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*Linear Independence of Infinite Products Generated by the Lucas Numbers*, 

**Abstract**  
The purpose of this paper is to give linear independence results for the infinite products  
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{q^n z}{q^{2n} + 1} \right),
\]  
where \(q (|q| > 1)\) and \(z\) are algebraic integers with suitable conditions.  
As an application, we derive that the ten numbers  
\[
1, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}}, \quad \prod_{n=1}^{\infty} \left( 1 \pm \frac{1}{L_{2n}} \right), \quad \prod_{n=1}^{\infty} \left( 1 \pm \frac{2}{L_{2n}} \right),
\]
\[
\prod_{n=1}^{\infty} \left( 1 \pm \frac{\Phi}{L_{2n}} \right), \quad \prod_{n=1}^{\infty} \left( 1 \pm \frac{\Phi^{-1}}{L_{2n}} \right)
\]
are linearly independent over \(\mathbb{Q}(\sqrt{5})\), where \(L_{2n}\) is the 2\(n\)-th Lucas number and \(\Phi\) is the golden ratio, and that  
\[
\sum_{n=1}^{\infty} \frac{1}{L_{2n} + a} \notin \mathbb{Q}(\sqrt{5})
\]
for any \(a = \pm 1, \pm 2, \pm \Phi, \pm \Phi^{-1}\).