Likewise, denote by B_r the isosceles trapezoid whose height, widths of upper and lower bases, and row numbers of the upper and lower bases are, respectively, $h_r = \frac{1}{2}(3^r - 1)$, $d_U = 2$, $d_L = 3^r - 1$, $n_U = \frac{1}{2}(3^r + 1)$, $n_L = 3^r - 1$, $r \geq 1$. In Figure 28 one can distinguish the triangles $A_0, \ldots, A_3$, and the trapezoids B_1, B_2.

Theorem 3.9. Let the row number of the base of the generalized Pascal triangle of order 3 be $n = (3^r - 1)/2$. Then for any natural number r, the number of trinomial coefficients not divisible by three is given by

$$Q_{1,2}(n) = 2^{r-1}(3^r + 1). \tag{3.12}$$

Proof: Consider the triangle A_r. The row $n = (3^{r-1} - 1)/2$ is inside A_r and is the base of the triangle A_{r-1}. The following row number, being one greater, has a ternary representation $(1\ 1 \ldots 1\ 2)_3$ which contains one $<2>_1$ block. From Theorem 3.3, it follows that in this row only four trinomial coefficients are not divisible by three: $\binom{n}{m}_3$ for $m = 0, 1$ and $m = 3^{r-1}$, $3^{r-1} + 1$; these generate two trapezoids B_{r-1} with lower bases on the row $n = 3^{r-1} - 1$. Now it follows from Theorem 3.3 that in row $n = 3^{r-1}$ there are only three nonzero coefficients (ones): $\binom{n}{m}_3$ for $m = 0$, 3^{r-1}, $2 \cdot 3^{r-1}$; these generate three triangles A_{r-1}, whose bases coincide with the base of A_r. It follows that A_r itself represents a geometric sum of four triangles A_{r-1} and two trapezoids B_{r-1} (Figure 29a). In like fashion, it may be shown that B_r is a geometric sum of four trapezoids B_{r-1} and two triangles A_{r-1} (Figure 29b). Denote by a_r the number of trinomial coefficients in A_r not divisible by three, and by b_r the same for B_r. By the preceding arguments we may form the system

$$\left.\begin{aligned} a_r &= 4a_{r-1} + 2b_{r-1} \\ b_r &= 2a_{r-1} + 4b_{r-1} \end{aligned}\right\}, \tag{3.13}$$

where $r \geq 2$ and the initial data is $a_1=4$, $b_1=2$. The solution of (3.13) is

$$a_r = 2^{r-1}(3^r+1), \quad b_r = 2^{r-1}(3^r-1), \tag{3.14}$$

the first of which gives $Q_{1,2}=a_r$, and the theorem is proved.

The total number of trinomial coefficients in A_r is $(3^r+1)^2/4$, and so the number of coefficients divisible by three is given by

$$Q_3((3^r-1)/2) = \frac{1}{4}(3^r+1)(3^r-2^{r+1}+1). \tag{3.15}$$

It is not difficult to see that from some n onward $Q_3(n) >> Q_{1,2}(n)$.

Theorem 3.10. For $n \to \infty$, $\lim Q_{1,2}(n)/Q_3(n)=0$.

Proof: Since $Q_{1,2}$ and Q_3 are nondecreasing functions of n, for $(3^r-1)/2 \leq n < (3^{r+1}-1)/2$ we have

$$[Q_{1,2}(n) / Q_3(n)] < \left[Q_{1,2}\left(\frac{3^{r+1}-1}{2}\right) \Big/ Q_3\left(\frac{3^r-1}{2}\right)\right].$$

Thus,

$$\lim_{n\to\infty} [Q_{1,2}(n) / Q_3(n)] < \lim_{r\to\infty} \left[Q_{1,2}\left(\frac{3^{r+1}-1}{2}\right) \Big/ Q_3\left(\frac{3^r-1}{2}\right)\right].$$

Using (3.12) and (3.15), we find

$$[Q_{1,2}(n) / Q_3(n)] < 12 \Big/ \left[\left(\frac{3}{2}\right)^r + \left(\frac{1}{2}\right)^r - 2\right].$$

But as $r \to \infty$, $\left(\frac{3}{2}\right)^r \to \infty$, $\left(\frac{1}{2}\right)^r \to 0$, and we have

$$\lim_{n\to\infty} [Q_{1,2}(n) / Q_3(n)] = 0,$$

which proves the theorem. The method of blocks introduced here may also be applied to finding distribution of the coefficients $\binom{n}{m}_s$ in the triangle of order s, for p=2,3,...; the calculations, of course, become increasingly complicated.

In Figures 30a and 30b we show the distributions of the trinomial coefficients with respect to the moduli 2^v and 3^v, respectively, i.e., the highest power which divides the given coefficients.

```
                              0
                            0 0 0
                          0 1 0 1 0
                        0 0 1 0 1 0 0
                      0 2 1 4 0 4 1 2 0
                    0 0 0 1 0 0 0 1 0 0 0
                  0 1 0 1 1 1 0 1 1 1 0 1 0
                0 0 2 0 0 1 0 0 0 1 0 0 2 0 0
              0 3 2 4 1 3 4 3 0 3 4 3 1 4 2 3 0
            0 0 0 2 1 1 1 8 0 0 0 8 1 1 1 2 0 0 0
          0 1 0 1 0 2 1 2 0 1 0 1 0 2 1 2 0 1 0 1 0
        0 0 1 0 3 0 0 1 0 0 1 0 2 0 0 1 0 0 3 0 1 0 0
      0 2 1 5 0 3 1 2 1 2 1 3 0 3 1 2 1 2 1 3 0 5 1 2 0
    0 0 0 1 0 0 0 1 5 1 2 1 0 0 0 1 2 1 5 1 0 0 0 1 0 0 0
  0 1 0 1 3 1 0 1 0 3 1 4 0 1 0 1 0 4 1 3 0 1 0 1 3 1 0 1 0
0 0 3 0 0 2 0 0 1 0 0 1 0 0 1 0 1 0 0 1 0 0 1 0 0 2 0 0 3 0 0
```

Figure 30a

```
0
0 0 0
0 0 1 0 0
0 1 1 0 1 1 0
0 0 0 0 0 0 0 0 0
0 0 1 1 2 1 2 1 1 0 0
0 1 1 0 2 2 1 2 2 0 1 1 0
0 0 0 0 0 0 1 1 1 0 0 0 0 0 0
0 0 2 0 0 2 0 0 2 0 0 2 0 0 2 0 0
0 2 2 1 2 2 1 2 2 0 2 2 1 2 2 1 2 2 0
0 0 0 1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 0 0 0
0 0 1 0 0 1 2 1 2 0 0 1 0 0 2 1 2 1 0 0 1 0 0
0 1 1 0 1 1 0 2 1 0 1 1 0 1 1 0 1 2 0 1 1 0 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 1 1 1 1 1 1 1 3 1 1 1 1 1 1 1 3 1 1 1 1 1 1 1 1 0 0
0 1 1 0 4 3 2 1 5 1 2 2 1 4 3 2 3 4 1 2 2 1 5 1 2 3 4 0 1 1 0
```

Figure 30b

3.2 DIVISIBILITY AND DISTRIBUTION MODULO p OF FIBONACCI, LUCAS, AND OTHER NUMBERS IN ARITHMETIC TRIANGLES

As we know, the Fibonacci numbers are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad F_1 = 1, \quad F_2 = 1. \tag{3.16}$$

Interpretations of these numbers may be found, for example, in [20], and they play an early part in number theory and combinatorial analysis. They form the sequence 1,1,2,3,5,8,13,..., and may be studied from the point of view of sequences, or may be constructed from their arithmetic triangle as in section 1.4.

Studies on questions of divisibility and the distribution modulo the prime p of the Fibonacci numbers may be found in [11, 62, 171, 199, 245, 286, 339, 387-389, 390]. We note below some of these results concerning the above topics, as well as periodicity modulo p.

The paper of V.E. Hoggatt and G.E. Bergum [199] gives several results:

(1) if $p \geq 2$ is a prime and F_n is divisible by p, then for $s \geq 1$, $F_{np^{s-1}}$ is divisible by p^s;

(2) if $n=3^m \cdot 2^{s+1}$, where $m,s \geq 1$, then F_n is divisible by n;

(3) if $n=3^m \cdot 2^{s+1} \cdot 5^r$, where $m,s,r \geq 1$, then F_n is divisible by n.

The sequence of numbers $\{x_n\}$, $n \geq 1$, is said to be uniformly distributed mod m, where $m \geq 2$ is a whole number, if

$$\lim_{n \to \infty} \frac{1}{n} A(n,m,r) = \frac{1}{m}$$

for each $r=0,1,\ldots,m-1$, where $A(n,m,r)$ is the number of terms of the sequence congruent to r (mod m). L. Kuipers and J.S. Shine [245] showed that:

(1) the Fibonacci sequence $\{F_n\}$ is uniformly distributed mod 5;

(2) the sequence $\{F_n\}$ is non-uniformly distributed mod p, where $p \geq 2$ is any prime except $p=5$;

(3) the sequence $\{F_n\}$ is non-uniformly distributed with respect to any composite modulus m, if $m \neq 5^k$, $k=3,4,\ldots$.

The sequence $\{L_n\}$, defined by the recurrence relation (3.16) but with initial conditions $L_1=1$, $L_2=3$, is discussed in [199], where it is shown that for $n=2 \cdot 3^k$, $k \geq 1$, L_n is divisible by n.

In [129], it is proved that the Catalan number $\binom{2n}{n}/(n+1)$ is odd if $n=2^r-1$, where r is a nonnegative whole number. And in [60], it is shown that for any prime $p>2$, the Catalan sequence $\{C_n\}$ may be decomposed into blocks of successive values C_n divisible by p (block B_k), and not divisible by p (block $\bar{B}_k$), with respective lengths l_k and $\bar{l}_k$; further

$$l_k = \frac{1}{2}(p^{m+1-\delta}-3), \quad \bar{l}_k = \frac{1}{2}(p+3+6\delta),$$

where m is the largest natural number for which k is divisible by $\left(\frac{p+1}{2}\right)^m$, $\delta=\delta_{3p}$, and

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}.$$

The present author in [11] studied the distribution modulo p of the elements in the Fibonacci and Lucas triangles. In Figure 31 are shown the distributions of the elements in the Fibonacci triangle for $p=2$ (31a) and $p=3$ (31b), and in Figure 32 these same distributions for the Lucas triangle.

Divisibility questions for the sequence of generalized Fibonacci numbers are discussed in [182].

We show also the distributions modulo $p=2$ or 3, as indicated in the figures, of the elements in the arithmetic triangles [366-369] composed of Gaussian binomial coefficients (Figure 33), Euler numbers (Figure 34), Stirling numbers of the first (Figure 35) and second (Figure 36) kinds, and (Figure 37) the distribution in the Pascal triangle of elements divisible by their row numbers, indicated by O's (divisible) and X's (not divisible) [177].

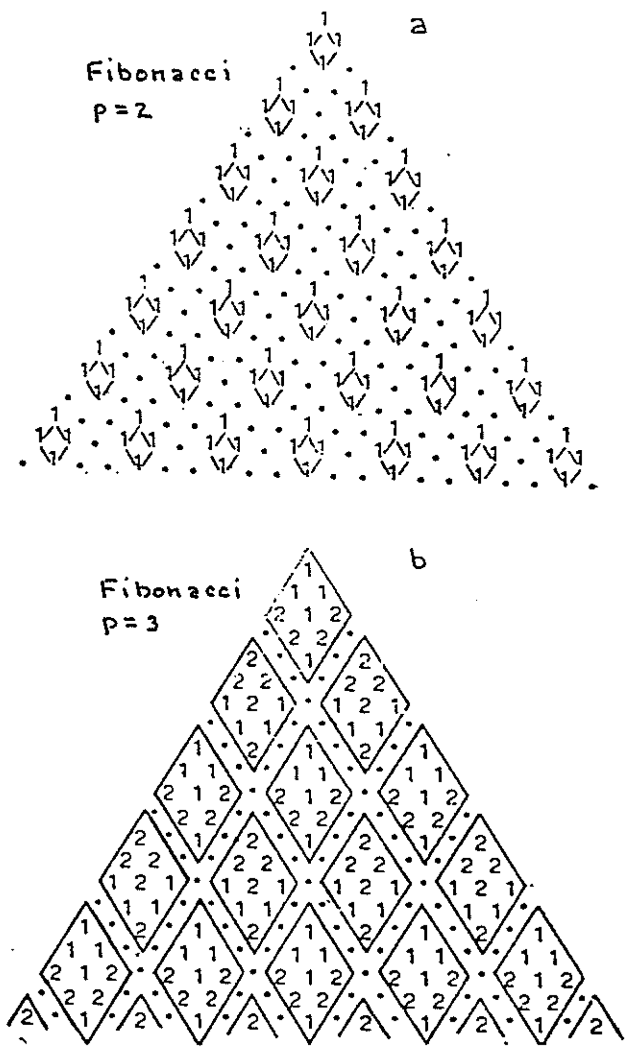

Figure 31

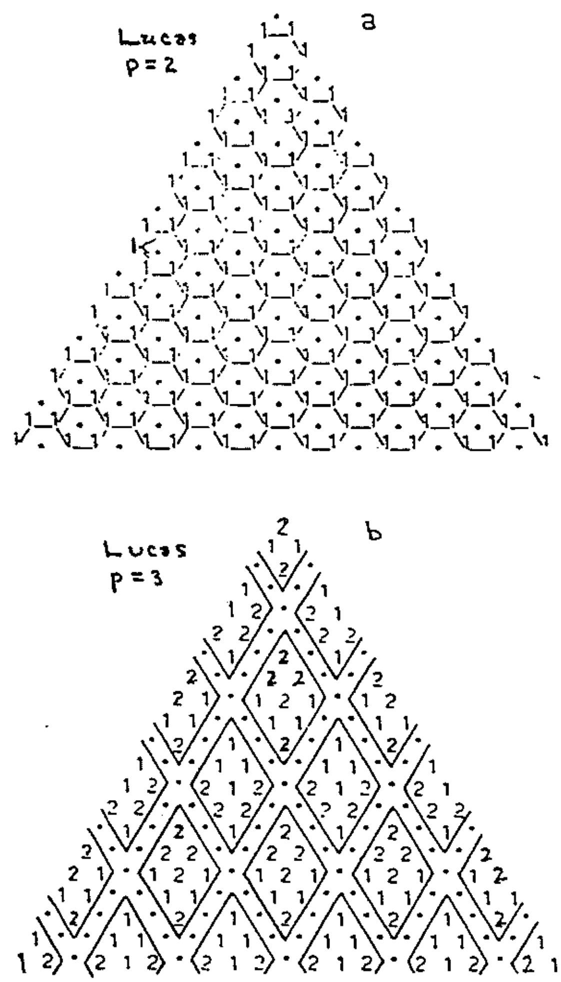

Figure 32

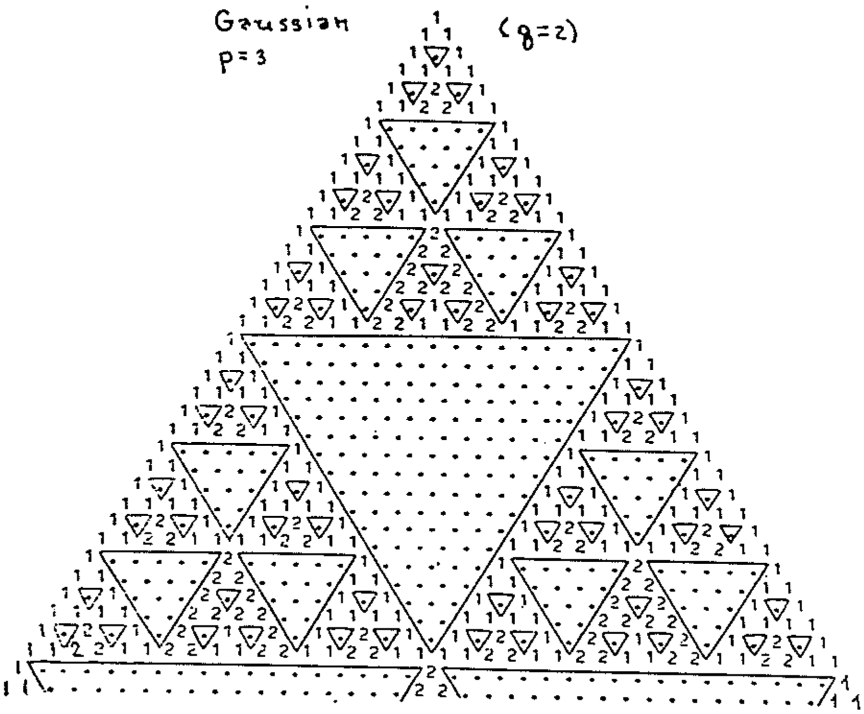

Figure 33

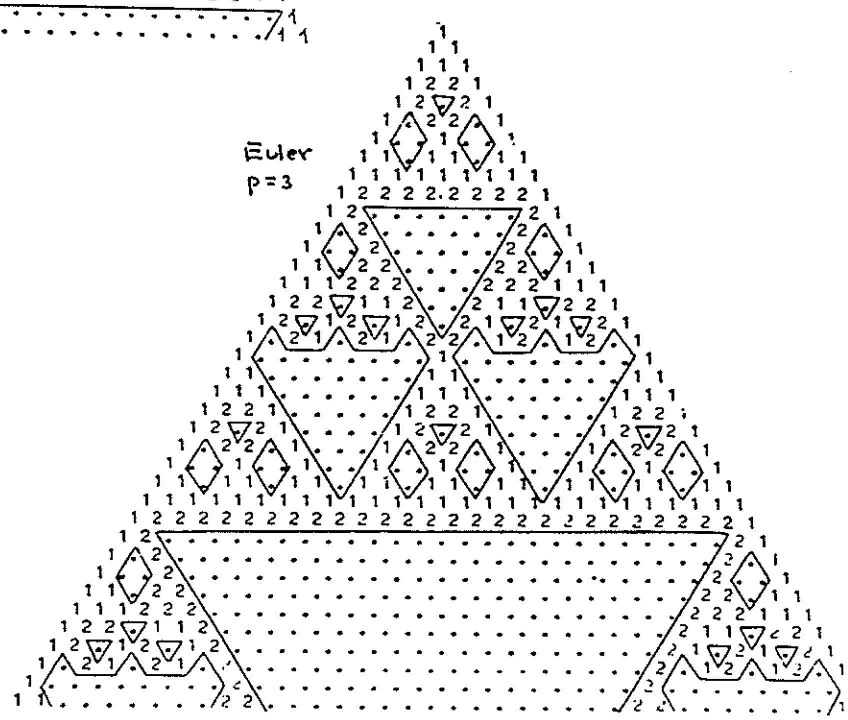

Figure 34

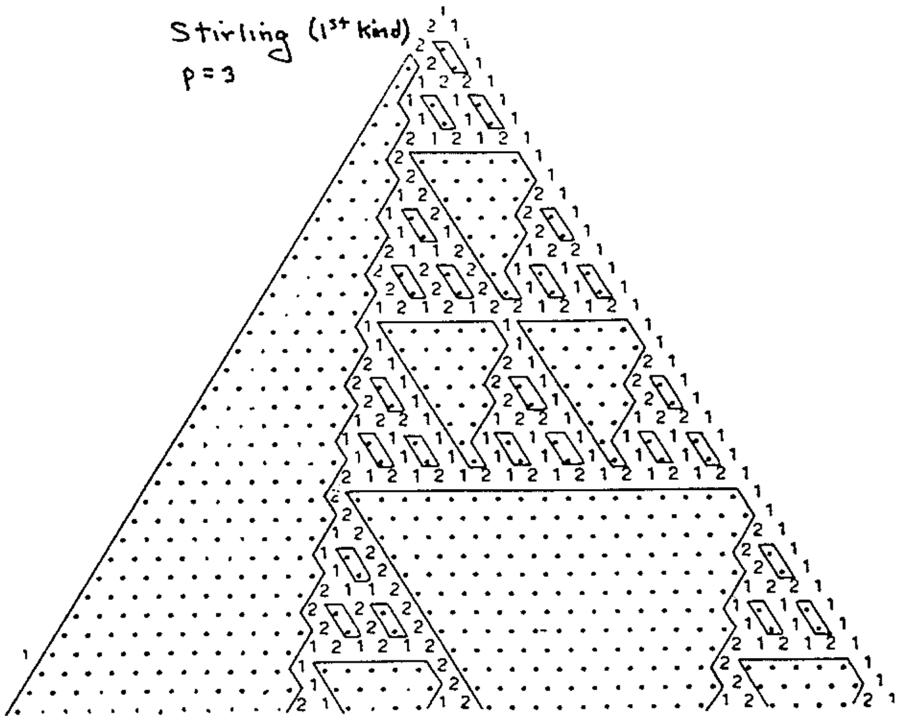

Figure 35

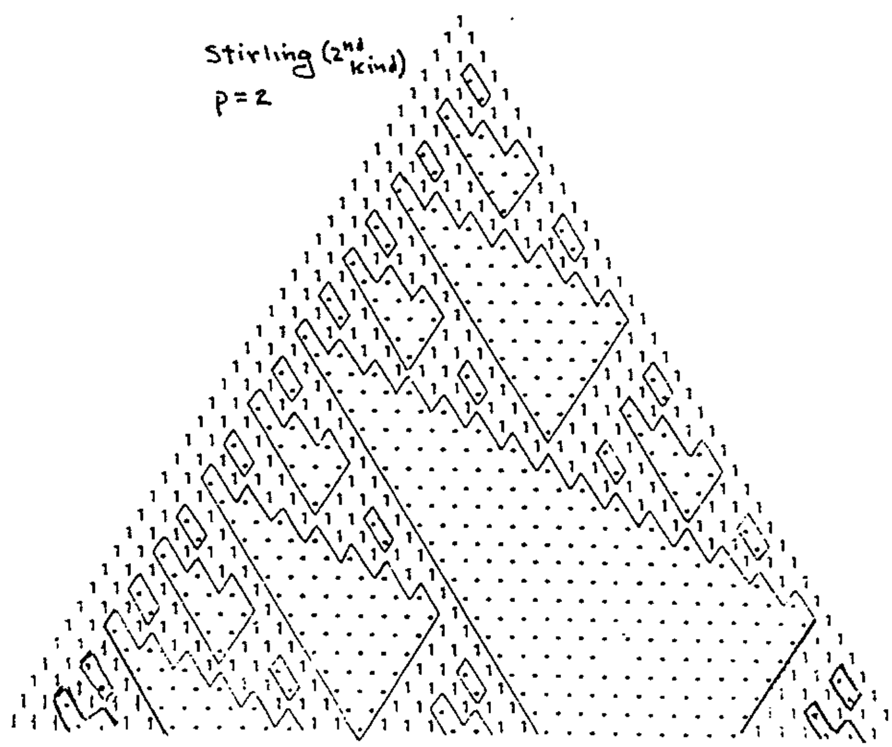

Figure 36

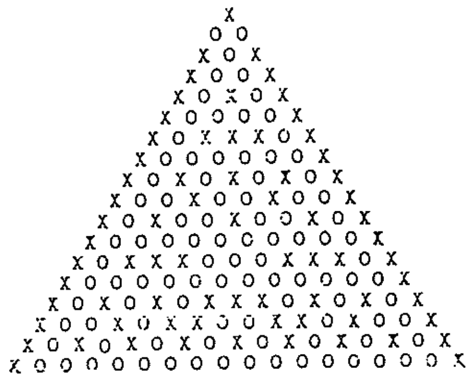

Figure 37

3.3 DISTRIBUTION OF SOME NUMERICAL SEQUENCES MODULO p

Material on questions of divisibility and distribution mod p of various numerical sequences appears in [68, 106, 107, 266, 299, 306-307, 332, 340, 377, 378, 386, 402, 404]. Some of these discuss sequences which are related one way or another to binomial coefficients, Fibonacci numbers, Lucas numbers, and other special numbers.

