Curtis Cooper, Steven Miller, Peter J. C. Moses, Murat Sahin, and Thotsaporn Thanatipanonda
On Identities of Ruggles, Horadam, Howard, and Young,
Fibonacci Quart. **55** (2017), no. 5, 52–65.

Abstract

Ruggles (1963) discovered that for integers $n \geq 0$ and $k \geq 1$

$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

Horadam (1965), Howard (2001), and Young (2003) each expanded this identity to generalized linear recurrence relations of orders 2, 3, and integers $r \geq 2$, respectively. In this paper we let $r \geq 2$ be an integer and w_0, w_1, ..., w_{r-1}, and p_1, p_2, ..., $p_r \neq 0$ be integers. For $n \geq r$ set

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r}.$$

We find identities like those of Ruggles, Horadam, Howard, and Young, of the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(1,r) w_{n+k} + R_k(0,r) w_n,$$

where, by a result of Young, $R_k(i,r)$ is a linear recurrence relation of order $\binom{r}{i}$ for $i = 0, 1, \ldots, r-1$. Our proof uses the Cayley-Hamilton theorem. Next, we find the recurrences $R_k(0,r)$ and $R_k(r-1,r)$ for arbitrary r. Finally, we explicitly find identities for orders $r = 3$, $r = 4$ and $r = 5$.