6. Shortcuts to Large $F_n$ and $L_n$

Since we shall be dealing extensively with inequalities in this section, we shall recall the following propositions.

For real numbers $a$, $b$, $c$, and $d$, we have:

(a) If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

(b) If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

(c) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

(d) If $0 < \frac{a}{b} < \frac{c}{d}$, then $\frac{b}{a} > \frac{d}{c}$, and conversely.

(e) If $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, then $\frac{a}{b} < \frac{e}{f}$.

(f) If $|b| < 1$, then $|b|^n < 1$ for $n = 1, 2, 3, \ldots$.

(g) If $a = b + c$ and $c > 0$, then $b < a$.

We shall also use an idea suggested by the following problem. Suppose that we have $s$ pounds of sugar (not necessarily an integer), and we ask how many one-pound sacks of sugar can be made from this quantity. Then we are interested in finding the greatest integer not greater than $s$. We denote this integer by $[s]$. Thus,

$$[7.2] = 7 \quad \text{and} \quad [7.9] = 7.$$ 

Similarly, we have

$$[-5.4] = -6 \quad \text{and} \quad [\frac{1}{2}] = 0.$$ 

We are now able to devise methods for finding values of $F_n$ and $L_n$ (or at least estimates of them) without doing all the additions from the beginning. The first theorem that we shall prove is the following.
THEOREM 1

\[ F_n = \left[ \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} \right] \quad \text{for} \quad n = 1, 2, 3, \ldots. \]

**Proof.** We have the Binet form

\[ F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad n = 1, 2, 3, \ldots, \]

where \( \alpha = \frac{1 + \sqrt{5}}{2} \approx 1 + \frac{2.363}{2} \approx 1.618 \) and \( \beta = \frac{1 - \sqrt{5}}{2} \approx -0.618. \)

We can write

\[ F_n = \frac{\alpha^n}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}} = \left( \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} \right) - \left( \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} \right). \]

Since \( 0 < |\beta| < 1, \) we find by (f) on page 30 that

\[ 0 < |\beta|^n < 1. \]

Since \( 1 < \frac{\sqrt{5}}{2}, \) we find by (e) that

\[ 0 < |\beta|^n < \frac{\sqrt{5}}{2}. \]

Since \( \sqrt{5} > 0, \) we find by (a) that

\[ 0 < \frac{|\beta|^n}{\sqrt{5}} < \frac{1}{2}. \]

(A)

Then by (c) we find (by adding \( \frac{1}{2} \)) that

\[ \frac{1}{2} < \frac{1}{2} + \frac{|\beta|^n}{\sqrt{5}} < 1. \]

Although \( \beta < 0, \) we have \( |\beta|^n = \beta^n \) when \( n \) is even, and so

(B) if \( n \) is even, \( \frac{1}{2} < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < 1. \)

On the other hand, we have \( |\beta|^n = -\beta^n \) when \( n \) is odd, and so from (A),

\[ -\frac{1}{2} < \frac{\beta^n}{\sqrt{5}} < 0. \]

Then by (c) we find (by adding \( \frac{1}{2} \)) that

(C) if \( n \) is odd, \( 0 < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < \frac{1}{2}. \)

Thus, from (B) and (C) we have, in general,

\[ 0 < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < 1. \]
But we saw on page 31 that \( F_n \) can be expressed as
\[
F_n = \left( \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} \right) - \left( \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} \right);
\]
and so, since we have shown (page 31) that \( \left( \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} \right) \) is positive and less than 1, we have
\[
F_n < \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} < F_n + 1,
\]
or
\[
F_n = \left[ \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} \right] \quad \text{for} \quad n = 1, 2, 3, \ldots,
\]
and the theorem is proved.

Similarly, the following theorem can be proved, but we shall not give the proof here.