As is known, if p is prime, the natural number n is divisible by $(p-1)$ if

$$r^n \equiv 1 \pmod{p}, \quad r = 1, 2, \ldots, p-1.$$

M. Bhaskaran [68] showed, for $p > 2$, that the positive whole number n is divisible by $(p+1)$ if

$$\sum_{k=0}^{\left[\frac{n-r}{p-1}\right]} (-1)^k \binom{n}{r+kp-k} \equiv 0 \pmod{p}, \quad r = 1, 3, \ldots, p-2.$$

A. Rotkiewicz proved in [332] that for any natural number $a>1$ there exists an n such that a^n+1 is divisible by n.

[106, 107, 404] are devoted to questions of divisibility of the sequence given by the recurrence

$$A_{n+2} = aA_{n+1} + bA_n,$$

for various choices of a,b and initial conditions A_0, A_1. Thus, in [106], $a=P$, $b=-Q$, P and Q whole numbers, and $A_0=0$, $A_1=1$; in [107], $a=p+2$, $b=-(p+1)$, p a prime, $A_0=0$, $A_1=1$, and it is shown that the sequence is uniformly distributed mod p^s.

In [378] there is discussed a sequence, and its properties, formed from the fifth column of the Pascal triangle.

In [386] the sequence of triangular numbers is discussed with respect to the modulus n and it is shown that the sequence is periodic mod n (n odd) and mod 2n (n even). The periodicity of m-gonal numbers mod n is also treated in [340].

A. Perelli and U. Zannier [306-307] studied arithmetic properties and periodicity mod p of some sequences; they proved, for example, that if

$$f(n+p) \equiv f(n) \pmod{p}, \quad p>p_0,\ n\in N,$$

then for certain specified conditions f must be a polynomial.

CHAPTER 4

FRACTAL PASCAL TRIANGLES AND OTHER ARITHMETIC TRIANGLES

In this chapter we use some results from Chapter 2 to form fractal Pascal, and generalized Pascal, triangles, as well as fractal arithmetic triangles whose elements are, e.g., Gaussian binomial coefficients, and Stirling and Euler numbers. We also give some interesting geometric figures containing elements of the Pascal triangle, these elements being related to one another by various arithmetic properties.

4.1 FRACTALS AND THEIR DIMENSIONS

The objects which today we call fractals, or describe as being fractal, were first studied in the early part of the present century, although the term "fractal" has only become established in the last decade. The term was introduced by the French mathematician B.B. Mandelbrot, and comes from the Latin adjective "fractus", connoting something fractional or cut up. The most complete descriptions of various classes of fractals in nature are in the books of B.B. Mandelbrot [270-272], H.-O. Peitgen and co-authors [304, 305], and in the collection of articles [51], and also in [27, 30]. The fractal property is possessed by many geographic features, among them coastlines, mountains, and valleys. There also exist many physical and chemical processes out of which arise complicated fractal structures.

Of considerable interest is the representation of fractal constructions arising from the Pascal triangle in the works of S. Wolfram [397,398], and O. Martin, A.M. Odlyzko, S. Wolfram [276], and their application to the study of cellular automata. A. Lakhtakia, et al., [213, 249-254], constructed a new class of fractals, the Pascal-Sierpinski gaskets, investigated their properties, and gave applications to various physical problems. An interesting class of fractals, connected with the Gaussian binomial coefficients, and the Stirling and Euler numbers, was constructed by M. Sved [368]. In [12, 16] are constructed fractal Pascal triangles, pyramids, generalized triangles, and Fibonacci and Lucas triangles. Fractal Pascal triangles also appear in the book of T.M. Green and C.L. Hamberg [162], as well as in [1, 2, 94, 147, 247, 344, 403].

Fractals as geometric objects possess a variety of properties, but fundamental among these are their fractional (non-integral) dimension and their self-similarity. Roughly speaking, a self-similar geometric figure is one which may be represented in the form of a finite number of figures similar among themselves. With such figures we may associate equilateral triangles and squares, the self-similarity of which is defined in a more complicated way. As examples, we mention the well-known self-similar geometric constructions in Figure 38, known as (a) the Sierpinski triangle curve, and (b) the Sierpinski carpet, after the Polish mathematician W. Sierpinski. The methods of construction are easily explained in terms of the figures: the triangle curve is obtained by repeatedly connecting the midpoints of the sides of the successively smaller equilateral triangles; the carpet is constructed by iterating the process of discarding the middle square from among the nine squares of the preceding stage. Figure 39 shows the so-called Koch triadic (snowflake) curve, whose construction begins with an equilateral triangle, each side of which is divided into three parts, with the middle part then replaced by two line segments of length equal to a third of the original side.

As we know, there exist various definitions of dimension, corresponding to quite different points of view. One of these ideas of dimension is related to the minimal number of coordinates necessary to unambiguously define the location of a point on a line, in a plane, and in space. Another, the concept of the topological dimension, is that the dimension of any set should be one greater than the dimension of the cut which separates it into two disconnected parts. We note here that these dimensions may only be integers; both definitions imply that a line is one-dimensional, a plane is two-dimensional, the usual geometric space is three-dimensional, and so on.

But there exist other concepts, as well, and one of these is that of the dimension of self-similarity. Let n be the number of identical parts into which a given self-similar object is decomposed when the size of the original parts has been reduced by a factor of m. Then the self-similar dimension is defined by the formula

$$D = \ln n / \ln m. \tag{4.1}$$

Using this introduced concept, we find that the dimension of the self-similar square obtained by successive division into four equal squares is ln 4/ln 2=2, that of the self-similar cube is ln 8/ln 2=3, and so on. But when we use (4.1) to calculate the dimensions of the objects in Figures 38 and 39, we find for the Sierpinski triangle curve that $D_1=\ln 3/\ln 2=1.5849$, for the Sierpinski carpet that $D_2=\ln 8/\ln 3=1.8727$, and for the Koch curve that $D_3=\ln 4/\ln 3=1.2618$. That is, the self-similar dimensions of these objects are non-integral. Non-integral dimensions are usually referred to as fractional dimensions, and in Figure 40 are some additional self-similar figures: fractals, having fractional dimensions [270].

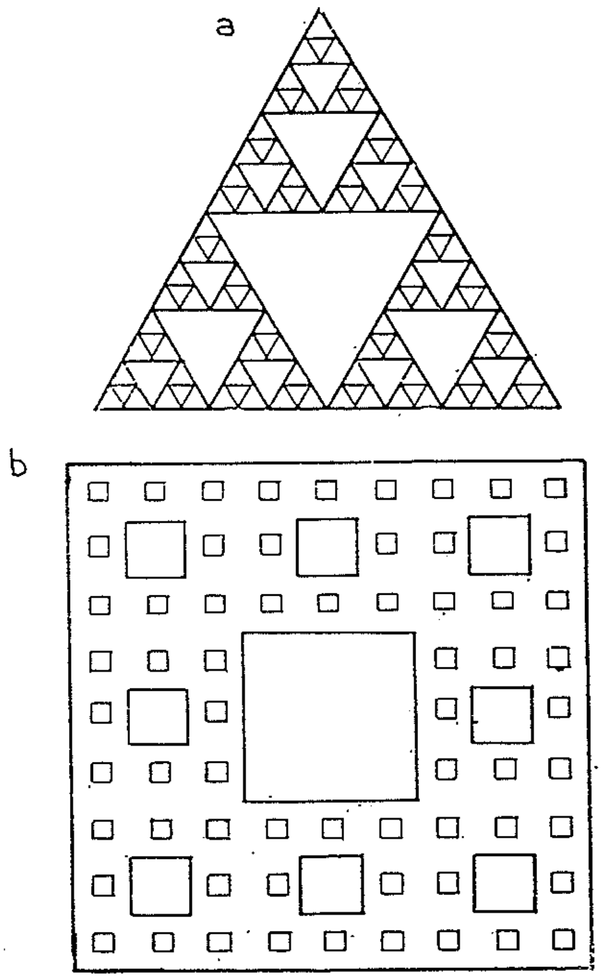

Figure 38

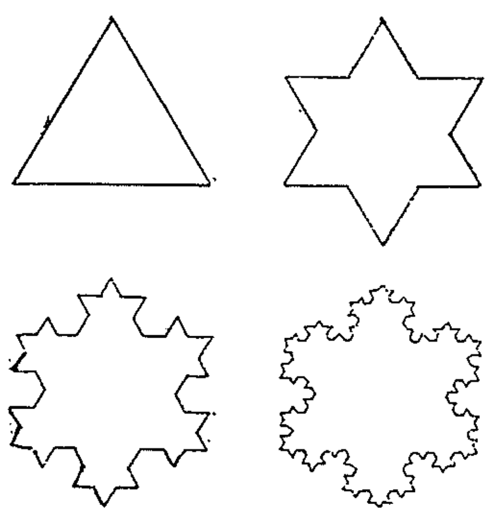

Figure 39

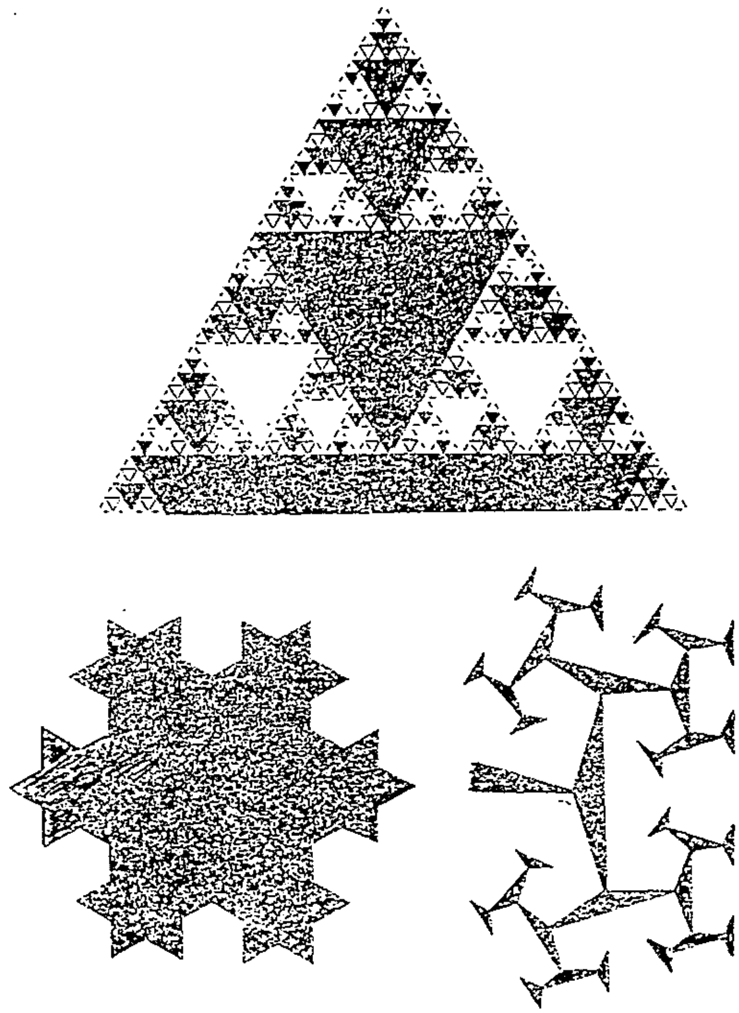

Figure 40

4.2 FRACTAL PASCAL TRIANGLES MODULO p

The elements of the Pascal triangle, reduced with respect to some modulus, in particular a prime modulus p, form various triangular geometric lattices - in fact, fractals, which played an early part in the analysis of these structures and processes. (In order to discover the properties of self-similar Pascal triangles mod p, we need to use a sufficiently large number of rows of the triangle.) A fundamental characteristic of these fractals formed from the Pascal triangle mod p is their fractional dimension D_p. Taking into account that for a prime p, the corresponding m and n (of 4.1) are $n=p(p+1)/2$ and $m=p$, it follows from (4.1) that

$$D_p = \ln \frac{p(p+1)}{2} \,/\, \ln p = 1 + \ln \frac{(p+1)}{2} \,/\, \ln p. \tag{4.2}$$

Then from (4.2) we find, for example,

$$D_2 \cong 1.585,\ D_3 \cong 1.631,\ D_5 \cong 1.683,\ \ldots.$$

And for $p \to \infty$,

$$\lim_{p\to\infty} D_p = 1 + \lim_{p\to\infty} \ln(p+1)/\ln p - \lim_{p\to\infty} \ln 2/\ln p = 2. \tag{4.3}$$

In Figures 41-43 we show the fractal Pascal triangles formed using the respective prime moduli p=2,3,5, and in Figures 44,45 using the composite moduli d=4,6 (in these latter two cases the self-similarity is of a more complicated kind, and (4.3) cannot be used to calculate the dimension). The dark ovals indicate points of the geometric lattice corresponding to the elements of the Pascal triangle not divisible by p=2,3,5 and d=4,6; the blanks indicate coefficients divisible by these moduli.

Applications of fractal Pascal triangles, mentioned earlier, are discussed in [12, 213, 249-254, 276, 395, 396].

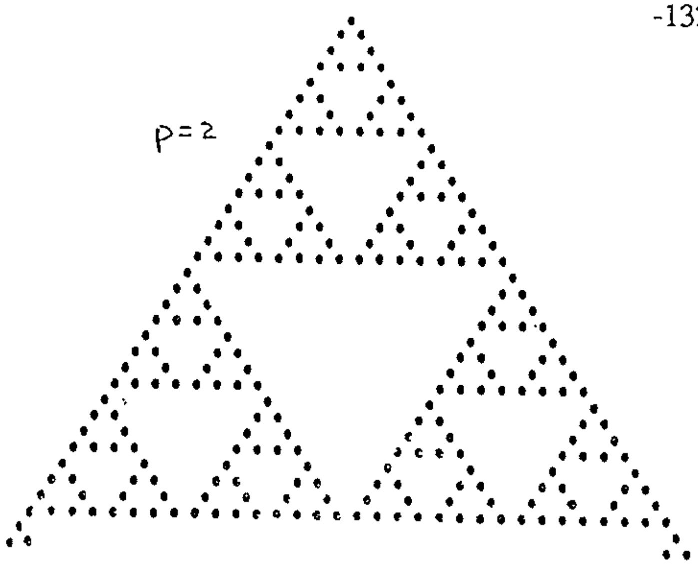

Figure 41

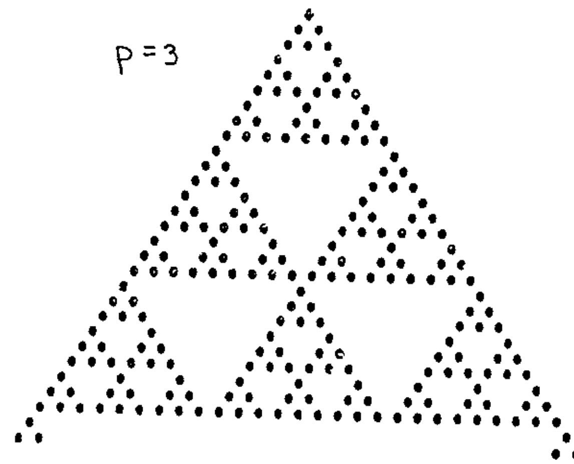

Figure 42

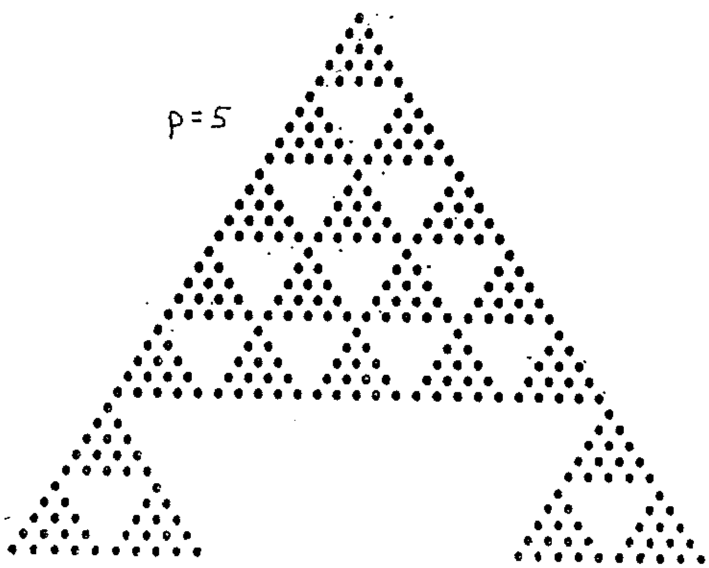

Figure 43

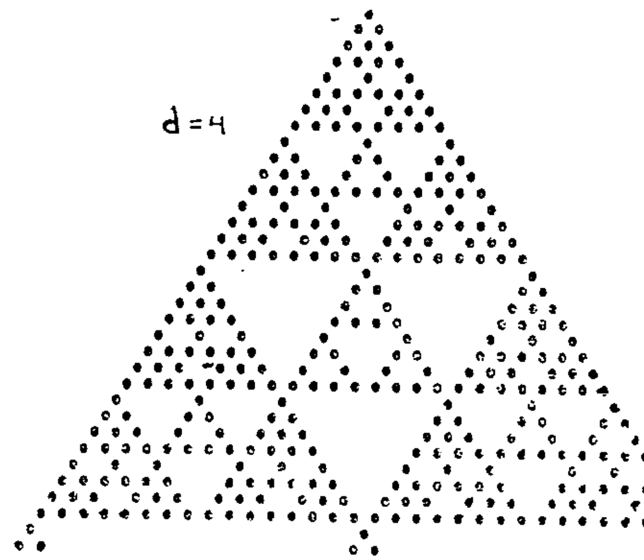

Figure 44

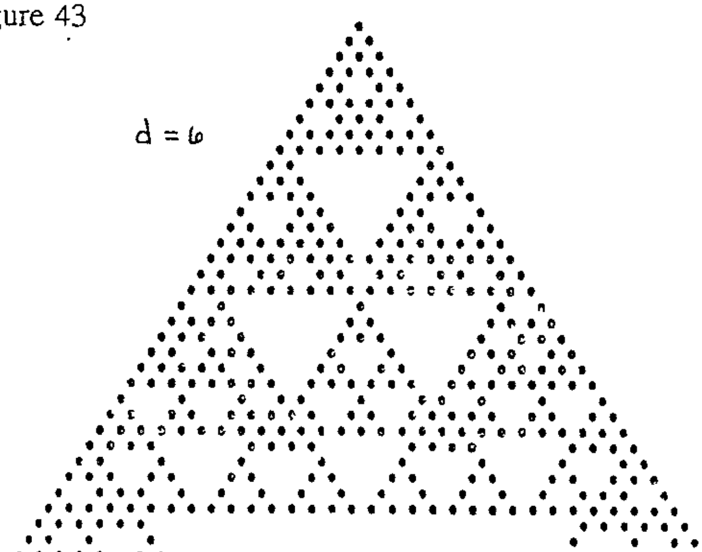

Figure 45

4.3 FRACTAL GENERALIZED PASCAL TRIANGLES AND OTHER ARITHMETIC TRIANGLES MODULO p

Fractal generalized Pascal triangles, with the coefficients $\binom{n}{m}_s$ reduced with respect to the prime moduli p=2,3 are given in [16]. M. Sved in [368] gives the constructions for fractal arithmetic triangles composed of Gaussian binomial coefficients, Stirling numbers of the first and second kind, and Euler numbers.

