Steven J. Miller, Dawn Nelson, Zhao Pan, and Huanzhong Xu
On the Asymptotic Behavior of Variance of PLRS Decompositions,
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Abstract

A positive linear recurrence sequence is of the form $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ with each $c_i \geq 0$ and $c_1 c_L > 0$, with appropriately chosen initial conditions. There is a notion of a legal decomposition (roughly, given a sum of terms in the sequence we cannot use the recurrence relation to reduce it) such that every positive integer has a unique legal decomposition using terms in the sequence; this generalizes the Zeckendorf decomposition, which states any positive integer can be written uniquely as a sum of non-adjacent Fibonacci numbers. Previous work proved not only that a decomposition exists, but that the number of summands $K_n(m)$ in legal decompositions of $m \in [H_n, H_{n+1})$ converges to a Gaussian. Using partial fractions and generating functions it is easy to show the mean and variance grow linearly in n: $an + b + o(1)$ and $Cn + d + o(1)$, respectively; the difficulty is proving a and C are positive. Previous approaches relied on delicate analysis of polynomials related to the generating functions and characteristic polynomials, and is algebraically cumbersome. We introduce new, elementary techniques that bypass these issues. The key insight is to use induction and bootstrap bounds through conditional probability expansions to show the variance is unbounded, and hence $C > 0$ (the mean is handled easily through a simple counting argument).