**THEOREM II**

\[
l_n = \left[ \alpha^n + \frac{1}{2} \right] \quad \text{for} \quad n = 2, 3, 4, \ldots.
\]

It is shown in *The Fibonacci Numbers* by N. N. Vorobiov* that \( F_n \) is the integer nearest to \( \frac{\alpha^n}{\sqrt{5}} \), that is,
\[
\left| F_n - \frac{\alpha^n}{\sqrt{5}} \right| < \frac{1}{2}.
\]

Using this result, \( F_n \) can be computed if logarithms to a sufficient number of places are used. A similar development can be given to show that
\[
| L_n - \alpha^n | < \frac{1}{2}.
\]

**PROBLEM 1**

Find \( F_{16} = \frac{\alpha^{16}}{\sqrt{5}} \).

*Solution.* \[
\log \frac{\alpha^{16}}{\sqrt{5}} = 16 \log \alpha - \log 2.236
\]

Since we are going to multiply \( \log \alpha \) by 16, we should find \( \log \alpha \) to more

* For one translation see the reference on page 23.
decimal places than we plan to use for the remainder of the computation.

\[
\alpha = \frac{1 + \sqrt{5}}{2} = \frac{1 + 2.23607}{2} = 1.6180
\]

From a five-place table of common, or base 10, logarithms we find

\[
\log \alpha \approx 0.20898.
\]

Thus, we have:

\[
\log \frac{\alpha^{16}}{\sqrt{5}} \approx 16(0.20898) - 0.3494 \approx 3.3437 - 0.3494 \approx 2.9943
\]

\[
\frac{\alpha^{16}}{\sqrt{5}} \approx 987.0
\]

Therefore, \( F_{16} = 987 \). You can check this answer in the list on page 83.

For larger indices we can find only the first three digits accurately if we are using four-place logarithms.

PROBLEM 2

Estimate \( F_{30} \).

*Solution.* \[
\log \frac{\alpha^{30}}{\sqrt{5}} \approx 30(0.20898) - 0.3494 \approx 5.9200
\]

\[
\frac{\alpha^{30}}{\sqrt{5}} \approx 831800
\]

Therefore, \( F_{30} \approx 832,000 \). You can find the exact value in the list on page 83.

PROBLEM 3

Find \( L_{14} \approx \alpha^{14} \).

*Solution.* \[
\log \alpha^{14} \approx 14(0.20898) \approx 2.9257
\]

\[
\alpha^{14} \approx 842.8
\]

Therefore, \( L_{14} = 843 \). You can check this value in the list on page 83.
We shall now develop methods for finding $F_{n+1}$ from $F_n$ and $L_{n+1}$ from $L_n$ even if we do not know the value of $n$. To achieve this, we shall prove the following new theorem.

**THEOREM III**

\[
F_{n+1} = [\alpha F_n + \frac{1}{2}], \quad n = 2, 3, 4, \ldots.
\]

**Proof.** Since

\[
F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},
\]

we have

\[
\alpha F_n = \frac{\alpha^{n+1} - \alpha \beta^n}{\sqrt{5}} = \frac{\alpha^{n+1} - \alpha \beta^n - \beta^{n+1} + \beta^{n+1}}{\sqrt{5}}
\]

\[
= \frac{\alpha^{n+1} - (\alpha \beta)^{n-1} - \beta^{n+1} + \beta^{n+1}}{\sqrt{5}}.
\]

Since $\alpha \beta = -1$, we have

\[
\alpha F_n = \frac{\alpha^{n+1} - \beta^{n+1} + \beta^{n+1} + \beta^{n-1}}{\sqrt{5}}
\]

\[
= F_{n+1} + \beta^{n-1} \left( \frac{\beta^2 + 1}{\sqrt{5}} \right)
\]

But

\[
\beta^2 + 1 = \beta + 2 = \frac{1 - \sqrt{5} + 4}{2} = \frac{5 - \sqrt{5}}{2} = \sqrt{5} \left( \frac{\sqrt{5}-1}{2} \right) = -\sqrt{5} \beta.
\]

Therefore,

\[
\alpha F_n = F_{n+1} + \beta^{n-1}(-\beta) = F_{n+1} - \beta^n,
\]

and

(A) \[
\alpha F_n + \frac{1}{2} = F_{n+1} + (\frac{1}{2} - \beta^n).
\]