The fractal triangles for $\binom{n}{m}_3$ mod 2 and mod 3 are shown in Figures 46,47; for the Gaussian binomial coefficients,q=2,mod 3 in Figure 48; for the Stirling numbers of the second kind mod 2 in Figure 49; and for the Euler numbers mod 3 in Figure 50.

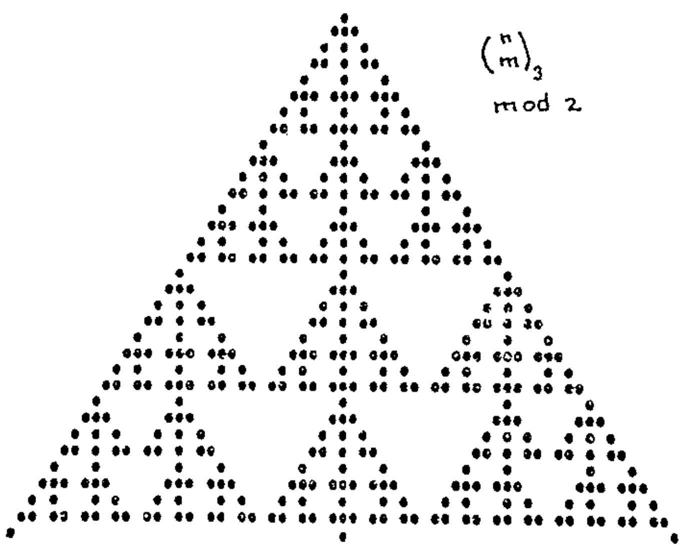

Figure 46

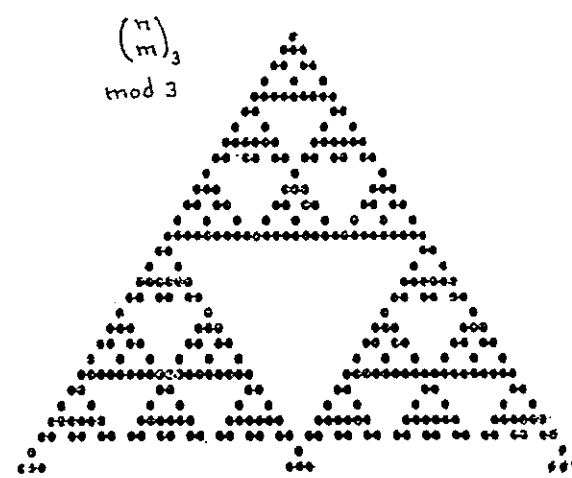

Figure 47

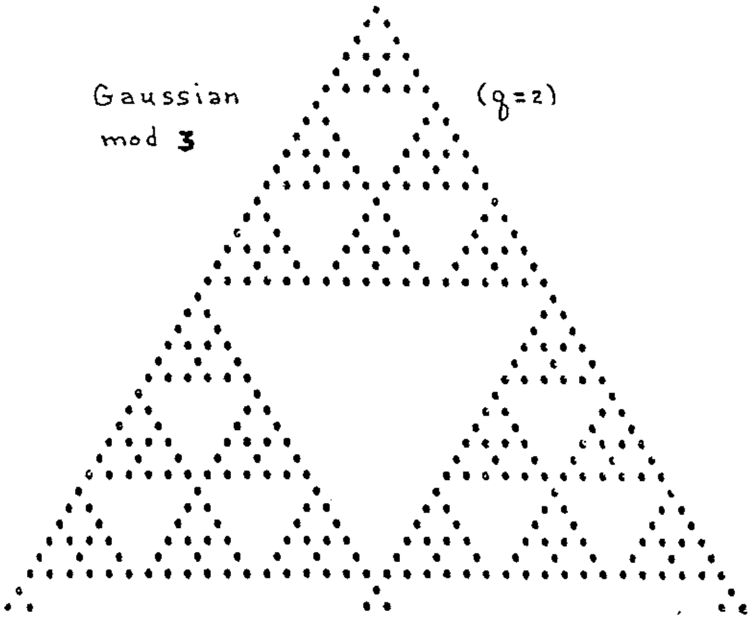

Figure 48

Figure 49

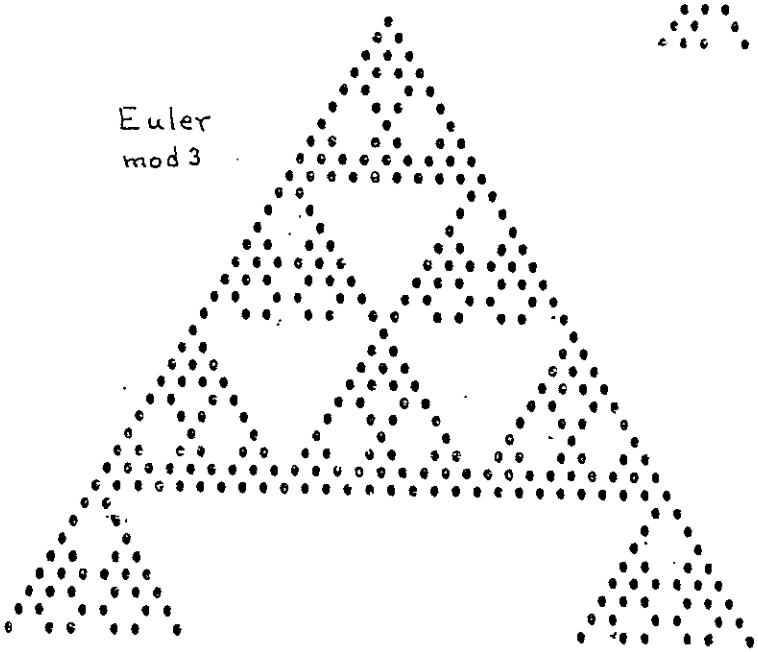

Figure 50

4.4 GEOMETRIC ARRANGEMENTS OF BINOMIAL COEFFICIENTS WHOSE PRODUCTS YIELD PERFECT POWERS

V.E. Hoggatt and W. Hansell [208] discovered an interesting property which can be described as follows. Let $\binom{n}{r}$ be an interior element of the Pascal triangle, and let this element be located at the center of a regular hexagon whose vertices consists of the neighboring binomial coefficients

$$\binom{n-1}{m-1}, \binom{n}{m+1}, \binom{n+1}{m}, \binom{n-1}{m}, \binom{n}{m-1}, \binom{n+1}{m+1}.$$

They showed that the product of these six binomial coefficients is a perfect square (of some number), and further that the product of the first three equals the product of the last three. Using a notation introduced in [384], denote the first three elements by O's and the last three by X's. Then the coefficients listed above form the figure shown in Figure 51a, and a specific example is shown in Figure 51b for the case n=8, m=2, where $7\cdot56\cdot36=8\cdot21\cdot84$.

V.E. Hoggatt and G.L. Alexanderson [196] extended this property to the case of multinomial coefficients. And H.W. Gould [157] showed there are arrangements of eight, and of ten, binomial coefficients in the Pascal triangle which also have this property; two of these are shown in Figure 51c, d.

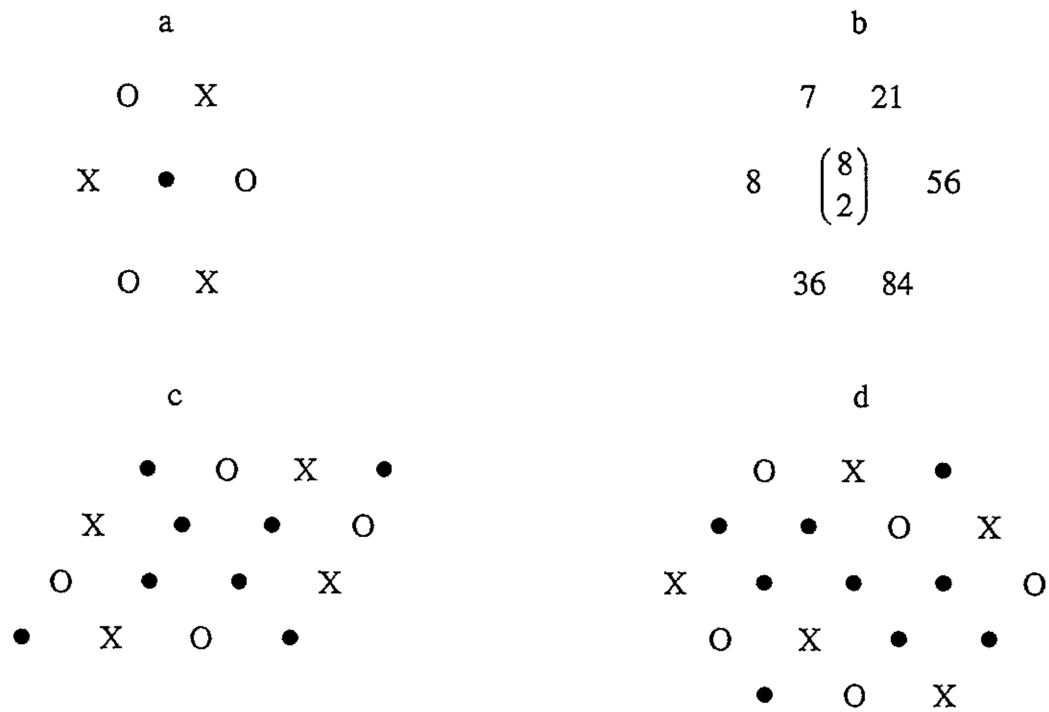

Figure 51

Further, if this property, for a given arrangement of binomial coefficients, does not depend on the choice of the values of m and n, then such a figure is said to be a perfect square pattern (PSP). Z. Usiskin [384] discussed the problem of conditions for the existence of PSP's, and showed that if the figure contains an even number of elements in each row and in its main diagonals then it belongs to the class of PSP's. As a result of this theorem, he constructed new PSP figures, shown in Figure 52a-d. He also discussed arrangements of binomial coefficients, the product of which is a third power. Thus, in Figure 52e, composed of three rhomboids, the products of the elements at the nodes denoted by O, X, Y are equal among themselves, and the product of all elements is a cube. This property extends also to the case of n^{th} powers, if the figure is composed of rhomboids whose sides consist of n elements.

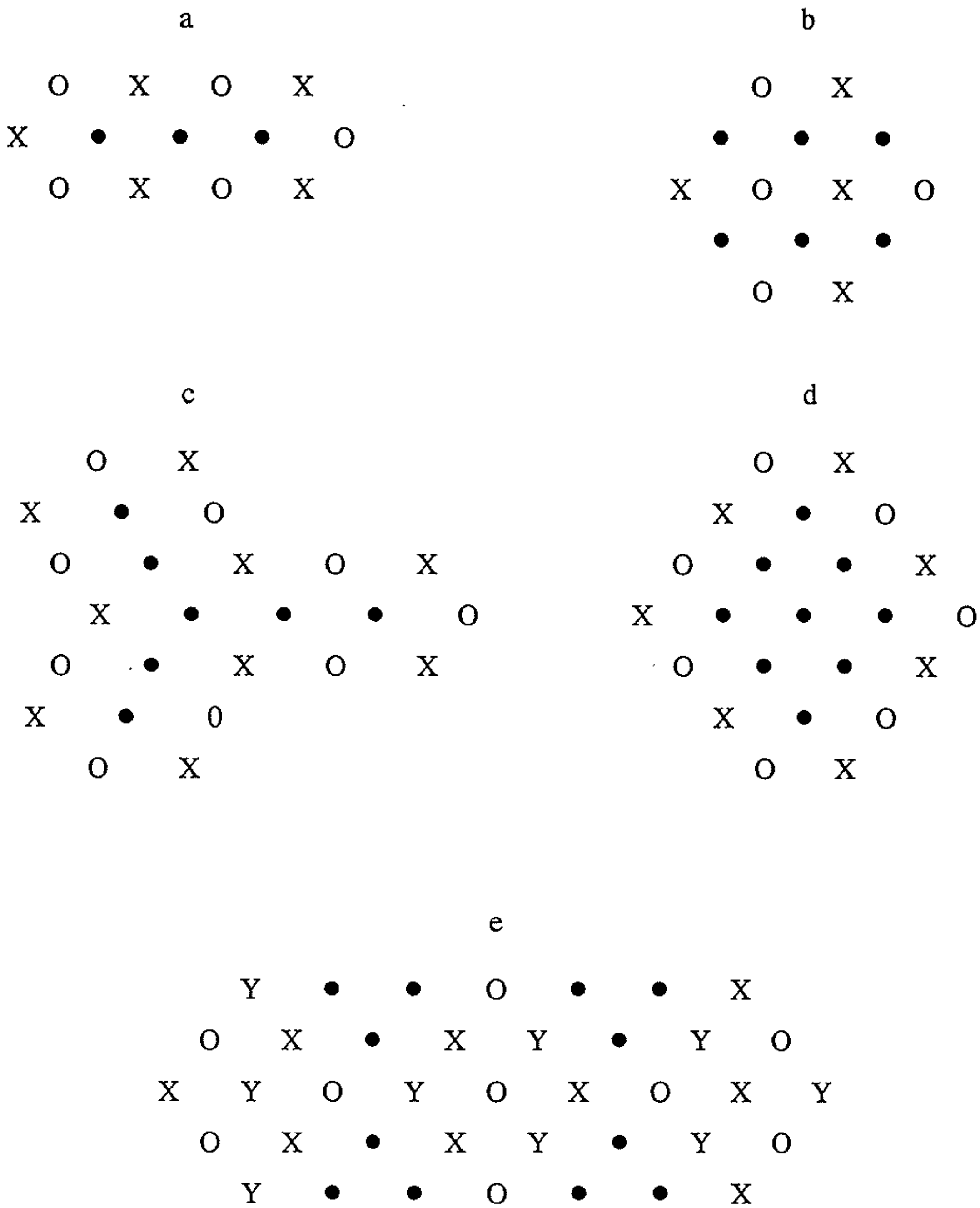

Figure 52

C.L. Moore [284] extended the results of [208] and established that for the binomial coefficients forming a regular hexagon whose sides lie along the horizontal rows and main diagonals of the Pascal triangle, and which contain $j+1$ elements, the product yields a perfect square if j is odd. Similar results are discussed by A.K. Gupta in [165].

C.T. Long [260], and C.T. Long and V.E. Hoggatt [264], using a lemma they proved, generalized known results and presented new geometric figures in the lattice of points forming the Pascal triangle (denoted by * in Figures 53, 54). They proved that the products of the binomial coefficients at the points of these figures form perfect squares. The assertion of their lemma is as follows: the product of the binomial coefficients at the vertices of the pair of parallelograms oriented as shown in Figure 53, is a perfect square. (If the parallelograms partially overlap then, in the total, vertices must be taken into account twice, or excluded, in the corresponding product.) Using the lemma, they proved a theorem which applies to many interesting geometric figures, some of which are shown in Figure 54. In effect, the theorem says that the products of the binomial coefficients located at the points on the contours of these figures, are perfect squares.

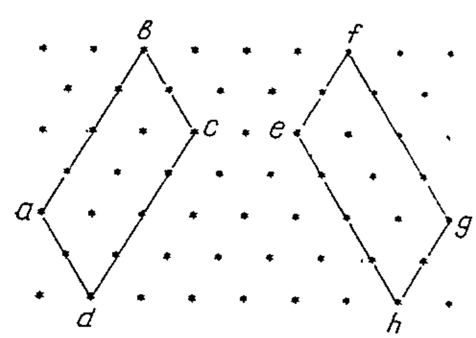

Figure 53

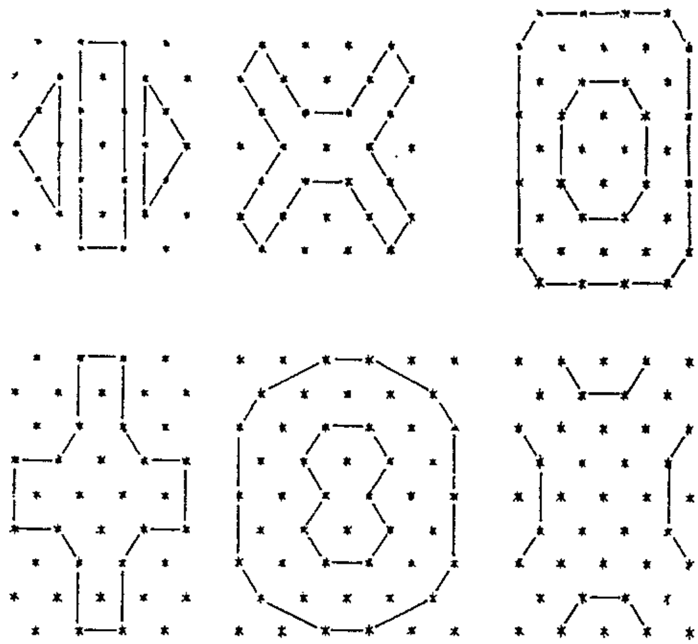

Figure 54

B. Gordon, D. Sato, and E. Straus [152] discussed P_k-sets of vertex points of the Pascal triangle lattice. They proved that the products of the binomial coefficients located at these points were k^{th} powers, and gave a description of these P_k-sets and a method for determining the minimum of f(k), where f(k) is the cardinal number of all P_k-sets. They also consider the problem of extending the results obtained to the case of the multinomial coefficients.

CHAPTER 5

GENERALIZED ARITHMETIC GRAPHS AND THEIR PROPERTIES

In this chapter we consider the properties of generalized arithmetic graphs, special cases of which are graphs modeled on generalized Pascal triangles. We will look at planar and spatial graphs of generalized Pascal triangles of order m, prove a theorem on their isomorphisms, and give an asymptotic formula for the number of paths in these graphs as $k \to \infty$. We also prove a theorem on the cross sections of spatial arithmetic graphs.

5.1 GENERALIZED ARITHMETIC GRAPHS

A technique often used in the solution of combinatorial problems is to interpret or reformulate the problem in the form of a graph, which provides a visual interpretation of the combinatorial object and may make possible the discovery of new properties. We might cite, for example, the use of the Ferrers graph for the representation of partitions [41], and the use of graphs in the study of partially ordered sets [39].

The interpretation in the form of a graph, of objects essentially combinatorial, allowed H. Hosoya [219, 220] to discover connections between the elements of the Pascal triangle and the Fibonacci sequence, and a number of forms of chemical structures. In [19] the possibility of using the Pascal triangle in building models of genetic codes is considered. The graph interpretation is also used in [29], in which a study of the properties of the graph suggests new ways and algorithms for the solution of combinatorial problems.

S.K. Das, N. Deo, and M.J. Quinn [113-116], and B.P. Sinha et al. [353], introduced and studied the properties of the Pascal graph; the adjacency matrix of the vertices of the graph coincides with the Pascal triangle mod 2. In particular, they introduced the Pascal matrix, which is a symmetric matrix with zeros on the main diagonal, and below (and above) it the Pascal triangle mod 2. They also studied and determined the properties of the Pascal planar graph.

We will introduce the idea of the generalized arithmetic graph and prove a theorem on the number of its paths; special cases of these graphs are the graphs modeled on various modifications of the Pascal triangle.

A generalized arithmetic graph G(F,X) (Figure 55) is a regular Berge graph [26] with a set X of vertices and mapping F which associates with each vertex $x \in X$ a subset (possibly empty) of X, i.e.,

$$F_x = \left\{ y \mid y \in X \wedge \vec{J}(x,y) \right\},$$

where $\vec{J}(x,y)$ is the edge running from the vertex $x \in X$ to the vertex $y \in X$.

