts of a linear composition of \( N \) as equivalent linear compositions, we may define a cyclic composition of \( N \). Cyclic compositions were originally studied by D. M. Y. Sommerville more than a century ago, who also considered symmetrical cyclic compositions of \( N \). In this paper, we prove that the equivalence class of every symmetrical cyclic composition of \( N \) with length \( K \) (excluding the one with all parts equal when \( K \) divides \( N \)) contains exactly two linear palindromic compositions of type I or II. Using this result, we derive generating functions for the cardinalities of classes of symmetrical cyclic compositions of \( N \) that avoid integers in a set \( A \). We then derive general recurrences for the cardinalities of these classes of symmetrical cyclic compositions. When \( A \) consists of all multiples of a positive integer \( r \), we use these recurrences to derive Fibonacci-type recurrences. We also indicate that the number of dihedral compositions of \( N \) with \( K \) parts in \( A \) is the average of the corresponding numbers of cyclic compositions and Sommerville’s symmetrical cyclic compositions.