Now, since $|\beta| < .62$, we have $|\beta|^2 < \frac{1}{2}$, and so

\[|\beta|^n < \frac{1}{2} \quad \text{for} \quad n \geq 2.\]

Also, $|\beta^n| = |\beta|^n$, and so

\[|\beta^n| < \frac{1}{2} \quad \text{or} \quad -\frac{1}{2} < \beta^n < \frac{1}{2}.
\]

Therefore, by (b) on page 30, we have

\[\frac{1}{2} > -\beta^n > -\frac{1}{2}, \quad \text{or} \quad -\frac{1}{2} < -\beta^n < \frac{1}{2},\]
and by (c) we have

\[ 0 < \frac{1}{2} - \beta^n < 1. \]

Now since \( \frac{1}{2} - \beta^n > 0 \), we find from equation (A) and (g) on page 30 that

\[ F_{n+1} < \alpha F_n + \frac{1}{2}. \]

Moreover, since \( \frac{1}{2} - \beta^n < 1 \), we have

\[ F_{n+1} + (\frac{1}{2} - \beta^n) < F_{n+1} + 1. \]

Applying these to equation (A), we have

\[ F_{n+1} < \alpha F_n + \frac{1}{2} < F_{n+1} + 1, \]

or

\[ F_{n+1} = \left\lfloor \alpha F_n + \frac{1}{2} \right\rfloor, \quad n = 2, 3, 4, \ldots, \]

and the theorem is proved.

**COROLLARY**

\[ F_{n+1} = \left\lfloor \frac{F_n + 1 + \sqrt{5}F_n^2}{2} \right\rfloor, \quad n \geq 2. \]

**Proof.** From Theorem III, we have

\[
F_{n+1} = \left\lfloor \alpha F_n + \frac{1}{2} \right\rfloor = \left[ F_n \left( \frac{1 + \sqrt{5}}{2} \right) + \frac{1}{2} \right]
= \left[ \frac{F_n + \sqrt{5}F_n + 1}{2} \right] = \left[ \frac{F_n + 1 + \sqrt{5}F_n^2}{2} \right].
\]

This corollary shows that we can compute \( F_{n+1} \) from \( F_n \) without using either \( n \) or \( \alpha \).

We could prove in a similar way the corresponding theorem and corollary for Lucas numbers.

**THEOREM IV**

\[ L_{n+1} = \left\lfloor \alpha L_n + \frac{1}{2} \right\rfloor, \quad n \geq 4. \]

**COROLLARY**

\[ L_{n+1} = \left\lfloor \frac{L_n + 1 + \sqrt{5}L_n^2}{2} \right\rfloor, \quad n \geq 4. \]

PROBLEM 4

Given that 610 is a Fibonacci number, use the Corollary to Theorem III to find the next one.

Solution.

\[
F_{n+1} = \frac{610 + 1 + \sqrt{5(610)^2}}{2} = \frac{611 + \sqrt{1860500}}{2} = \frac{611 + 1364.0}{2} = \frac{1975.0}{2} = 987
\]

You can check this value in the list on page 83. Alternatively, you may compute as follows:

\[
F_{n+1} = \frac{610 + 1 + \sqrt{5(610)}}{2} = \frac{611 + (2.236)(610)}{2} = \frac{611 + 1363.96}{2} = \frac{1974.96}{2} = 987
\]

For larger Fibonacci numbers you may need to find \(\sqrt{5}\) to more decimal places.

EXERCISES

In Exercises 1–4, use four-place tables of logarithms except for \(\log \alpha = 0.20898\).

1. Find \(F_{12}\), using the method of Problem 1.

2. Find \(L_{12}\), using the method of Problem 3.

3. Find the first three digits of \(F_{34}\), using the method of Problem 2.

4. Find the first three digits of \(L_{33}\).

5. Given that 1597 is a Fibonacci number, find the next one.

6. Given that 2207 is a Lucas number, find the next one.