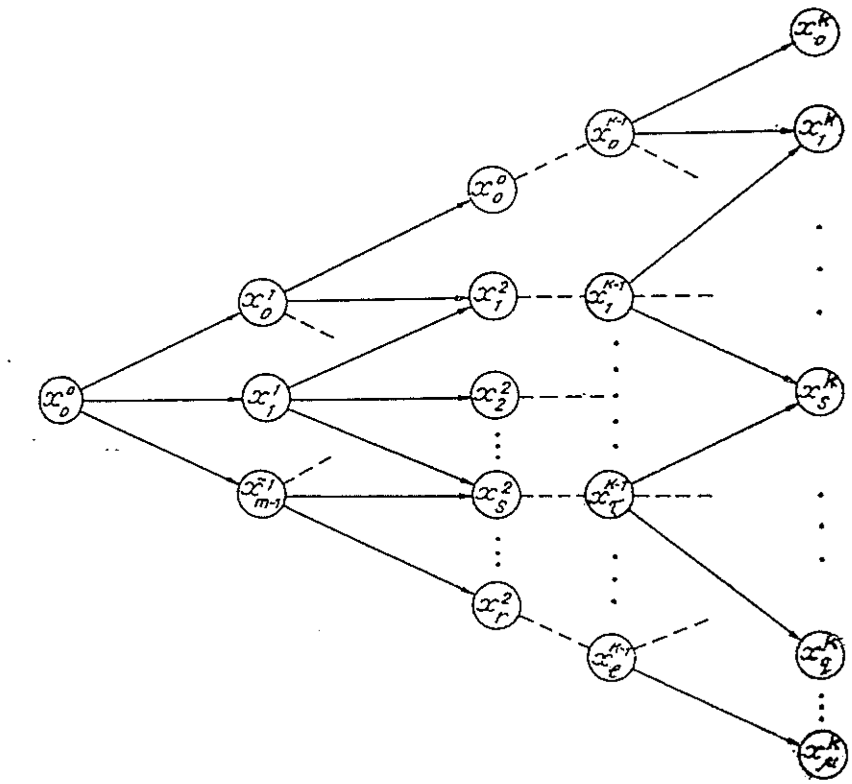

Figure 55

Definition 5.1. The notation

$$B_0=\{\alpha_0^0\},\ B_1=\{\alpha_0^1,\ \alpha_1^1,\ \ldots,\ \alpha_{m_1-1}^1\},\ B_2=\{\alpha_0^2,\ \alpha_1^2,\ \ldots,\ \alpha_{m_2-1}^2\},\ \ldots,\ B_k=\{\alpha_0^k,\ \alpha_1^k,\ \ldots,\ \alpha_{m_k-1}^k\}$$

will denote a basis (set) for the generalized arithmetic graph, with the cardinality of the indexed sets given by

$$|B_0| = 1, \quad |B_1| = m_1, \quad |B_2| = m_2, \ldots, \quad |B_k| = m_k.$$

Definition 5.2. A generalized arithmetic graph is an oriented graph, for which the following conditions hold:

(a) $X = \bigcup_{i=0}^{k} X_i$, and $X_i \cap X_j = \varnothing$, i.e., subsets of vertices at different levels have no common vertices;

(b) $\exists! x_0 \in X[Fx_0 = X_1 \wedge X_1 = B_1 \wedge F^{-1}x_0 = \varnothing]$, i.e., there exists a unique vertex $x_0 \in X$ for which the assertion is true, and this vertex is the root of the graph G(F,X);

(c) $\forall x_s \in X_{i-1}\left[Fx_s \subseteq X_i \rightarrow Fx_s = \left\{x_s + \alpha_0^i,\ x_s + \alpha_1^i,\ \ldots,\ x_s + \alpha_{m_i - 1}^i\right\}\right]$, i.e., from each vertex $x_s \in X_{i-1}$ there issue exactly $|B_i|$ edges;

(d) $\forall x_s \in X_k[Fx_s = \varnothing]$, i.e., any vertex $x_s \in X_k$ at level k is terminal, and the subset X_k is terminal.

Lemma 5.1. In the graph G(F,X), if the vertex $x_s \in X_j$ is reachable from $x_j \in X_i$, then these vertices are joined by a path of length $(j-i)$.

Proof. Since the graph is a Berge graph it contains no loops or inconsistently oriented edges, i.e.,

$$\forall x_q \in X_i[Fx_q \in X_{i+1} \wedge x_q \overline{\in} Fx_q].$$

If $x \in X_i$, then the edge $\vec{J}(x,y)$ connects the vertex $x \in X_i$ with the vertex $y \in X_{i+1}$, and this edge is uniquely determined (Definition 5.2(c)), and, we may assert that the graph G(F,X) is of increasing character. Thus if the vertex $x_s \in X_j$ is reachable from the vertex $x_q \in X_i$, then the vertices are connected by a path of length $(j-i)$, which is what was to be shown.

It is known that any graph is completely determined, up to isomorphisms, by the adjacency matrix of its vertices. We will discuss below a number of properties of the graph G(F,X) in terms of the properties of its adjacency matrix.

Theorem 5.1. The number of paths connecting the vertex $x_0 \in X_0$ with the reachable vertex $x_s \in X_k$ of the generalized arithmetic graph is determined by the relations

$$r^0_{x_0,x_0} = 1,$$

$$r^1_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1 \\ 0, \text{ if } x_s \overline{\in} X_1, \end{cases}$$

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s-\alpha_0^{k-1}} + r^{k-1}_{x_0,x_s-\alpha_1^{k-1}} + \cdots + r^{k-1}_{x_0,x_s-\alpha_{m_k-1}^{k-1}},$$

where $r^\ell_{i,j}$ is the (i,j) element of R^ℓ, the ℓ^{th} power of the adjacency matrix, $\ell=0,1,2,\ldots,k$; $i=x_0$; $j=x_s \in X_k$.

Proof. By Lemma 5.1, the paths joining $x_0 \in X$ to $x_s \in X_k$ have a unique length k, and their number may be calculated with the help of the adjacency matrix R of the graph G(F,X). To do this we need to calculate the k^{th} power of the matrix $R=[r_{ij}]$, where we impose on the semi-ring K generated, the conditions [26] $\xi\eta=1$, $\eta\xi=\theta^2=\zeta=0$. The distributivity condition must be fulfilled for elements of the form $n=1+1+\cdots+1$, where the number is composed of units, and so k contains as a sub-semi-ring k′ the nonnegative whole numbers with ordinary addition and multiplication.

The construction of the matrix R of the graph G(F,X) takes the following form: we consider the subsets in the order $X_0, X_1, \ldots, X_k$, and the elements within a subset in order of increasing element number, and then index the rows and columns in the corresponding order. From Definition 5.2(b) it follows that

$$r^1_{x_0,\alpha_0^1} = r^1_{x_0,\alpha_1^1} = \cdots = r^1_{x_0,\alpha_{m_1-1}^1} = 1, \qquad (5.1)$$

i.e., the vertex x_0 (the root) is connected to the vertices of $X_1=B_1$ by paths of length one. We adopt the convention that x_0 is joined to itself by a path of length one, i.e., $r^1_{x_0,x_0}=1$.

Then we can write (5.1) in the form

$$r^1_{x_0,x_s} = \begin{cases} 1, & \text{if } x_s \in X_1, \\ 0, & \text{if } x_s \bar{\in} X_1. \end{cases}$$

The elements of the matrix $R^2=r^2_{i,j}$ are defined as the number of paths of length two, connecting $x_0 \in X_0$ with $j \in X_2$. To determine the elements of R^2 we use the formula

$$r^2_{i,j} = \sum_{q \in X} r^1_{i,q}\, r^1_{q,i}. \tag{5.2}$$

Since the vertex $j \in X_2$, by Definition 5.2(c), is reachable only from the vertices $j-\alpha^2_0$, $j-\alpha^2_1$, ..., $j-\alpha^2_{m_2-1}$, and not from the remaining vertices of X_1,

$$r^1_{i,j-\alpha^1_i} \cdot r^1_{j-\alpha^1_i,j} = \begin{cases} r^1_{j-\alpha^1_{i,j}} = 1 \text{ and } r^1_{i,j-\alpha^1_i} \neq 0, & \text{if } (j-\alpha^1_i) \in X_1; \\ r^1_{j-\alpha^1_{i,j}} = 0 \text{ and } r^1_{i,j-\alpha^1_i} = 0, & \text{if } (j-\alpha^1_i) \bar{\in} X_1. \end{cases}$$

Thus, (5.2) may be written as

$$r^2_{i,j} = r^1_{i,j-\alpha^1_0} + r^1_{i,j-\alpha^1_1} + \cdots + r^1_{i,j-\alpha^1_{m_1-1}}. \tag{5.3}$$

Setting $i=x_0$ and $j=x_s$ in (5.3), and using Theorem 5.1 for the case k=2, we get

$$r^2_{x_0,x_s} = r^1_{x_0,x_s-\alpha^1_0} + r^1_{x_0,x_s-\alpha^1_1} + \cdots + r^1_{x_0,x_s-\alpha^1_{m_1-1}}. \tag{5.4}$$

Now, we assume that the conditions of the theorem are true for the (k-1)st power of the matrix R, and use induction to show their correctness for the kth power, $R^k=[r^k_{i,j}]$. The elements $r^k_{i,j}$ are determined by the formula

$$r^k_{i,j} = \sum_{q \in X} r^{k-1}_{i,q}\, r^1_{q,j}. \tag{5.5}$$

If we take into account that the vertex $j \in X_k$ is reachable (by paths of length one) only from the vertices

$$\left(j-\alpha^{k-1}_0\right),\ \left(j-\alpha^{k-1}_1\right),\ \dots,\ \left(j-\alpha^{k-1}_{m_{k-1}-1}\right) \in X_{k-1},$$

we must have that the corresponding elements $r^1_{q,j}$ equal one for the subscript j, and equal zero for other vertices. Then, (5.5) may be written as

$$r^k_{i,j} = r^{k-1}_{i,j-\alpha^{k-1}_0} + r^{k-1}_{i,j-\alpha^{k-1}_1} + \cdots + r^{k-1}_{\ell,j-\alpha^{k-1}_{m_k-1}},$$

and from this it follows, setting $i=x_0 \in X_0$ and $j=x_s \in X$, that

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s-\alpha^{k-1}_0} + r^{k-1}_{x_0,x_s-\alpha^{k-1}_1} + \cdots + r^{k-1}_{x_0,x_s-\alpha^{k-1}_{m_k-1}}, \tag{5.6}$$

which is the assertion of the theorem.

Of particular interest in the solution of many combinatorial problems and complicated system design problems is the special case of the generalized arithmetic graph - the generalized m-arithmetic graph, with the basis $<\alpha_0,\alpha_1,\dots,\alpha_{m-1}>$. It follows from Definition 5.2 that the generalized m-arithmetic graph may be obtained from the generalized arithmetic graph G(F,X) if the following conditions are satisfied:

- the generalized m-arithmetic graph has a unique basis $B=\{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}\}$, where

$$B_1 = B_2 = \cdots = B_k = B \text{ and } |B| = m,$$

- the elements of the subsets $X_0,X_1,\ldots,X_k \subseteq X$, of vertices of the m-arithmetic graph are related among themselves by

$$\forall x_s \in X_{i-1}\left[Fx_s \subseteq X_i \rightarrow Fx_s = \{x_s+\alpha_0,\ x_s+\alpha_1,\ \ldots,\ x_s+\alpha_{m-1}\}\right].$$

The number of paths joining the (root) vertex $x_0 \in X_0$ with the reachable vertex $x_s \in X_k$ of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ satisfies the recurrence relations

$$r^{0}_{x_0,x_0} = 1, \qquad (5.7)$$

$$r^{1}_{x_0,x_0} = \begin{cases} 1, \text{ if } x_s \in X_1, \\ 0, \text{ if } x_s \overline{\in} X_1, \end{cases} \qquad (5.8)$$

$$r^{k}_{x_0,x_s} = r^{k-1}_{x_0,x_s-\alpha_0} + r^{k-1}_{x_0,x_s-\alpha_1} + \cdots + r^{k-1}_{x_0,x_s-\alpha_{m-1}}, \qquad (5.9)$$

where (5.7) - (5.9) are obtained from the corresponding relation in Theorem 5.1, if we consider that the graph in this case has a unique basis $B=\{\alpha_0,\ldots,\alpha_{m-1}\}$.

5.2 THE SPECIAL CASE OF THE GENERALIZED m-ARITHMETIC GRAPH

The graph interpretation of the generalized Pascal triangle of order m is discussed in [3, 33-37]. Graphs of this type have been successfully used in decision problems arising in multi-stage discrete processes. Using the properties of the generalized arithmetic graph to construct interactive systems for technological decision making, it is possible to treat decision problems involving choices of equipment to achieve flexible industrial systems. References [33-35] give some algorithms for recognizing paths in the generalized arithmetic graph.

We consider now the special case of the generalized m-arithmetic graph with the basis $<\alpha_0,\alpha_1,\ldots,\alpha_{m-1}>$, which is a widely used model in recent studies of analogs of the Pascal triangle.

Let the elements of the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ of the generalized arithmetic graph be defined in the following way:

$$\alpha_m = \alpha_{m-1} + p = \alpha_0 + mp = m,$$

where $\alpha_0=0$, $\alpha_1=1$.

<u>Definition 5.3</u>. The generalized m-arithmetic graph with the basis $<0,1,\ldots,m-1>$ is said to be a graph model of the generalized Pascal triangle of order m.

The recurrence formulas determining the number of paths in the m-arithmetic graph (Theorem 5.1), taking into account the properties of the basis, coincide with the recurrence formulas for the elements of the triangle of order m:

$$r^0_{x_0,x_0} = 1,$$

$$r^1_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1, \\ 0, \text{ if } x_s \overline{\in} X_1, \end{cases}$$

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s} + r^{k-1}_{x_0,x_{s-1}} + \cdots + r^{k-1}_{x_0,x_{s-m+1}}.$$

Example. Let m=3 and k=2, so that $X_0=\{0\}$, $X_1=\{0,1,2\}$, $X_2=\{0,1,2,3,4\}$. In this case the elements of the adjacency matrix, $r^2_{x_0,x_s}$, coincide with the coefficients generated by the function $A(t)=(1+t+t^2)^2$, i.e.,

$$A(t)=\sum_{x_s\in X_2} r^2_{x_0,x_s}t^{x_s}.$$

The geometric interpretation of the m-arithmetic graph corresponding to the generalized Pascal triangle of order 3 is shown in Figure 56a.

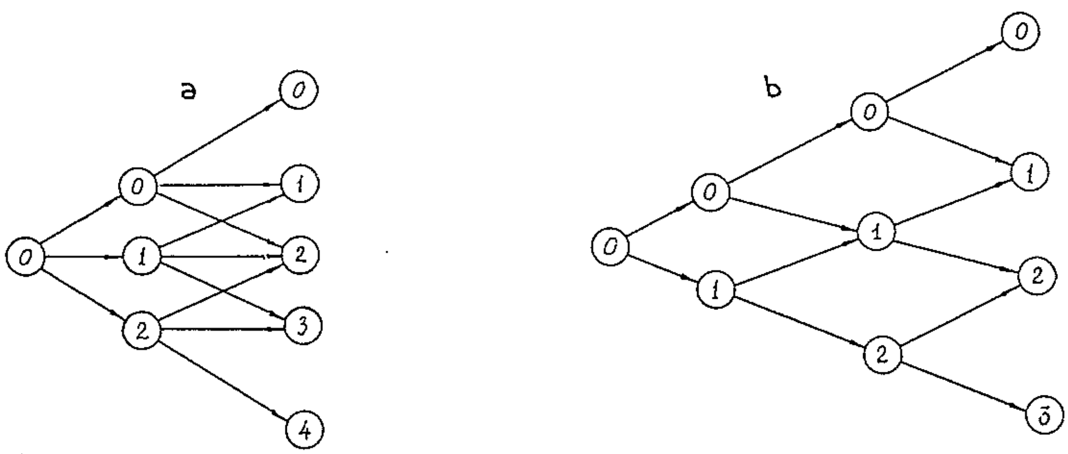

Figure 56

Definition 5.4. The generalized m-arithmetic graph with the basis <0,1> is said to be the graph of the Pascal triangle.

Theorem 5.2. The number of paths connecting the (root) vertex $x_0\in X_0$ to the vertex $x_s\in X_k$ in the graph of the Pascal triangle is given by the binomial coefficient $\binom{k}{x_s}$.

Proof. From Theorem 5.1, we have $r^0_{x_0,x_0}=\binom{0}{x_0}=1$. For k=1 the vertices in the subset $X_1=\{0,1\}$ are connected with x_0 by paths of length one; the numbers of paths are given by

$$r^1_{x_0,x_s} = \begin{cases} 1, \text{ if } x_s \in X_1 \\ 0, \text{ if } x_s \overline{\in} X_1 \end{cases}. \tag{5.10}$$

It follows from (5.10) that for $x_s=0$ the formula $r^1_{x_0,0}=1$ is correct, and we may write here $\binom{1}{0}$ in place of unity. For $x_s=1$, we obtain $r^1_{x_0,x_1}=1$, which we write as $\binom{1}{1}$. That is, the statement of Theorem 5.2 is true for k=0,1. Suppose that the assertion of the theorem is satisfied for vertices in the subset X_{k-1}, i.e., numbers of paths connecting $x_0 \in X_0$ with the vertices $x_s \in X_{k-1}=\{0,1,\ldots,x_s-0,x_s-1,\ldots,k-1\}$ are given by

$$\left\{\binom{k-1}{0}, \binom{k-1}{1}, \ldots, \binom{k-1}{x_s-0}, \binom{k-1}{x_s-1}, \ldots, \binom{k-1}{k-1}\right\}.$$

Now, from Theorem 5.1 we find for the Pascal triangle graph

$$r^k_{x_0,x_s} = r^{k-1}_{x_0,x_s-0} + r^{k-1}_{x_0,x_s-1}, \tag{5.11}$$

and by the induction hypothesis we can substitute $\binom{k-1}{x_0-s}$ for the first term, and $\binom{k-1}{x_s-1}$ for the second, whence

$$r^k_{x_0,x_s} = \binom{k-1}{x_s-0} + \binom{k-1}{x_s-1} = \binom{k}{x_s},$$

and so the result is true for k also, and the theorem is proved. The graph of the Pascal triangle with basis <0,1> for k=3 is shown in Figure 56b.

5.3 AN ASYMPTOTIC FORMULA FOR THE NUMBER OF PATHS IN THE GENERALIZED m-ARITHMETIC GRAPH

The method of determining the number of paths connecting $x_0 \in X_0$ with $x_s \in X_k$ by means of the adjacency matrix of the graph G(X,F) is convenient for small k, but involves computational difficulties if k is large. This situation can be avoided if it is possible to find an asymptotic formula to estimate the number of paths connecting x_0 with $x_s \in X_k$ as $k \to \infty$.

From the definition of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ it follows that from each vertex, independent of its value (except terminal vertices), there originate exactly m edges, and each of these corresponds uniquely to one of the basis values. On the other hand, the graph G(X,F) consists of a set of paths of length k of the form $s=(x_0,x_1,\ldots,x_s)$, where $x_0 \in X_0, x_1 \in X_1, \ldots, x_s \in X_k$. As a simple example, take the value of the root to be $x_0=0$, for which we then have

$$x_1 = x_0 + \beta_1; \quad x_2 = x_1 + \beta_2; \quad \ldots; \quad x_s = x_{s-1} + \beta_s,$$

where $\beta_j \in \{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}\}$. With $x_0=0$, we can write $x_s=\beta_1+\beta_2+\cdots+\beta_k$.

Lemma 5.2. In the generalized m-arithmetic triangle with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$, the number of all possible paths of the form s is $N=m^k$.

Proof. The component β_1 of the path s may take on any of the m basis elements as value. The component β_2 may take on the same values independently of β_1, and so on for each $\beta_j \in B$. Using the "product rule" of combinatorial analysis [44], we have

$$N = |\ \{\beta_1,\beta_2,\ldots,\beta_k\},\ \beta_j \in B\ | = m^k.$$

In terms of the elements of the adjacency matrix, it follows from the lemma that the number of paths of the form s may be written as

$$\sum_{x_s \in X_k} r^k_{x_0, x_s} = m^k.$$

Then each vertex $x_s \in X_k$ may be assigned the probability

$$p(x_s) = p(x_s = \beta_1 + \beta_2 + \cdots + \beta_k) = \frac{r^k_{x_0, x_s}}{m^k},$$

where

$$\sum_{x_s \in X_k} p(x_s) = 1.$$

The individual terms of the expression $x_s = \beta_1 + \cdots + \beta_k$ may be considered independent random variables taking on as values the basis elements. The expected value of the sum of independent random variables equals the sum of the expected values, and so we have

$$a_k = ka_1 = k\left(\alpha_0 + p\frac{m-1}{2}\right), \tag{5.12}$$

where a_1 is the expected value of $\beta_1 \in X_1$.

The variance in this case is determined by the formula

$$\sigma_k^2 = k\sigma_1^2 + \frac{kp^2(m+1)}{6m}, \tag{5.13}$$

where σ_1 is the variance of $\beta_1 \in X_1$.

Using a known limit theorem [38] from probability theory for $k\to\infty$, and (5.12) and (5.13), we obtain the asymptotic formula for the number of paths connecting $x_0 \in X_0$ with $x_s \in X_k$:

$$r^k_{x_0,x_s} = \frac{m^k}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(x_s-a_k)^2}{2\sigma_k^2}\right] + O\left(\frac{m^{k-1}}{\sqrt{k}}\right). \tag{5.14}$$

The result (5.14) could also be used to obtain approximate values of the elements of the generalized Pascal triangle of order m, for large values of k.

5.4 GENERALIZED m-ARITHMETIC GRAPHS AND SPATIAL ISOMORPHISMS

We consider now the spatial representation of the m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$, and a theorem on the isomorphism of this graph with the planar graph.

Definition 5.5. The module-graph of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ is the graph $G_{x_p}(X',F')$ defined in the following way:

(a) $X' = X'_0 \cup X'_1$ and $X'_0 \cap X'_1 = \varnothing$, where $X'_0 = \{x_p\}$, $X'_1 = \{x_p+\alpha_0, x_p+\alpha_1, \ldots, x_p+\alpha_{m-1}\}$;

(b) $\exists! x_p \in X'_0 \left[F'_{x_p} = X'_1 \wedge (F')^{-1}x_p = \varnothing\right]$, i.e., the vertex $x_p \in X'_0$ is the root of the graph $G_{x_p}(X',F')$;

(c) $\forall x_s \in X'_1 [Fx_s = \varnothing]$.

It follows from the definition of the module-graph $G_{x_p}(X',F')$ that this graph is a subgraph of G(X,F), i.e.,

$$G_{x_p}(X',F') \subseteq G(X,F).$$

The module-graph $G_{x_p}(X',F')$ with $x_p=0$ and $X_1'=\{\alpha_0,\alpha_1,\alpha_2\}$ is shown in Figure 57a.

The notion of orientedness of a graph need not be arbitrarily imposed on the edges and vertices of the graph $G_{x_p}(X',F')$ generated here. That is, the edges may have any geometric length and direction without violating the properties of incidence and connectedness. Further, without violating the conditions of Definition 5.5, in the graph $G_{x_p}(X',F')$ we may:

(1) identify the root $x_p \in X_0'$ with the origin of coordinates in m-dimensional space;

(2) take each edge $\vec{J}(x_p,x_i)$, where $x_i \in X_1'$, to have the direction and magnitude corresponding to the unit vectors $\vec{e}_0$, $\vec{e}_1$, ..., $\vec{e}_{m-1}$ of m-dimensional space.

If in fact these identifications are made, we obtain the spatial representation of the module-graph $G_{x_p}(X',F')$, coinciding, up to isomorphisms, with the usual m-dimensional coordinate space. The vertex x_p is at the origin, and the edges $\vec{J}(x_p,x_p+\alpha_0),\ldots,\vec{J}(x_p,x_p+\alpha_{m-1})$ coincide with the unit vectors. Such a representation for $x_p=0$ and $X_1'=\{\alpha_0,\alpha_1,\alpha_2\}$ is shown in Figure 57b.

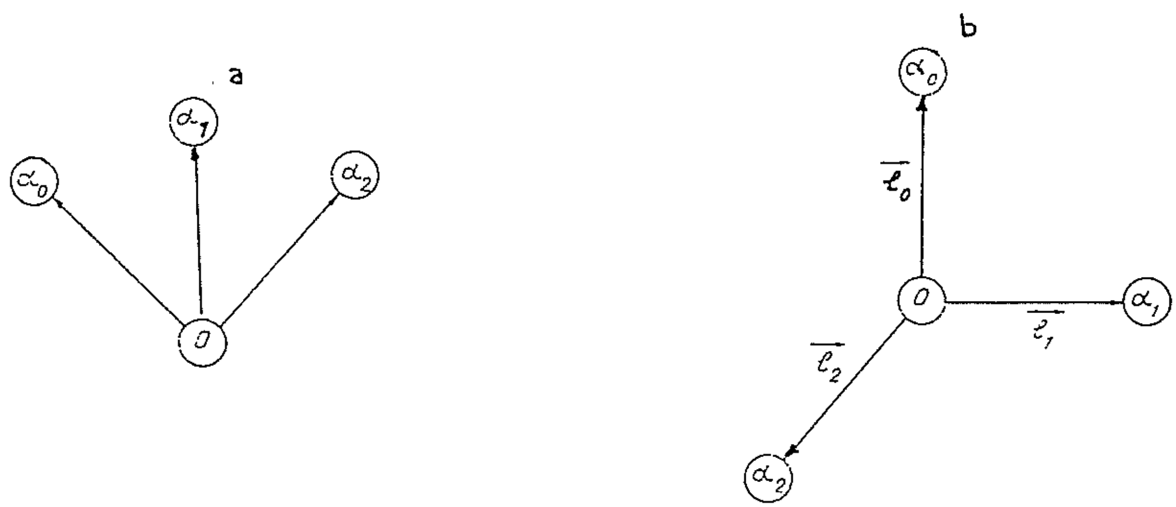

Figure 57

In what follows, when we speak of a module-graph we will be referring to its spatial representation.

Lemma 5.3. The generalized m-arithmetic graph with the usual basis may be represented by means of its module-graphs in m-space, i.e.,

$$G(X,F) = \bigcup_{x_p} G_{x_p}(X', F'),$$

where $x_p \in X_0 \cup X_1 \cup X_2 \cup \ldots \cup X_{k-1}$.

Proof. For k=1, the assertion is easy to show, i.e.,

$$G(X,F) = \bigcup_{x_0} G_{x_p}(X',F') = G_{x_0}(X',F'),$$

where $X_0 = X_0' = \{x_0\}$ and $X_1 = X_1'$.

As for the mappings F and F', F'=F will fulfill the condition. It follows, then, that for k=1 the generalized m-arithmetic graph G(F,X) with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$ coincides, up to isomorphisms, with its module-graph.

Now suppose the assertion of the lemma is valid for the m-arithmetic graph with the set of vertices $X''=X_0 \cup X_1 \cup \cdots \cup X_{k-1}$ (path lengths $k-1$), and denote this graph by $G_{k-1}(X'',F'')$. For this graph, (5.15) has the form

$$G_{k-1}(X'',F'') = \bigcup_{x_p} G_{x_p}(X',F'),$$

where $x_p \in X_0 \cup X_1 \cup \cdots \cup X_{k-2}$.

Then, we will have for the graph G(X,F)

$$G(X,F) = G_{k-1}(X'',F'') \bigcup_{x_p \in X_{k-1}} G_{x_p}(X',F'), \tag{5.16}$$

and from Definition 5.2 and Definition 5.5 it follows that

$$\forall x_p \in X_{k-1} \left[Fx_p = \{x_p+\alpha_0, x_p+\alpha_1, \ldots, x_p+\alpha_{m-1}\} \wedge F'x_p = Fx_p \right].$$

And for the subset of vertices X_k, we have that

$$X_k = \bigcup_{x_p \in X_{k-1}} F'x_p,$$

according to which, if $G_{k-1}(X'',F'')$ may be represented by its module-graph, then for G(X,F) in (5.16) it follows that

$$G(X,F) = \bigcup_{x_p \in Q} G_{x_p}(X',F') \bigcup_{x_p \in X_{k-1}} G_{x_p}(X',F') = \bigcup_{x_p \in D} G_{x_p}(X',F'),$$

where $Q=X_0 \cup X_1, \ldots, X_{k-2}$ and $D=X_0 \cup X_1 \cup \cdots \cup X_{k-1}$, which is the assertion of the lemma.

Definition 5.6. The graph obtained as the union of module-graphs by (5.15) is said to be the spatial representation of the generalized m-arithmetic graph with the basis $<\alpha_0,\ldots,\alpha_{m-1}>$, or the spatial arithmetic graph, and is denoted by $G_S(X,F)$.

Figure 58 shows the spatial arithmetic graph with the basis $<1,3,5>$ and k=3; the vertices are shown as points, and the edges are not oriented.

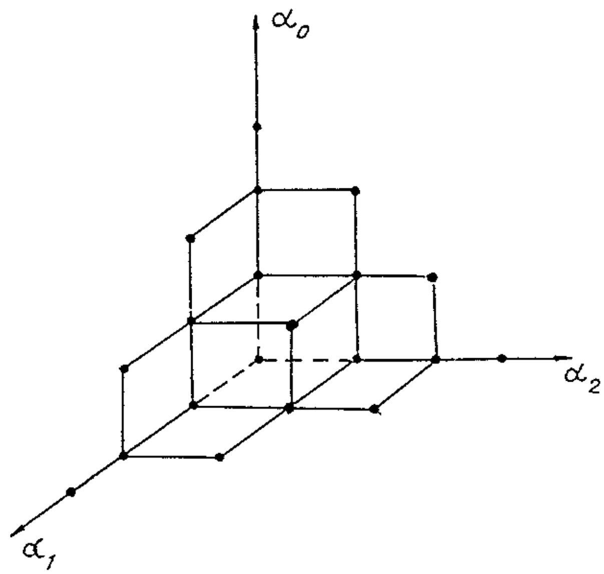

Figure 58

Theorem 5.3. The spatial arithmetic graph $G_S(X,F)$ is isomorphic to the generalized m-arithmetic graph G(X,F).

Proof. The spatial graph may be obtained as the union of the module-graphs $G_{x_p}(X',F')$, the edges of which have defined directions, and coincide with the direction and magnitude of the unit vectors. This last condition does not violate the properties of incidence and connectedness of the vertices of G(X,F), and by a known theorem [32] this is a necessary and sufficient condition for the isomorphism of G(X,F) and $G_S(X,F)$.

5.5 PROPERTIES OF THE CROSS SECTIONS OF THE SPATIAL ARITHMETIC GRAPH

We discuss now some properties of the cross sections of the spatial m-arithmetic graph, from which we might derive some general methods for obtaining recurrent sequences from various analogs of the Pascal triangle.

For each vertex $x_s \in X_k$ of the spatial arithmetic graph $G_S(X,F)$, we have the valid representation $x_s = x_0 + \beta_1 + \beta_2 + \cdots + \beta_k$, where $\beta_j \in \{\alpha_0, \cdots, \alpha_{m-1}\}$, and if we take $x_0 = 0$, then $x_s = \beta_1 + \cdots + \beta_k$. On the other hand, we can also represent x_s by the linear expression

$$x_s = \mu_0 \alpha_0 + \mu_1 \alpha_1 + \cdots + \mu_{m-1} \alpha_{m-1},$$

where μ_i is the number of components β_j taking on the value α_i.

Definition 5.7. The numbers $\mu_0, \mu_1, \cdots, \mu_{m-1}$ are said to be the coordinates of the vertex $x_s \in X_k$.

The number of paths connecting $x_0 \in X_0$ with $x_s \in X_k$, i.e., $r^k_{x_0,x_s}$, may be determined by the multinomial formula

$$r^k_{x_0,x_s} = \frac{(\mu_0 + \mu_1 + \cdots + \mu_{m-1})!}{\mu_0! \mu_1! \cdots \mu_{m-1}!}, \tag{5.18}$$

where $\mu_0 + \cdots + \mu_{m-1} = k$.

Definition 5.8. By a cross section of the spatial arithmetic graph $G_S(X,F)$, we will mean the sum of the numbers of paths connecting the origin of coordinates $x_0 \in X_0$ with the vertices x_s of this graph which lie in the hyperplane

$$a_0 y_0 + a_1 y_1 + \cdots + a_{m-1} y_{m-1} = n \tag{5.19}$$

where the a_i are positive whole numbers.

Denote this cross section by T_n. Then, since the coordinates $(\mu_0,\ldots,\mu_{m-1})$ of the vertices $x_s \in X$ of the graph $G_S(X,F)$ must satisfy (5.19), i.e.,

$$a_0\mu_0 + a_1\mu_1 + \cdots + a_{m-1}\mu_{m-1} \equiv n, \tag{5.20}$$

T_n is determined by the equation

$$T_n = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1})!}{\mu_0!\mu_1!\cdots\mu_{m-1}!}. \tag{5.21}$$

<u>Theorem 5.4</u>. The cross section T_n of the graph $G_S(X,F)$ satisfies the recurrence formula

$$T_n = T_{n-a_0} + T_{n-a_1} + \cdots + T_{n-a_{m-1}}, \tag{5.22}$$

where $T_0=1$ and $T_k=0$, if $k<0$.

<u>Proof</u>. Subtracting a_0 from each side of (5.19) we have

$$a_0(y_0-1) + a_1y_1 + \cdots + a_{m-1}y_{m-1} = n-a_0.$$

From (5.20) and (5.21), the cross section T_{n-a_0} is given by

$$T_{n-a_0} = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{(\mu_0-1)!\mu_1!\cdots\mu_{m-1}!}.$$

Likewise, for a_1 we will have

$$T_{n-a_1} = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\cdots+\mu_{m-1}-1)!}{\mu_0!(\mu_1-1)!\cdots\mu_{m-1}!},$$

and so on, up to a_{m-1}:

$$T_{n-a_{m-1}} = \sum_{x_s \in X} \frac{(\mu_0+\mu_1+\dots+\mu_{m-1}-1)!}{\mu_0!\mu_1!\cdots(\mu_{m-1}-1)!}.$$

Substituting these expressions for the T's on the right side of (5.22), we have

$$\sum_{x_s \in X} \left[\frac{(\mu_0+\mu_1+\dots+\mu_{m-1}-1)!}{(\mu_0-1)!\mu_1!\cdots\mu_{m-1}!} + \frac{(\mu_0+\mu_1+\dots+\mu_{m-1}-1)!}{\mu_0!(\mu_1-1)!\cdots\mu_{m-1}!} + \dots + \frac{(\mu_0+\mu_1+\dots+\mu_{m-1}-1)!}{\mu_0!\mu_1!\cdots(\mu_{m-1}-1)!} \right]. \tag{5.23}$$

But after an elementary transformation, (5.23) may be written in the form

$$\sum_{x_s \in X} \frac{(\mu_0+\mu_1+\dots+\mu_{m-1})!}{\mu_0!\mu_1!\cdots\mu_{m-1}!},$$

which is just T_n, and this completes the proof.

For n=0, the plane (5.19) passes through the origin of coordinates; in this case we take $T_0=1$. For values of $n<0$, $T_n=0$, since $G_S(X,F)$ has no vertices in the negative half-space. Theorem 5.4 says, in effect, that among the cross sections by parallel planes we have the recurrence relation (5.22). Thus, we can think of the spatial graph of the generalized Pascal triangle of order m as the source of an infinite number of recurrence relations of the type (5.22).

CHAPTER 6

MATRICES AND DETERMINANTS OF BINOMIAL AND GENERALIZED BINOMIAL COEFFICIENTS AND OTHER NUMBERS

Matrices and determinants composed of binomial and generalized binomial coefficients, and Fibonacci, Lucas, and Catalan numbers arise in the solution of specific systems of algebraic and difference equations, and may have interesting properties. The problems of evaluating determinants composed of binomial coefficients arranged in some specific form in the Pascal triangle have long been known; some of these may be found in [292]. Material on the construction of matrices and determinants of elements of the Pascal triangle, its generalizations, and other numbers, may be found in the papers of M. Bicknell and V.E. Hoggatt [72, 74, 75, 204-206], and in [92, 93, 105, 150, 258, 269, 282, 285, 326, 333, 342, 375].

6.1 MATRICES AND DETERMINANTS OF ELEMENTS OF THE PASCAL TRIANGLE

It turns out that there are many ways of choosing a square matrix of elements of the Pascal triangle so that the determinant is unity, or may be evaluated in terms of the corresponding binomial coefficients by some explicit formula. In [74] there are specific examples of choices of a square matrix from the Pascal triangle so that the determinant is unity.

With the Pascal triangle written in rectangular form, let us choose from it the $n \times n$ matrix $A=(a_{ik})$ as shown,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 1 & 2 & 3 & 4 & 5 & . & . & . \\ 1 & 3 & 6 & 10 & 15 & . & . & . \\ 1 & 4 & 10 & 20 & 35 & . & . & . \\ 1 & 5 & 15 & 35 & 70 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n},$$

where a_{ik} is the i^{th} row, k^{th} column element, $1 \le i,k \le n$. It is not difficult to show that

$$a_{ik} = \binom{i+k-2}{k-1}.$$

In [74] it is shown that the determinant of any submatrix consisting of the first m rows and columns of A has a value of one; for example

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1.$$

With the Pascal triangle in right triangular form, let us choose from it the $n\times n$ matrix B as shown,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 1 & 2 & 1 & 0 & 0 & . & . & . \\ 1 & 3 & 3 & 1 & 0 & . & . & . \\ 1 & 4 & 6 & 4 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n\times n}, \text{ where } b_{ik} = \binom{i-1}{k-1}.$$

Again in [74], it is shown that the determinant of any submatrix of B which contains the column of 1's, has value one; for example

$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = 1.$$

Along with the matrices A and B in [74], there are also some other matrices, the determinants of which have simple evaluations. Thus, let R(m,r) be the $m \times m$ submatrix of A which has as its first row the second row of A, and as its first column the r^{th} column of A; for example,

$$R(4,3) = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{bmatrix}.$$

The authors show that the determinant of R(m,r) equals the binomial coefficient $\binom{m+r-1}{m}$, and also satisfies the recurrence formula

$$\det R(m,r) = \det R(m-1,r) + \det R(m,r-1).$$

For some specific values of m,r we find that

$$\det R(m,1) = 1, \quad \det R(1,r) = r, \quad \det R(m,2) = m+1,$$

$$\det R(2,r) = \frac{1}{2}r(r+1), \quad \det R(4,3) = 15.$$

From Netto's book on combinatorics [292], and omitting the proofs, here are some interesting determinants D_k composed of binomial coefficients.

$$D_1 = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+1} & \cdots & \binom{n}{m+r} \\ \binom{n+1}{m} & \binom{n+1}{m+1} & \cdots & \binom{n+1}{m+r} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+1} & \cdots & \binom{n+r}{m+r} \end{vmatrix}$$

$$= \frac{\binom{n}{m}\binom{n+1}{m}\cdots\binom{n+r}{m}}{\binom{m+r}{r}\binom{m+r-1}{r-1}\cdots\binom{m+1}{1}}, \; m \le n,$$

$$D_2 = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+2} & \cdots & \binom{n}{m+2r} \\ \binom{n+1}{m} & \binom{n+1}{m+2} & \cdots & \binom{n+1}{m+2r} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+2} & \cdots & \binom{n+r}{m+2r} \end{vmatrix}$$

$$= 2^{\frac{1}{2}r(r-1)} \frac{\binom{n}{m+r}\binom{n+1}{m+r-1}\cdots\binom{n+r}{m}}{\binom{m+2r}{r}\binom{m+2r-2}{r-2}\cdots\binom{m+2}{1}}, \; m \le n-r,$$

$$D_3 = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+3} & \cdots & \binom{n}{m+3r} \\ \binom{n+1}{m} & \binom{n+1}{m+3} & \cdots & \binom{n+1}{m+3r} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+3} & \cdots & \binom{n+r}{m+3r} \end{vmatrix}$$

$$= 3^{\frac{1}{2}r(r+1)} \frac{\binom{n}{m+2r}\binom{n+1}{m+2r-2}\cdots\binom{n+r}{m}}{\binom{m+3r}{r}\binom{m+3r-3}{r-1}\cdots\binom{m+3}{1}}, \quad m \le n-2r.$$

The determinants D_1, D_2, D_3 may be extended to the case of any whole number k, the result being

$$D_k = \begin{vmatrix} \binom{n}{m} & \binom{n}{m+k} & \cdots & \binom{n}{m+kr} \\ \binom{n+1}{m} & \binom{n+1}{m+k} & \cdots & \binom{n+1}{m+kr} \\ \vdots & \vdots & & \vdots \\ \binom{n+r}{m} & \binom{n+r}{m+k} & \cdots & \binom{n+r}{m+kr} \end{vmatrix}$$

$$= k^{\frac{1}{2}r[r-(-1)^k]} \frac{\binom{n}{m+(k-1)r}\binom{n+1}{m+(k-1)(r-1}\cdots\binom{n+r}{m}}{\binom{m+kr}{r}\binom{m+k(r-1)}{r-1}\cdots\binom{m+k}{1}}, \quad m \le n-(k-1)r.$$

Matrices and determinants composed of binomial coefficients are also discussed in [61, 72, 74, 150, 269, 285].

6.2 MATRICES AND DETERMINANTS OF ELEMENTS OF GENERALIZED PASCAL TRIANGLES

Consider the $n \times n$ matrix A_3 composed of trinomial coefficients, of the form

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 0 & 1 & 2 & 3 & 4 & . & . & . \\ 0 & 1 & 3 & 6 & 10 & . & . & . \\ 0 & 0 & 2 & 7 & 16 & . & . & . \\ 0 & 0 & 1 & 6 & 19 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n \times n} .$$

It is not hard to show that A_3 may be represented in the form of a product of matrices composed of binomial coefficients

$$A_3 = F_1 A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & 3 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n \times n} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 0 & 1 & 2 & 3 & 4 & . & . & . \\ 0 & 0 & 1 & 3 & 6 & . & . & . \\ 0 & 0 & 0 & 1 & 4 & . & . & . \\ 0 & 0 & 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n \times n} .$$

In [74, 75] it is shown that, in general, the matrix A_s of generalized binomial coefficients $\binom{n}{m}_s$ may also be represented as a product $A_s = F_{s-2} A^T$. Using this result, the authors prove that:

(1) the determinant of the $n \times n$ matrix A_s' whose first row and first column coincide with those of A_s, has the value one;

(2) the determinant of the $n \times n$ matrix A_s'' whose first column is the first column of A_s, and whose first row is the r^{th} row of A_s, has the value $\binom{n+r-1}{n}$.

Suppose we form the matrix of coefficients in the expansions of the elements in the sequence 1, 1+x, (1+x)(2+x), (1+x)(2+x)(3+x), ...:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 2 & 3 & 1 & 0 & 0 & . & . & . \\ 6 & 11 & 6 & 1 & 0 & . & . & . \\ 24 & 50 & 35 & 10 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}_{n \times n},$$

where the coefficients $a_{r,s}$ are determined by the recurrence

$$a_{n,m} = na_{n-1,m} + a_{n-1,m-1}, \quad a_{0,0} = 1.$$

Now form from the elements of C, the n^{th} order determinant

$$D_{n,m} = \begin{bmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,m-1} \\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,m-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1,0} & a_{n+m-1,1} & \cdots & a_{n+m-1,m-1} \end{bmatrix}.$$

Then, according to [333],

$$D_{n,m} = (n!)^m.$$

6.3 MATRICES AND DETERMINANTS OF ELEMENTS OF OTHER ARITHMETIC TRIANGLES

As discussed in Section 1.4, arithmetic triangles composed of Gaussian binomial coefficients, or Fibonacci, Lucas, Stirling, Euler, and other special numbers may form lower triangular, upper triangular, or other kinds of matrices. Studies of such matrices and their applications may be found in [92, 93, 204-206, 258, 282, 326]. We mention a few of these results.

D.A. Lind [258] formed the matrix of Fibonomial coefficients

$$\begin{bmatrix} m \\ r \end{bmatrix} = F_m F_{m-1} \cdots F_{m-r+1} / (F_1, F_2 \cdots F_r),\ r \geq 1,$$

where $\begin{bmatrix} m \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} m \\ r \end{bmatrix} = 0$ for $0 \leq m \leq r-1$ or $r < 0$. Let $D_{n,k}$ be the determinant of the $n \times n$ matrix (a_{rs}), where

$$a_{rs} = -(-1)^{(s-r+1)(s-r+2)/2} \begin{bmatrix} k+1 \\ s-r+1 \end{bmatrix},\quad r,s = 1,2,\ldots,n.$$

Then

$$D_{n,k} = \begin{bmatrix} n+k \\ k \end{bmatrix}.$$

We also note the values of some other determinants given in [258]; thus

$$\begin{bmatrix} 1 & 1 & 0 & 0 & . & . & . \\ -1 & 1 & 1 & 0 & . & . & . \\ 0 & -1 & 1 & 1 & . & . & . \\ 0 & 0 & -1 & 1 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = F_{n+1}, \qquad \begin{bmatrix} 2 & -1 & 0 & 0 & . & . & . \\ -1 & 2 & -1 & 0 & . & . & . \\ 0 & -1 & 2 & -1 & . & . & . \\ 0 & 0 & -1 & 2 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = n+1,$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & . & . & . \\ -1 & 2 & 2 & -1 & . & . & . \\ 0 & -1 & 2 & 2 & . & . & . \\ 0 & 0 & -1 & 2 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = F_{n+1}F_{n+2}, \qquad \begin{bmatrix} 3 & -3 & 1 & 0 & . & . & . \\ -1 & 3 & -3 & 0 & . & . & . \\ 0 & -1 & 3 & -3 & . & . & . \\ 0 & 0 & -1 & 3 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n} = \frac{1}{2}(n+1)(n+2).$$

T.A. Brennan in [92, 93] studied the properties of the $n\times n$ matrix P_n, formed from a version of the Pascal triangle,

$$P_n = \begin{bmatrix} . & . & . & 0 & 0 & 0 & 1 \\ . & . & . & 0 & 0 & 1 & 1 \\ . & . & . & 0 & 1 & 2 & 1 \\ . & . & . & . & . & . & . \end{bmatrix}_{n\times n}.$$

He proved that the characteristic polynomial is given by

$$|xJ-Q_n| = \sum_{r=0}^{n+1} (-1)^{\frac{r(r+1)}{2}} \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n-r+1},$$

where Q_n is the matrix obtained from P_n by transposing P_n about its secondary diagonal, and $\begin{bmatrix} n \\ r \end{bmatrix}$ is the Fibonomial coefficient mentioned earlier.

CHAPTER 7

COMBINATORIAL ALGORITHMS FOR CONSTRUCTING GENERALIZED-HOMOGENEOUS POLYNOMIALS. SOME CLASSES OF NON-ORTHOGONAL POLYNOMIALS

In this chapter, based on some new concepts and definitions - in particular the generalized power of a variable, and of a monomial - we introduce the so-called generalized-homogeneous polynomial. We will establish a formula for the maximal number of terms of the generalized-homogeneous polynomial and find for these polynomials the analog of the Euler formula. We also introduce the notion of the factorial power, and use it to construct and study factorial polynomials. For these, we establish differentiation and integration formulas, and other relations.

We will introduce also the so-called exponent matrix and coefficient matrix, from which we may develop combinatorial algorithms for the construction of basis systems of polynomial solutions for generalized polyharmonic equations, and equations with mixed derivatives; various forms of the Pascal triangle are used in these algorithms. We will consider polynomial solutions for equations with third-order partial derivatives; more detailed studies may be found in [5-10, 13, 14] and in [82-87].

7.1 GENERALIZED-HOMOGENEOUS POLYNOMIALS. EULER'S FORMULA

As is known, a polynomial is said to be homogeneous if every term is of the same degree. We generalize this basic notion, which will allow the possibility of constructing basis systems of polynomial solutions for a wide class of differential equations, including those with mixed derivatives; we use the methods of combinatorial analysis in carrying out this development. For greater clarity, we will consider the case of three variables; the extension to any number of variables is presented in [14].

Before we consider any new concepts, let us consider the vectors p, q, r:

$$p = (p_1,p_2,p_3), \quad q = (q_1,q_2,q_3), \quad r = (r_1,r_2,r_3),$$

where p and q are nonnegative whole numbers, and r is a positive whole number. We will assume that $r_1 \le r_2 \le r_3$, and introduce the transformation

$$\left.\begin{aligned} q_1 &= r_3\left[\frac{p_1}{r_1}\right] + r_1\left\{\frac{p_1}{r_1}\right\}, \\ q_2 &= r_3\left[\frac{p_2}{r_2}\right] + r_2\left\{\frac{p_2}{r_2}\right\}, \\ q_3 &= r_3\left[\frac{p_3}{r_3}\right] + r_3\left\{\frac{p_3}{r_3}\right\} = p_3, \end{aligned}\right\} \qquad (7.1)$$

where [A] denotes the integer part of A, and {A} the fractional part. It is not hard to see that (7.1) is unique, and takes nonnegative whole numbers into nonnegative whole numbers. That is, suppose that two distinct values p_k' and p_k'' correspond to the same value q_k. We can then put $p_k' = n$, and $p_k'' = n+m$, with n,m nonnegative whole numbers, and we will have

$$r_3\left[\frac{n+m}{r_k}\right] + r_k\left\{\frac{n+m}{r_k}\right\} = r_3\left[\frac{n}{r_k}\right] + r_k\left\{\frac{n}{r_k}\right\}, \quad k = 1,\ 2,\ 3.$$

We write this in the form

$$(r_3 - r_k)\left[\frac{n+m}{r_k}\right] + r_k\left[\frac{n+m}{r_k}\right] + r_k\left\{\frac{n+m}{r_k}\right\}$$

$$= (r_3 - r_k)\left[\frac{n}{r_k}\right] + r_k\left[\frac{n}{r_k}\right] + r_k\left\{\frac{n}{r_k}\right\},$$

and note that for any nonnegative whole number a and natural number b it is true that

$$b\left[\frac{a}{b}\right] + b\left\{\frac{a}{b}\right\} = a;$$

then

$$(r_3 - r_k)\left(\left[\frac{n+m}{r_k}\right] - \left[\frac{n}{r_k}\right]\right) + m = 0,$$

and since $r_3 - r_k \geq 0$, $\left[\frac{n+m}{r_k}\right] - \left[\frac{n}{r_k}\right] \geq 0$, it follows that the left hand side reduces to zero only for $m=0$, and so $p_k' = p_k''$ and (7.1) is unique.

Consider also the transformation

$$\left.\begin{aligned} p_1 &= r_1\left[\frac{q_1}{r_3}\right] + r_3\left\{\frac{q_1}{r_3}\right\}, \\ p_2 &= r_2\left[\frac{q_2}{r_3}\right] + r_3\left\{\frac{q_2}{r_3}\right\}, \\ p_3 &= r_3\left[\frac{q_3}{r_3}\right] + r_3\left\{\frac{q_3}{r_3}\right\} = q_3. \end{aligned}\right\} \qquad (7.2)$$

Let us represent $q_k(k=1,2)$ in the form $q_k=m_k r_3+s_k$, where $s_k=0,1,\ldots,r_3-1$. It is not hard to see that (7.2) is unique for $s_k=0,1,\ldots,r_k-1$, but not for $s_k=r_k,r_{k+1},\ldots,r_3-1$. Also, let $x=(x_1,x_2,x_3)$, with p_k the power of x_k.

Definition 7.1. The number q_k, obtained from the transformation (7.1), is said to be the generalized power (or degree) of the variable x_k relative to the pair (r_k,r_3).

If $r_1=r_2=r_3$, then the generalized power reduces to the ordinary power.

Consider now the monomial $x^p=x_1^{p_1}x_2^{p_2}x_3^{p_3}$.

Definition 7.2. The number $n=|q|=q_1+q_2+q_3$, where q_k is the generalized power of the variable x_k relative to (r_k,r_3), is said to be the generalized power (or degree) of the monomial relative to $r=(r_1,r_2,r_3)$.

Monomials having equal generalized powers relative to the same r are said to be generalized-homogeneous.

Definition 7.3. A polynomial whose terms have equal generalized powers relative to some $r=(r_1,r_2,r_3)$ is said to be generalized-homogeneous, and the number $n=|q|=q_1+q_2+q_3$ is its generalized degree.

As we know, the maximal number of distinct homogeneous monomials in three variables with total degree n is $N_n=(n+1)(n+2)/2$. Noting this, and using the method of combinatorial analysis as in [14, 15, 41], we find that the number of generalized-homogeneous monomials with degree n relative to $r=(r_1,r_2,r_3)$ is given by

$$N_n(r_1,r_2,r_3) = \frac{1}{2}m(m+1)r_1r_2 + (m+1)f_s(r_1,r_2,r_3) \quad (n=mr_3+s), \tag{7.3}$$

where

$$f_s(r) = \begin{cases} \frac{1}{2}(s+1)(s+2), \; s = 0,1,\ldots,r_1-1, \\ \frac{1}{2}r_1(r_1+1) + r_1(s-r_1+1), \; s = r_1,r_1+1,\ldots,r_2-1, \\ \frac{1}{2}r_1(r_2-r_1+1) + \frac{1}{2}(2r_1+r_2-s+2)(s-r_1+1), \; s = r_2,\ldots,r_1+r_2-2, \\ \frac{1}{2}r_1r_2, \; s = r_1+r_2-1, r_1+r_2,\ldots,r_3-1. \end{cases}$$

If $r_1=r_2=r_3=1$, then $m=n$, $s=0$, $f_0(1,1,1)=1$, and $N_n(1,1,1)=(n+1)(n+2)/2$, so that in this case the generalized-homogeneous monomial of degree n reduces to an ordinary monomial of degree n.

It can be shown from (7.3) that for $n\to\infty$,

$$N_n(r_1,r_2,r_3) \cong r_1r_2(n+1)(n+2)/2r_3^2. \tag{7.4}$$

This asymptotic formula allows us to determine (approximate) the number of distinct generalized-homogeneous monomials for large n and arbitrary r_1,r_2,r_3.

As is known, a homogeneous function of degree λ, continuously differentiable in its domain of definition, satisfies

$$f(x_1,x_2,x_3) = \frac{1}{\lambda}\sum_{k=1}^{3} x_k \frac{\partial}{\partial x_k} f(x_1,x_2,x_3),$$

which is Euler's formula, and is widely used in various parts of mathematics. If $f(x_1,x_2,x_3)$ is a polynomial $P^n(x_1,x_2,x_3)$, the formula takes the form

$$P^n(x_1,x_2,x_3) = \frac{1}{n}\left(x_1\frac{\partial}{\partial x_1}P^n + x_2\frac{\partial}{\partial x_2}P^n + x_3\frac{\partial}{\partial x_3}P^n\right). \tag{7.5}$$

Let $P_r^n(x_1,x_2,x_3)$ be a generalized-homogeneous polynomial of degree n relative to $r=(r_1,r_2,r_3)$, with generalized degree $n=|q|=q_1+q_2+q_3$. It may be shown [10, 14] that any generalized-homogeneous polynomial may be written in the form

$$P_r^n(x_1,x_2,x_3) = \sum_{\alpha_1=0}^{r_1-1}\sum_{\alpha_2=0}^{r_2-1} P_{r;\alpha}^n(x_1,x_2,x_3),$$

where $\alpha=(\alpha_1,\alpha_2)$. As the result of a transformation [10, 14] we obtain the analog of Euler's formula for generalized-homogeneous polynomials:

$$P^n(x_1,x_2,x_3) = \sum_{k=1}^{3}\frac{r_3}{r_k}\sum_{\alpha_1=0}^{r_1-1}\sum_{\alpha_2=0}^{r_2-1}\frac{x_k}{n+\delta(r;\alpha)}\frac{\partial}{\partial x_k}P_{r;\alpha}^n(x_1,x_2,x_3), \tag{7.6}$$

where

$$\delta(r,\alpha) = \sum_{j=1}^{2}(r_3-r_j)\frac{\alpha_i}{r_j}.$$

If $r_1=r_2=r_3$, then $\alpha_1=\alpha_2=0$ and $\delta(r,\alpha)=0$, and (7.6) reduces to the ordinary Euler formula. The generalized formula (7.6) is used to derive recurrence and other relations for generalized-homogeneous polynomials.

7.2 FACTORIAL POLYNOMIALS. ALGORITHMS FOR CONSTRUCTING "EXPONENT MATRICES"

Consider the simple case of an arbitrary polynomial in one variable

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n, \tag{7.7}$$

where the a_i are real numbers. Various operations performed on such polynomials - differentiation, integration, arithmetic operations - may be considerably simplified by introducing the factorial powers. We transform the polynomial $P_n(x)$ to a factorial polynomial by using the factorial power of the variable x, i.e., we set

$$x^{m,!} = \frac{x^m}{m!} = x(m).$$

The result, which we indicate by $P_n(x) = P_{n,!}(x)$, is that

$$P_{n,!}(x) = b_0 x^{n,!} + b_1 x^{n-1,!} + \dots + b_{n-1} x^{1,!} + b_n, \tag{7.8}$$

where $b_k = (n-k)!\, a_k$. With any polynomial (7.8), which is said to be the basic factorial polynomial associated with the polynomial (7.7), we may relate two systems of so-called associated polynomials

$$P_{n-m,!}(x) \quad (m=1,2,\dots,n) \text{ and } P_{n+m,!}(x) \quad (m=1,2,\dots),$$

where these are defined by:

$$P_{n-m,!}(x) = b_0 x^{n-m,!} + b_1 x^{n-m-1,!} + \dots + b_{n-m+1} x^{1,!} + b_{n-m}, \tag{7.9}$$

$$P_{n+m,!}(x) = b_0 x^{n+m,!} + b_1 x^{n+m-1,!} + \dots + b_{n-1} x^{m+1,!} + b_n x^{m,!}$$
$$+ c_{n+1} x^{m-1,!} + c_{n+2} x^{m-2,!} + \dots + c_{n+m-1} x^{1,!} + c_{n+m}, \quad (7.10)$$

and $c_{n+1}, c_{n+2}, \dots, c_{n+m}$ are arbitrary constants.

We introduce the associated polynomial to simplify differentiation and integration; thus,

$$\frac{d^k}{dx^k} P_{n,!}(x) = P_{n-k,!}(x), \quad (7.11)$$

$$\int \cdots \int P_{n,!}(x)\, dx \dots dx = P_{n+k,!}(x). \quad (7.12)$$

After carrying out such operations on $P_{n,!}(x)$, the result may again be transformed to an ordinary polynomial, with the coefficients being related by $a_k = b_k/(n-k)!$. The operations of addition and subtraction of factorial polynomials are as for ordinary polynomials; for finding products and quotients we use the formulas

$$x^{n,!} x^{m,!} = \binom{n+m}{m} x^{n+m,!}, \quad (7.13)$$

$$\frac{x^{n,!}}{x^{m,!}} = \binom{n}{m}^{-1} x^{n-m,!}, \quad m<n. \quad (7.14)$$

It is not hard to extend the idea of the factorial polynomial to the case of more than one variable and obtain the relations corresponding to (7.7)-(7.14). The factorial powers, monomials, and polynomials, as established in [10, 14] and other works, are introduced not only to simplify forms; they also are important in obtaining solutions, invariant with respect to the order, of partial differential equations of high order.

We turn now to the construction of generalized-homogeneous monomials for a given n and $r=(r_1,r_2,r_3)$. First, we construct the "exponent matrix" for the degree n of the homogeneous monomial; then, from these elements we exclude any with at least one variable which has a non-unique inverse transformation. Finally, from the remaining elements we produce the ordinary powers by using the inverse transformation. An example should clarify these matters, and we choose the values $n=8$, $r_1=2$, $r_2=3$, $r_3=4$. The first step is to write the "exponent matrix", whose elements are the possible triples $\{q_1q_2q_3\}$ of generalized powers of the variables x_1,x_2,x_3:

$$A_8 = \begin{bmatrix} 008 \\ 107 & 017 \\ 206 & 116 & 026 \\ 305 & 215 & 125 & 035 \\ 404 & 314 & 224 & 134 & 044 \\ 503 & 413 & 323 & 233 & 143 & 053 \\ 602 & 512 & 422 & 332 & 242 & 152 & 062 \\ 701 & 611 & 521 & 431 & 341 & 251 & 161 & 071 \\ 800 & 710 & 620 & 530 & 440 & 350 & 260 & 170 & 080 \end{bmatrix}$$

From the elements of A_8 we exclude those for which there is at least one $q_k=m_kr_3+s_k$, with $s_k \geq r_k$. It is not hard to see that these will be the elements with $q_1=2,3,6,7$ or $q_2=3,7$. Denoting these by {xxx}, we obtain the matrix

$$B_8 = \left[\begin{array}{lllllllll} 008 \\ 107 & 017 \\ xxx & 116 & 026 \\ xxx & xxx & 125 & xxx \\ 404 & xxx & xxx & xxx & 044 \\ 503 & 413 & xxx & xxx & 143 & 053 \\ xxx & 512 & 422 & xxx & xxx & 152 & 062 \\ xxx & xxx & 521 & xxx & xxx & xxx & 161 & xxx \\ 800 & xxx & xxx & xxx & 440 & xxx & xxx & xxx & 080 \end{array}\right]$$

Then, using the inverse transformation we find the matrix with elements $\{p_1p_2p_3\}$

$$C_8 = \left[\begin{array}{lllllllll} 008 \\ 107 & 017 \\ xxx & 116 & 026 \\ xxx & xxx & 125 & xxx \\ 204 & xxx & xxx & xxx & 034 \\ 303 & 213 & xxx & xxx & 133 & 043 \\ xxx & 312 & 222 & xxx & xxx & 142 & 052 \\ xxx & xxx & 321 & xxx & xxx & xxx & 151 & xxx \\ 400 & xxx & xxx & xxx & 230 & xxx & xxx & xxx & 060 \end{array}\right]$$

Except for the excluded elements, each element of C_8 corresponds to a generalized-homogeneous monomial of (total) degree eight relative to r=(2,3,4); that is, the monomial $x_1^{p_1}x_2^{p_2}x_3^{p_3}$. The number of such terms is given by formula (7.3), which in this case produces the value 21. (In the special case when the polynomial is homogeneous, the matrices A_8, B_8, C_8 will all be the same.)

For practical applications, we may also need certain submatrices of C_n, also called "exponent matrices" and generated from "generating elements" (which are elements of C_n) as follows. For $\alpha_1=0,1,\ldots,r_1$, and $\alpha_2=0,1,\ldots,r_2$, the generating elements are the triples $\{\alpha_1,\alpha_2,n-\alpha_1-\alpha_2\}$ (these would occur in C_n). For each generating element $\{\alpha_1,\alpha_2,n-\alpha_1-\alpha_2\}$ construct the left triangular "exponent matrix" $C_n(\alpha_1,\alpha_2)$ as follows:

(a) $(\alpha_1,\alpha_2,n-\alpha_1-\alpha_2)$ is the upper left corner element;

(b) in succeeding rows and columns,

  (1) the first exponent of the triple increases by r_1 from row to row and decreases by r_1 from column to column,

  (2) the second exponent of the triple increases by r_2 from column to column,

  (3) the third exponent of the triple deceases by r_3 from row to row;

(c) the decreases in (b_1) and (b_3) determine the size of the matrix, since all exponents must be nonnegative.

Thus, for the current example $n=8$, $r=(2,3,4)$, the six generating elements are {008}, {017}, {026}, {107}, {116}, {125}, and the corresponding matrices $C_8(\alpha_1,\alpha_2)$ are

$$C_8(0,\ 0) = \begin{bmatrix} 008 & & \\ 204 & 034 & \\ 400 & 230 & 060 \end{bmatrix} \qquad C_8(1,\ 0) = \begin{bmatrix} 107 & \\ 303 & 133 \end{bmatrix}$$

$$C_8(0,\ 1) = \begin{bmatrix} 017 & \\ 213 & 043 \end{bmatrix} \qquad C_8(1,\ 1) = \begin{bmatrix} 116 & \\ 312 & 142 \end{bmatrix}$$

$$C_8(0,\ 2) = \begin{bmatrix} 026 & \\ 222 & 052 \end{bmatrix} \qquad C_8(1,\ 2) = \begin{bmatrix} 125 & \\ 321 & 151 \end{bmatrix} .$$

Along with the left triangular matrices A_n, B_n, C_n, and $C_n(\alpha_1, \alpha_2)$, we may also construct the right triangular matrices A_n', B_n', C_n', and $C_n'(\alpha_1, \alpha_2)$. Thus, C_8' has the form

$$C_8' = \begin{bmatrix} & & & & & & & & 008 \\ & & & & & & & 107 & 017 \\ & & & & & & xxx & 116 & 026 \\ & & & & & xxx & xxx & 125 & xxx \\ & & & & 204 & xxx & xxx & xxx & 034 \\ & & & 303 & 213 & xxx & xxx & 133 & 043 \\ & & xxx & 312 & 222 & xxx & xxx & 142 & 052 \\ & xxx & xxx & 321 & xxx & xxx & xxx & 151 & xxx \\ 400 & xxx & xxx & xxx & 230 & xxx & xxx & xxx & 060 \end{bmatrix}$$

the generating elements are as before, and (with the construction appropriately modified) the $C_8(\alpha_1, \alpha_2)$ are

$$C_8^{\prime}(0,\ 0) = \left[\begin{array}{ccc} & & 008 \\ & 204 & 034 \\ 400 & 230 & 060 \end{array}\right] \qquad C_8^{\prime}(1,\ 0) = \left[\begin{array}{cc} & 107 \\ 303 & 133 \end{array}\right]$$

$$C_8^{\prime}(0,\ 1) = \left[\begin{array}{cc} & 017 \\ 213 & 043 \end{array}\right] \qquad C_8^{\prime}(1,\ 1) = \left[\begin{array}{cc} & 116 \\ 312 & 142 \end{array}\right]$$

$$C_8^{\prime}(0,\ 2) = \left[\begin{array}{cc} & 026 \\ 222 & 052 \end{array}\right] \qquad C_8^{\prime}(1,\ 2) = \left[\begin{array}{cc} & 125 \\ 321 & 151 \end{array}\right] .$$

The elements of these matrices may represent not only the ordinary but also the factorial powers of the variables, as when, e.g., {abc} corresponds to $x^{a,!}y^{b,!}z^{c,!}$. The introduction of factorial powers makes it possible to develop algorithms for the construction of basis systems of polynomial solutions, invariant relative to r_1,r_2,r_3, of partial differential equations for any n. Both the left and right "exponent matrices" will be used in Section 7.3 for the construction of basis systems of polynomial solutions by means of algorithms in which the Pascal triangle is used.

7.3 COMBINATORIAL ALGORITHMS FOR THE CONSTRUCTION OF "COEFFICIENT MATRICES" AND BASIS SYSTEMS OF POLYNOMIAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS.

Algorithms for constructing basis systems of polynomial solutions are often applied to various classes of partial differential equations, among which are the polyharmonic, polywave, and polycaloric equations of Euler-Poisson-Darboux and Beltrami. Here we present combinatorial algorithms for constructing basis systems of polynomial solutions for more complicated equations, in particular:

$$\Delta_r u(x) \equiv \left(D_1^{r_1} + D_2^{r_2} + D_3^{r_3}\right) u(x) = 0, \tag{7.15}$$

the generalized Laplace equation, and

$$\nabla_r v(x) \equiv \left(D_1^{r_1} D_2^{r_2} + D_1^{r_1} D_3^{r_3} + D_2^{r_2} D_3^{r_3}\right) v(x) = 0, \tag{7.16}$$

the generalized equation of Mangeron type [82-87]. In (7.15), (7.16), $x=(x_1,x_2,x_3)$, $D_k=\frac{\partial}{\partial x_k}$, $r=(r_1,r_2,r_3)$, where $r_1 \le r_2 \le r_3$ are natural numbers.

The author in [14] constructed and studied basis systems of polynomial solutions of (7.15), the general representation of these solutions being of the form

$$H^{n,m,0}_{r;\alpha_1,\alpha_2}(x) = n! \sum_{i=0}^{l_\alpha - m} \sum_{k=0}^{m} (-1)^{i+k} \binom{i+k}{k} \times \frac{x_1^{r_1 i + r_1 k + \alpha_1}}{(r_1 i + r_1 k + \alpha_1)!} \frac{x_2^{r_2 m - r_2 k + \alpha_2}}{(r_2 m - r_2 k + \alpha_2)!} \frac{x_3^{n - r_3 m - r_3 i - \alpha_1 - \alpha_2}}{(n - r_3 m - r_3 i - \alpha_1 - \alpha_2)!}, \tag{7.17}$$

where n, m, α_1, α_2 are nonnegative numbers, and $l_\alpha = \left[\frac{n-\alpha_1-\alpha_2}{r_3}\right]$. In [14] the following theorems are proved.

Theorem 7.1. For $\alpha_1=0,1,\ldots,r_1-1$, $\alpha_2=0,1,\ldots,r_2-1$, and $m=0,1,\ldots,l_\alpha$, the generalized harmonic polynomials (7.17) form a basis system of

$$N_1(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (l_\alpha + 1) \tag{7.18}$$

linearly independent polynomials satisfying (7.15).

A basis system of polynomial solutions of (7.16) may be constructed from two general representations [14] of the form

$$P^{n,m,0}_{r;\alpha_1,\alpha_2}(x) = n! \sum_{i=0}^{l_\alpha - m} (-1)^i \sum_{k=0}^{i} \binom{i}{k} \times x_1^{r_1 i - r_1 k + r_1 m + \alpha_1, !} x_2^{r_2 k + \alpha_2, !} x_3^{n - r_3 m - r_3 i - \alpha_1 - \alpha_2, !}, \tag{7.19}$$

$$Q^{n,m,0}_{r;\alpha_1,\alpha_2}(x) = n! \sum_{i=0}^{l_\alpha - m} (-1)^i \sum_{k=0}^{i} \binom{i}{k} \times x_1^{r_1 k + \alpha_1, !} x_2^{r_2 i - r_2 k + r_2 m + \alpha_2, !} x_3^{n - r_3 m - r_3 i - \alpha_1 - \alpha_2, !}. \tag{7.20}$$

It may be shown that for m=0, $P^{0,0,0}_{r;\alpha_1,\alpha_2} \equiv Q^{0,0,0}_{r;\alpha_1,\alpha_2}$, and for $m=1,2,\ldots,l_\alpha$, (7.19) and (7.20) are linearly independent.

Theorem 7.2. With α_1, α_2,m as in Theorem 7.1, the polynomials (7.19) and (7.20) form a basis system of

$$N_{1,2}(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (2l_\alpha+1) \tag{7.21}$$

linearly independent solutions of (7.16).

We note also that in [14] the iterated versions of (7.15), (7.16) are discussed, i.e., $\Delta_r^N(x)=0$, $\nabla_r^N(x)=0$. By introducing the normalizing factor $\binom{i}{p}$ and taking account of p in the exponent, the formulas (7.17), (7.19), (7.20) can be generalized to give polynomial representations of the solutions of $\Delta_r^N(x)=0$, $\nabla_r^N(x)=0$, of the form $H^{n,m,p}_{r;\alpha_1,\alpha_2}(x)$; $P^{n,m,p}_{r;\alpha_1,\alpha_2}(x)$ and $Q^{n,m,p}_{r;\alpha_1,\alpha_2}(x)$, in which $p=0,1,\ldots,N-1$.

The formulas (7.17), (7.19), (7.20) could be used directly to construct polynomial solutions, but in the forms given they entail a good deal of calculation. In this connection, we suggest below a combinatorial algorithm for the construction of basis systems of polynomial solutions to (7.15) and (7.16), based on the introduction of the triangular matrices of factorial exponents and "coefficient matrices" in the form of the Pascal triangle, modified so that coefficients in odd-numbered rows are negative.

First, we construct the left and right triangular matrices of factorial exponents for any $n, r=(r_1,r_2,r_3)$, and $\alpha_1=0,1,\ldots,r_1-1$, $\alpha_2=0,1,\ldots,r_2-1$. Denote the left triangular matrices by $C_n(\alpha_1,\alpha_2)$ and the right by $C_n'(\alpha_1,\alpha_2)$, and their elements by the symbol $(i,k)_\alpha$, where $0 \le i,k \le l_\alpha$. These matrices then appear as

$$C_n(\alpha_1,\alpha_2) = \left[\begin{array}{lllll}
(0,\,0)_\alpha & & & & \\
(1,\,0)_\alpha & (0,\,1)_\alpha & & & \\
(2,\,0)_\alpha & (1,\,1)_\alpha & (0,\,2)_\alpha & & \\
\vdots & \vdots & \vdots & \ddots & \\
(l_\alpha,\,0)_\alpha & (l_\alpha-1,\,1)_\alpha & (l_\alpha-2,\,2)_\alpha & \cdots & (0,\,l_a)_\alpha
\end{array}\right],$$

$$C_n'(\alpha_1,\alpha_2) = \left[\begin{array}{rrrrr}
 & & & & (0,\,0)_\alpha \\
 & & & (1,\,0)_\alpha & (0,\,1)_\alpha \\
 & & (2,\,0)_\alpha & (1,\,1)_\alpha & (0,\,2)_\alpha \\
 & \iddots & \vdots & \vdots & \vdots \\
(l_\alpha,\,0)_\alpha & \cdots & (2,\,l_\alpha-2)_\alpha & (1,\,l_\alpha-1)_\alpha & (0,\,l_\alpha)_\alpha
\end{array}\right].$$

In these matrices, each of the elements $(i,k)_\alpha$ corresponds to a triple of factorial powers by virtue of the relation

$$(i,\ k)_\alpha \rightarrow \{r_1 i+\alpha_1,\ r_2 k+\alpha_2,\ n-r_3 i-r_3 k-\alpha_1-\alpha_2\}, \tag{7.22}$$

and the matrices consist of $(l_\alpha+1)$ rows, where $l_\alpha = \left[\frac{n-\alpha_1-\alpha_2}{r_3}\right]$.

Thus, for example, let $n=15$, $r=(2,3,4)$, and take $\alpha_1=1, \alpha_2=0$. It is simple to construct $C_{15}(1,0)$ and $C'_{15}(1,0)$, and the resulting matrices are

$$C_{15}(1,0) = \left[\begin{array}{cccccccccccc} 1 & 0 & 14 & & & & & & & & & \\ 3 & 0 & 10 & 1 & 3 & 10 & & & & & & \\ 5 & 0 & 6 & 3 & 3 & 6 & 1 & 6 & 6 & & & \\ 7 & 0 & 2 & 5 & 3 & 2 & 3 & 6 & 2 & 1 & 9 & 2 \end{array}\right],$$

$$C'_{15}(1,0) = \left[\begin{array}{cccccccccccc} & & & & & & & & & 1 & 0 & 14 \\ & & & & & & 3 & 0 & 10 & 1 & 3 & 10 \\ & & & 5 & 0 & 6 & 3 & 3 & 6 & 1 & 6 & 6 \\ 7 & 0 & 2 & 5 & 3 & 2 & 3 & 6 & 2 & 1 & 9 & 2 \end{array}\right].$$

The other matrices are likewise easily constructed, and there will be a total of $r_1 r_2 = 2\cdot 3 = 6$ distinct matrices of either kind.

We describe now a scheme for constructing a basis system of polynomial solutions for arbitrary n, first for (7.15) and then for (7.16). For (7.15), only the left triangular matrices $C_n(\alpha_1,\alpha_2)$ are needed, along with the Pascal triangle whose odd rows have negative elements.

To shorten the description, we discuss the case in which $C_n(\alpha_1,\alpha_2)$ has five rows. Since the algorithm is invariant with respect to the values of the factorial exponents, all elements of this triangle will be denoted simply by {xxx}. We write this symbolic array and the Pascal triangle in the form

$$\begin{array}{cccccccccc}
1 & \underline{1} & 1 & \underline{1} & 1 & xxx & & & & \\
 & 4 & \underline{3} & 2 & \underline{1} & xxx & xxx & & & \\
 & & 6 & \underline{3} & 1 & xxx & xxx & xxx & & \\
 & & & 4 & \underline{1} & xxx & xxx & xxx & xxx & \\
 & & & & 1 & xxx & xxx & xxx & xxx & xxx ,
\end{array}$$

where the negative elements are underlined. Whatever the values of the elements in the matrix of factorial exponents, the coefficients of the monomials are determined by successive superpositions of the Pascal triangle. We show the scheme for the (in this case) five positions of the triangles relative to one another.

$$\begin{array}{ccccccccc}
1 & \underline{1} & 1 & \underline{1} & 1\cdot xxx & & & & \\
 & 4 & \underline{3} & 2 & \underline{1}\cdot xxx & xxx & & & \\
 & & 6 & \underline{3} & 1\cdot xxx & xxx & xxx & & \\
 & & & 4 & \underline{1}\cdot xxx & xxx & xxx & xxx & \\
 & & & & 1\cdot xxx & xxx & xxx & xxx & xxx
\end{array}$$

$$\begin{array}{cccccccc}
 & & & xxx & & & & \\
1 & \underline{1} & 1 & \underline{1}\cdot xxx & 1\cdot xxx & & & \\
 & 4 & \underline{3} & 2\cdot xxx & \underline{1}\cdot xxx & xxx & & \\
 & & 6 & \underline{3}\cdot xxx & 1\cdot xxx & xxx & xxx & \\
 & & & 4\cdot xxx & \underline{1}\cdot xxx & xxx & xxx & xxx \\
 & & & & 1 & & &
\end{array}$$

		xxx				
		xxx	*xxx*			
1	$\underline{1}$	1·*xxx*	$\underline{1}$·*xxx*	1·*xxx*		
	4	$\underline{3}$·*xxx*	2·*xxx*	$\underline{1}$·*xxx*	*xxx*	
		6·*xxx*	$\underline{3}$·*xxx*	1·*xxx*	*xxx*	*xxx*
			4	$\underline{1}$		
				1		

	xxx				
	xxx	*xxx*			
	xxx	*xxx*	*xxx*		
1	$\underline{1}$·*xxx*	1·*xxx*	$\underline{1}$·*xxx*	1·*xxx*	
	4·*xxx*	$\underline{3}$·*xxx*	2·*xxx*	$\underline{1}$·*xxx*	*xxx*
		6	$\underline{3}$	1	
			4	$\underline{1}$	
				1	

xxx				
xxx	*xxx*			
xxx	*xxx*	*xxx*		
xxx	*xxx*	*xxx*	*xxx*	
1·*xxx*	$\underline{1}$·*xxx*	1·*xxx*	$\underline{1}$·*xxx*	1·*xxx*
	4	$\underline{3}$	2	$\underline{1}$
		6	$\underline{3}$	1
			4	$\underline{1}$
				1

If $C_n(\alpha_1,\alpha_2)$ contains a greater, or lesser, number of rows, the corresponding Pascal triangle must consist of the same number of rows. And, carrying out this superposition scheme with all of the $r_1 \cdot r_2$ matrices $C_n(\alpha_1,\alpha_2)$, we will form a total of

$$N_1(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (l_\alpha+1)$$

linearly independent polynomial solutions of (7.15).

To illustrate the procedure described, we form the basis system of generalized-homogeneous polynomial solutions of (7.15) of degree n=8, with $r_1=2, r_2=3, r_3=4$. The six matrices $C_8(\alpha_1,\alpha_2)$ were given in the example of Section 7.2, and we will need the Pascal triangle in the form

$$\begin{bmatrix} 1 & 1 & 1 \\ & 2 & 1 \\ & & 1 \end{bmatrix}$$

for those matrices having three rows. As a result of applying the algorithm and converting to factorial form, we will have (by Theorem 7.1) 13 linearly independent polynomial solutions of (7.15):

$$H^{8,0,0}_{4;0,0}(x) = 8!\left(x_3^{8,!} - x_1^{2,!}x_3^{4,!} + x_1^{4,!}\right),$$

$$H^{8,1,0}_{4;0,0}(x) = 8!\left(-x_1^{2,!}x_3^{4,!} + 2x_1^{4,!} + x_2^{3,!}x_3^{4,!} - x_1^{2,!}x_2^{3,!}\right),$$

$$H^{8,2,0}_{4;0,0}(x) = 8!\left(x_1^{4,!} - x_1^{2,!}x_2^{3,!} + x_2^{6,!}\right),$$

$$H^{8,0,0}_{4;1,0}(x) = 8!\left(x_1^{1,!}x_2^{7,!} - x_1^{3,!}x_3^{3,!}\right),$$

$$H^{8,1,0}_{4;1,0}(x) = 8!\left(-x_1^{3,!}x_3^{3,!} + x_1^{1,!}x_2^{3,!}x_3^{3,!}\right),$$

$$H^{8,0,0}_{4;0,1}(x) = 8!\left(x_2^{1,!}x_3^{7,!} - x_1^{2,!}x_3^{1,!}x_4^{3,!}\right),$$

$$H^{8,1,0}_{4;0,1}(x) = 8!\left(-x_1^{2,!}x_2^{1,!}x_3^{3,!} + x_2^{4,!}x_3^{3,!}\right),$$

$$H^{8,0,0}_{4;1,1}(x) = 8!\left(x_1^{1,!}x_2^{1,!}x_3^{6,!} - x_1^{3,!}x_2^{1,!}x_3^{2,!}\right),$$

$$H^{8,1,0}_{4;1,1}(x) = 8!\left(-x_1^{3,!}x_2^{1,!}x_3^{2,!} + x_1^{1,!}x_2^{4,!}x_3^{2,!}\right),$$

$$H^{8,0,0}_{4;0,2}(x) = 8!\left(x_2^{2,!}x_3^{6,!} - x_1^{2,!}x_2^{2,!}x_3^{2,!}\right),$$

$$H^{8,1,0}_{4;0,2}(x) = 8!\left(-x_1^{2,!}x_2^{2,!}x_3^{2,!} + x_2^{5,!}x_3^{2,!}\right),$$

$$H^{8,0,0}_{4;1,2}(x) = 8!\left(x_1^{1,!}x_2^{2,!}x_3^{5,!} - x_1^{3,!}x_2^{2,!}x_3^{1,!}\right),$$

$$H^{8,1,0}_{4;1,2}(x) = 8!\left(-x_1^{3,!}x_2^{2,!}x_3^{1,!} + x_1^{1,!}x_2^{5,!}x_3^{1,!}\right).$$

If necessary, any or all of the 13 polynomials may be written in ordinary form; thus, e.g.,

$$H^{8,0,0}_{4;0,0}(x) = x_3^8 - 840x_1^2x_3^4 + 1680x_1^4.$$

We turn now to the algorithm for constructing a basis system of polynomial solutions of degree n for (7.16). In this case we will need both the left and right triangular matrices $C_n(\alpha_1, \alpha_2)$ and $C_n'(\alpha_1, \alpha_2)$, and also modified (odd rows have negative coefficients) left and right triangular forms of the Pascal triangle.

For brevity, we again discuss the case for matrices with five rows, first for the part of the algorithm which uses the left triangular matrices $C_n(\alpha_1, \alpha_2)$. Denoting, as before, the elements of the matrix by {xxx}, we write the symbolic and Pascal triangles in the array

1					xxx				
1	1				xxx	xxx			
1	2	1			xxx	xxx	xxx		
1	3	3	1		xxx	xxx	xxx	xxx	
1	4	6	4	1	xxx	xxx	xxx	xxx	xxx .

The algorithm for this case is similar to, but differs in detail from, the algorithm for the previous case. We show the scheme for the five superpositions of the triangles relative to one another, and which give the constructions for the polynomials $P^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$:

1·xxx				
1·xxx	1·xxx			
1·xxx	2·xxx	1·xxx		
1·xxx	3·xxx	3·xxx	1·xxx	
1·xxx	4·xxx	6·xxx	4·xxx	1·xxx

xxx				
1·xxx	xxx			
1·xxx	1·xxx	xxx		
1·xxx	2·xxx	1·xxx	xxx	
1·xxx	3·xxx	3·xxx	1·xxx	xxx
1	4	6	4	1

xxx				
xxx	xxx			
1·xxx	xxx	xxx		
1·xxx	1·xxx	xxx	xxx	
1·xxx	2·xxx	1·xxx	xxx	xxx
1	3	3	1	
1	4	6	4	1

xxx				
xxx	xxx			
xxx	xxx	xxx		
1·xxx	xxx	xxx	xxx	
1·xxx	1·xxx	xxx	xxx	xxx
1	2	1		
1	3	3	1	
1	4	6	4	1

xxx				
xxx	xxx			
xxx	xxx	xxx		
xxx	xxx	xxx	xxx	
1·xxx	xxx	xxx	xxx	xxx
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

The first part of the algorithm gives us the polynomials $P^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$, for $m=0,1,2,\ldots,l_\alpha$. For the construction of the polynomials $Q^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$, $m=1,2,\ldots,l_\alpha$, we write the symbolic triangle and the Pascal triangle in the form

				1					xxx
			1	1				xxx	xxx
		1	2	1			xxx	xxx	xxx
	1	3	3	1		xxx	xxx	xxx	xxx
1	4	6	4	1	xxx	xxx	xxx	xxx	xxx .

We show the scheme for the four (m begins at 1) superpositions of the triangles relative to one another, and which give the constructions for the polynomials $Q^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$:

				xxx
			xxx	1·xxx
		xxx	1·xxx	1·xxx
	xxx	1·xxx	2·xxx	1·xxx
xxx	1·xxx	3·xxx	3·xxx	1·xxx
1	4	6	4	1

				xxx
			xxx	xxx
		xxx	xxx	1·xxx
	xxx	xxx	1·xxx	1·xxx
xxx	xxx	1·xxx	2·xxx	1·xxx
	1	3	3	1
1	4	6	4	1

				xxx
			xxx	xxx
		xxx	xxx	xxx
	xxx	xxx	xxx	1·xxx
xxx	xxx	xxx	1·xxx	1·xxx
		1	2	1
	1	3	3	1
1	4	6	4	1

				xxx
			xxx	xxx
		xxx	xxx	xxx
	xxx	xxx	xxx	xxx
xxx	xxx	xxx	xxx	1·xxx
			1	1
		1	2	1
	1	3	3	1
1	4	6	4	1

In practical applications of this Pascal triangle algorithm one must write out the starting form clearly, and carry out carefully these patterns using the elements of $C_n(\alpha_1,\alpha_2)$ and $C_n'(\alpha_1,\alpha_2)$.

Writing out all the matrices $C_n(\alpha_1,\alpha_2)$ and $C_n'(\alpha_1,\alpha_2)$, a total of $2\tau_1\tau_2$, and applying the algorithm, we will, by construction, have a total of

$$N_{1,2}(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (2l_\alpha + 1)$$

linearly independent polynomial solutions of (7.16).

As an example, we form the basis system of polynomial solutions of (7.16) of degree n=8, with $r_1=2, r_2=3, r_4=4$. The Pascal triangles needed are

$$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} & 1 \\ 1 & 1 \end{bmatrix}.$$

The first of these (left triangular) is used in connection with the matrices $C_8(\alpha_1,\alpha_2)$, and the second with the matrices $C_8'(\alpha_1,\alpha_2)$, constructed in Section 7.2:

$$C_8(0,0),\ C_8(1,0),\ C_8(0,1),\ C_8(1,1),\ C_8(0,2),\ C_8(1,2),$$

and

$$C_8'(0,0),\ C_8'(1,0),\ C_8'(0,1),\ C_8'(0,2),\ C_8'(1,2).$$

As a result of the first part of the algorithm, we will have 13 linearly independent polynomials $P^{8,m,0}_{4;\alpha_1,\alpha_2}(x)$:

$$
\begin{aligned}
P_{4;0,0}^{8,0,0}(x) &= 8!\left(x_3^{8,!} - x_1^{2,!}x_3^{4,!} - x_2^{3!}x_3^{4,!} + x_1^{4,!} + 2x_1^{2,!}x_2^{3,!} + x_2^{6,!}\right),\\
P_{4;0,0}^{8,1,0}(x) &= 8!\left(x_1^{2,!}x_3^{4,!} - x_1^{4,!} - x_1^{2,!}x_2^{3,!}\right),\\
P_{4;0,0}^{8,2,0}(x) &= 8!\,x_1^{4,!},\\
P_{4;1,0}^{8,0,0}(x) &= 8!\left(x_1^{1,!}x_3^{7,!} - x_1^{3,!}x_3^{3,!} - x_1^{1,!}x_2^{3,!}x_3^{3,!}\right),\\
P_{4;1,0}^{8,1,0}(x) &= 8!\,x_1^{3,!}x_3^{3,!},\\
P_{4;0,1}^{8,0,0}(x) &= 8!\left(x_2^{1,!}x_3^{7,!} - x_1^{2,!}x_2^{1,!}x_3^{3,!} - x_2^{4,!}x_3^{3,!}\right),\\
P_{4;0,1}^{8,1,0}(x) &= 8!\,x_1^{2,!}x_2^{1,!}x_3^{3,!},\\
P_{4;1,1}^{8,0,0}(x) &= 8!\left(x_1^{1,!}x_2^{1,!}x_3^{6,!} - x_1^{3,!}x_2^{1,!}x_3^{2,!} - x_1^{1,!}x_2^{4,!}x_3^{2,!}\right),\\
P_{4;1,1}^{8,1,0}(x) &= 8!\,x_1^{3,!}x_2^{1,!}x_3^{2,!},\\
P_{4;0,2}^{8,0,0}(x) &= 8!\left(x_2^{2,!}x_3^{6,!} - x_1^{2,!}x_2^{2,!}x_3^{2,!} - x_2^{5,!}x_3^{2,!}\right),\\
P_{4;0,2}^{8,1,0}(x) &= 8!\,x_1^{2,!}x_2^{2,!}x_3^{2,!},\\
P_{4;1,2}^{8,0,0}(x) &= 8!\left(x_1^{1,!}x_2^{2,!}x_3^{5,!} - x_1^{3,!}x_2^{2,!}x_3^{1,!} - x_1^{1,!}x_2^{5,!}x_3^{1,!}\right),\\
P_{4;1,2}^{8,1,0}(x) &= 8!\,x_1^{3,!}x_2^{2,!}x_3^{1,!}.
\end{aligned}
$$

Applying the second part of the algorithm results in seven linearly independent polynomials $Q_{4;\alpha_1,\alpha_2}^{8,m,0}(x)$:

$$
\begin{aligned}
Q^{8,1,0}_{4;0,0}(x) &= 8!\left(x_2^{3,!}x_3^{4,!}-x_1^{2,!}x_2^{3,!}-x_2^{6,!}\right),\\
Q^{8,2,0}_{4;0,0}(x) &= 8!\,x_2^{6,!},\\
Q^{8,1,0}_{4;1,0}(x) &= 8!\,x_1^{1,!}x_2^{3,!}x_3^{3,!},\\
Q^{8,1,0}_{4;1,1}(x) &= 8!\,x_1^{1,!}x_2^{4,!}x_3^{2,!},\\
Q^{8,1,0}_{4;0,1}(x) &= 8!\,x_2^{4,!}x_3^{3,!},\\
Q^{8,1,0}_{4;0,2}(x) &= 8!\,x_2^{5,!}x_3^{2,!},\\
Q^{8,1,0}_{4;1,2}(x) &= 8!\,x_1^{1,!}x_2^{5,!}x_3^{1,!}.
\end{aligned}
$$

In all, we obtain 13 polynomials $P^{8,m,0}_{4;\alpha_1,\alpha_2}(x)$ and 7 polynomials $Q^{8,m,0}_{4;\alpha_1,\alpha_2}(x)$, for a total of 20 linearly independent generalized-homogeneous polynomial solutions of degree eight for (7.16); this agrees with the number given by

$$N_{1,2}(r,8) = \sum_{\alpha_1=0}^{1}\sum_{\alpha_2=0}^{2}(2l_\alpha + 1),$$

where

$$l_\alpha = \left[\frac{8-\alpha_1-\alpha_2}{4}\right],$$

and a simple calculation gives $N_{1,2}(r,8)=20$. As before, we note that any of these polynomials may also be represented as a polynomial with ordinary powers of the variables.

7.4 POLYNOMIALS OF BINOMIAL TYPE AND RELATED POLYNOMIALS

A sequence of polynomials $\{f_n\}$ is said to be a sequence of binomial type if

$$f_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} f_i(x) f_{n-i}(y), \tag{7.23}$$

where $n=0,1,\ldots$, and $f_0=1$. Classical polynomial sequences of binomial type are, e.g., x^n, $(x)_n=x(x-1)\cdots(x-n+1)$, and $x(x-na)^{n-1}$. The theory of polynomial sequences of binomial type was developed by G.-C. Rota and R. Mullin [331], who gave a complete characterization of these sequences, worked out algorithms for determining their corresponding constants, and discussed some enumeration problems. R.B. Brown [95] gave an example of considerable combinatorial interest in this connection; he also generalized some of the results in [331] and considered the ring structure of sequences of binomial type. L. Brand [91] discussed functions of binomial type for negative factorials, i.e., sequences of the form $(x)_{-n}=[(x+1)(x+2)\cdots(x+n)]^{-1}$. Using polynomial sequences of binomial type, the author [7-10] constructed and studied binomial and trinomial polynomials.

Let r be a natural number, and let $s=0,1,\ldots,r-1$. Then the polynomial of binomial type

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^n y^{n-i}$$

may be written in the form

$$(x+y)^n = \sum_{s=0}^{r-1} \sum_{k=0}^{\left[\frac{n-s}{r}\right]} \binom{n}{rk+s} x^{rk+s} y^{n-rk-s}. \tag{7.24}$$

The right side of (7.24) may be considered as a sum of r polynomials, corresponding to s=0,1,...,r−1.

Using factorial powers, and taking the n! outside the summation, we write out two classes of polynomials:

$$G_{r,s}^{n,0}(x,y) = n! \sum_{k=0}^{\left[\frac{n-s}{r}\right]} x^{rk+s,1} y^{n-rk-s,1},$$

$$H_{r,s}^{n,0}(x,y) = n! \sum_{k=0}^{\left[\frac{n-s}{r}\right]} (-1)^k x^{rk+s,1} y^{n-rk-s,1}.$$

Then, putting in the summation the normalizing factor $\binom{k}{p}$, where $p=0,1,\ldots,\left[\frac{n-s}{r}\right]$, we obtain

$$G_{r,s}^{n,p}(x,y) = n! \sum_{k=p}^{\left[\frac{n-s}{r}\right]} \binom{k}{p} x^{rk+s,1} y^{n-rk-s,1}, \tag{7.25}$$

$$H_{r,s}^{n,p}(x,y) = n! \sum_{k=p}^{\left[\frac{n-s}{r}\right]} (-1)^k \binom{k}{p} x^{rk+s,1} y^{n-rk-s,1}. \tag{7.26}$$

The polynomials (7.26), (7.27) are called p-binomials. Omitting the details of the derivations of the numerous properties of these polynomials, we give their formulas for differentiation, indefinite integration, and some recurrence relations.

Differentiation formulas for the p-binomials with respect to the variables x and y are of the form

$$\left.\begin{aligned}
\frac{\partial}{\partial x} G_{r,s}^{n,p}(x,y) &= nG_{r,s-1}^{n-1,p}(x,y), \quad s\neq 0,\\
\frac{\partial}{\partial x} G_{r,0}^{n,p}(x,y) &= nG_{r,r-1}^{n-1,p}(x,y) + nG_{r,r-1}^{n-1,p-1}(x,y),\\
\frac{\partial}{\partial y} G_{r,s}^{n,p}(x,y) &= nG_{r,s}^{n-1,p}(x,y), \quad s=0,1,\dots,r-1,
\end{aligned}\right\} \tag{7.27}$$

$$\left.\begin{aligned}
\frac{\partial}{\partial x} H_{r,s}^{n,p}(x,y) &= nH_{r,s-1}^{n-1,p}(x,y), \quad s\neq 0,\\
\frac{\partial}{\partial x} H_{r,0}^{n,p}(x,y) &= -nH_{r,r-1}^{n-1,p}(x,y) - nH_{r,r-1}^{n-1,p-1}(x,y),\\
\frac{\partial}{\partial y} H_{r,s}^{n,p}(x,y) &= nH_{r,s}^{n-1,p}(x,y), \quad s=0,1,\dots,r-1.
\end{aligned}\right\} \tag{7.28}$$

For p=0 in (7.27), (7.28) we get

$$G_{r,r-1}^{n-1,-1}(x,y) = H_{r,r-1}^{n-1,-1}(x,y) = 0.$$

Replacing in (7.27), (7.28) the power n by (n+1), and the index s by (s+1), and integrating both sides with respect to the differentiation variables, we obtain (to within an arbitrary function independent of the variable of integration) the indefinite integrals of the p-binomials:

$$\left.\begin{aligned}
\int G_{r,s}^{n,p}(x,y)\,dx &= \frac{1}{n+1} G_{r,s+1}^{n+1,p}(x,y),\\
\int G_{r,s}^{n,p}(x,y)\,dy &= \frac{1}{n+1} G_{r,s}^{n+1,p}(x,y),
\end{aligned}\right\} \tag{7.29}$$

$$\left.\begin{aligned}\int H_{r,s}^{n,p}(x,y)\,dx &= \frac{1}{n+1}H_{r,s+1}^{n+1,p}(x,y),\\ \int H_{r,s}^{n,p}(x,y)\,dy &= \frac{1}{n+1}H_{r,s}^{n+1,p}(x,y).\end{aligned}\right\} \qquad (7.30)$$

From the differentiation and indefinite integration formulas, it is not difficult to obtain by integration by parts formulas for the indefinite integral of a product of two p-binomials; the values of definite integrals may also be calculated.

Using Euler's formula for a homogeneous function of degree n,

$$F(x,y) = \frac{x}{n}\frac{\partial}{\partial x}F^n(x,y) + \frac{y}{n}\frac{\partial}{\partial y}F^n(x,y),$$

and (7.27), (7.28), we can derive the recurrence relations

$$\left.\begin{aligned}G_{r,s}^{n,p}(x,y) &= xG_{r,s-1}^{n-1,p}(x,y) + yG_{r,s}^{n-1,p}(x,y), \quad s\neq 0,\\ G_{r,0}^{n,p}(x,y) &= xG_{r,r-1}^{n-1,p}(x,y) + xG_{r,r-1}^{n-1,p-1}(x,y) + yG_{r,0}^{n-1}(x,y),\end{aligned}\right\} \qquad (7.31)$$

$$\left.\begin{aligned}H_{r,s}^{n,p}(x,y) &= xH_{r,s-1}^{n-1,p}(x,y) + yH_{r,s}^{n-1,p}(x,y), \quad s\neq 0,\\ H_{r,0}^{n,p}(x,y) &= -xH_{r,r-1}^{n-1,p}(x,y) - xH_{r,r-1}^{n-1,p-1}(x,y) + yH_{r,0}^{n-1,p}(x,y).\end{aligned}\right\} \qquad (7.32)$$

Again from (7.27), (7.28) we may obtain differential equations whose solutions are p-binomials. Using the first two formulas in (7.27) and calculating the r^{th}-order derivative of the function $G_{r,s}^{n,p}(x,y)$ we find

$$\begin{aligned}\frac{\partial^r}{\partial x^r} G_{r,s}^{n,p}(x,y) &= \frac{\partial^{r-s}}{\partial x^{r-s}}\left(\frac{\partial^s}{\partial x^s} G_{r,s}^{n,p}(x,y)\right) \\ &= (n-s+1,1)_s \, \frac{\partial^{r-s-1}}{\partial x^{r-s-1}}\left(\frac{\partial}{\partial x} G_{r,0}^{n-s,p}(x,y)\right) \\ &= (n-s,1)_{s+1} \, \frac{\partial^{r-s-1}}{\partial x^{r-s-1}}\left(G_{r,r-1}^{n-s-1,p}(x,y) + G_{r,r-1}^{n-s-1,p-1}(x,y)\right) \\ &= (n-r+1,1)_r \left(G_{r,s}^{n-r,p}(x,y) + G_{r,s}^{n-r,p-1}(x,y)\right),\end{aligned}$$

from which we have

$$\frac{\partial^r}{\partial x^r} G_{r,s}^{n,p}(x,y) = (n-r+1,1)_r G_{r,s}^{n-r,}(x,y) + (n-r+1,1)_r G_{r,s}^{n-r,p-1}(x,y).$$

And, using the third formula of (7.27), we find

$$\frac{\partial^r}{\partial y^r} G_{r,s}^{n,p}(x,y) = (n-r+1,1)_r G_{r,s}^{n-r,p}(x,y),$$

and subtracting this equation from the previous one we obtain

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right) G_{r,s}^{n,p}(x,y) = (n-r+1,1)_r G_{r,s}^{n-r,p-1}(x,y).$$

Here we assume that $n-r \geq 0$, $p \geq 1$. If $n < r$, then the right hand side is certainly zero. Let $n > pr$; then from this formula we will have

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right)^p G_{r,s}^{n,p}(x,y) = (n-pr+1,1)_{pr} G_{r,s}^{n-pr,0}(x,y), \quad p\neq 0,$$

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right) G_{r,s}^{n,0}(x,y) = 0.$$

If $n<pr$, the right side of the first equation is certainly zero. Thus, the p-binomial $G_{r,s}^{n,p}(x,y)$ is a polynomial solution of

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right)^{p+1} u(x,y) = 0,$$

for $n\geq pr$, $s=0,1,\ldots,r-1$.

A similar argument shows that the p-binomial $H_{r,s}^{n,p}(x,y)$, for $n\geq pr$, $s=0,1,\ldots,r-1$, gives a polynomial solution of

$$\left(\frac{\partial^r}{\partial x^r} + \frac{\partial^r}{\partial y^r}\right)^{r+1} v(x,y) = 0.$$

If $r=2$, $s=0,1$, then we obtain a basis system of $2p+2$ polywave polynomials $G_{2,s}^{n,p}(x,y)$, and the same number of polyharmonic polynomials $H_{2,s}^{n,p}(x,y)$, which for $p=0$ reduce to the corresponding wave and harmonic polynomials $G_{2,s}^{n,0}(x,y)$ and $H_{2,s}^{n,0}(x,y)$.

Note that by taking linear combinations of p-binomial polynomials, we can obtain polynomial solutions of various classes of differential equations. Consider, for example, the differential equation of order 2m

$$\left(\frac{\partial^{2m}}{\partial x^m \partial y^m} + \frac{\partial^{2m}}{\partial x^{2m}} + \frac{\partial^{2m}}{\partial y^{2m}}\right) u(x,y) = 0; \tag{7.33}$$

we will construct a basis system of polynomial solutions using linear combinations of the p-binomial polynomials $G_{r,s}^{n,0}(x,y)$. Omitting the details of the choice of these combinations, it can be shown that the linearly independent polynomials

$$P_{2m,s}^{n,0}(x,y) = G_{3m,s}^{n,0}(x,y) - G_{3m,m+s}^{n,0}(x,y), \quad s = 0,1,2,\dots,2m-1, \quad n>2m,$$

form a basis system of 2m linearly independent polynomial solutions of (7.33). In fact, let $s \le m-1$; then

$$\frac{\partial^{2m}}{\partial x^m \partial y^m} G_{3m,s}^{n,0}(x,y) = (n-2m+1,\ 1)_{2m} G_{3m,2m+s}^{n-2m,0}(x,y),$$

$$\frac{\partial^{2m}}{\partial x^{2m}} G_{3m,s}^{n,0}(x,y) = (n-2m+1,\ 1)_{2m} G_{3m,m+s}^{n-2m,0}(x,y),$$

$$\frac{\partial^{2m}}{\partial y^{2m}} G_{3m,s}^{n,0}(x,y) = (n-2m+1,\ 1)_{2m} G_{3m,s}^{n-2m,0}(x,y),$$

$$\frac{\partial^{2m}}{\partial x^m \partial y^m} G_{3m,m+s}^{n,0}(x,y) = (n-2m+1,\ 1)_{2m} G_{3m,s}^{n-2m,0}(x,y),$$

$$\frac{\partial^{2m}}{\partial x^{2m}} G_{3m,m+s}^{n,0}(x,y) = (n-2m+1,\ 1)_{2m} G_{3m,2m+s}^{n-2m,0}(x,y),$$

$$\frac{\partial^{2m}}{\partial y^{2m}} G_{3m,m+s}^{n,0}(x,y) = (n-2m+1,\ 1)_{2m} G_{2m,m+s}^{n-2m,0}(x,y).$$

Thus, it follows that the polynomials $P_{2m,s}^{n,0}(x,y)$ satisfy (7.33) for $s=0,1,\dots,m-1$. In a similar way, it can be shown that these polynomials also satisfy (7.33) for $s=m, m+1, \dots, 2m-1$.

To construct p-trinomial polynomials we use the expansion of a trinomial in the form

$$(x+y+z)^n = \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k, i-k} x^k y^{i-k} z^{n-i}$$

$$= \sum_{s_1=0}^{r-1} \sum_{s_2=0}^{r-1} \sum_{i=0}^{\left[\frac{n-s_1-s_2}{r}\right]} \sum_{k=0}^{i} \binom{n}{rk+s_1,\ ri-rk+s_2}$$

$$\times\ x^{rk+s_1} y^{ri-rk+s_2} z^{n-ri-s_1-s_2}.$$

Using factorial powers of the variables and inserting the normalization, we distinguish two classes of p-trinomial polynomials:

$$G^{n,p}_{r;s_1,s_2}(x,y,z) = n! \sum_{i=p}^{\left[\frac{n-s_1-s_2}{r}\right]} \sum_{k=0}^{i} \binom{i}{p} \tag{7.34}$$

$$\times\ x^{rk+s_1,1} y^{ri-rk+s_2,1} z^{n-ri-s_1-s_2,1},$$

$$H^{n,p}_{r;s_1,s_2}(x,y,z) = n! \sum_{i=p}^{\left[\frac{n-s_1-s_2}{r}\right]} \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{p} \tag{7.35}$$

$$\times\ x^{rk+s_1,1} y^{ri-rk+s_2,1} z^{n-ri-s_1-s_2,1}.$$

In these expressions, $s_1=0,1,\ldots,r-1$, $s_2=0,1,\ldots,r-1$, and $p=0,1,\ldots,[(n-s_1-s_2)/r]$. Their differentiation formulas are:

$$\frac{\partial}{\partial x} G^{n,p}_{r;s_1,s_2}(x,y,z) = nG^{n-1,p}_{r;s_1-1,s_2}(x,y,z), \quad s_1 \neq 0,$$

$$\frac{\partial}{\partial x} G^{n,p}_{r;0,s_2}(x,y,z) = nG^{n-1,p}_{r;r-1,s_2}(x,y,z) + nG^{n-1,p-1}_{r;r-1,s_2}(x,y,z),$$

$$\frac{\partial}{\partial y} G^{n,p}_{r;s_1,s_2}(x,y,z) = nG^{n-1,p}_{r;s_1,s_2-1}(x,y,z), \quad s_2 \neq 0,$$

$$\frac{\partial}{\partial y} G^{n,p}_{r;s_1,0}(x,y,z) = nG^{n-1,p}_{r;s_1,r-1}(x,y,z) + nG^{n-1,p-1}_{r;s_1,r-1}(x,y,z),$$

$$\frac{\partial}{\partial z} G^{n,p}_{r;s_1,s_2}(x,y,z) = nG^{n-1,p}_{r;s_1,s_2}(x,y,z),$$

$$\frac{\partial}{\partial x} H^{n,p}_{r;s_1,s_2}(x,y,z) = nH^{n-1,p}_{r;s_1-1,s_2}(x,y,z), \quad s_1 \neq 0,$$

$$\frac{\partial}{\partial x} H^{n,p}_{r;0,s_2}(x,y,z) = -nH^{n-1,p}_{r;r-1,s_2}(x,y,z) - nH^{n-1,p-1}_{r;r-1,s_2}(x,y,z),$$

$$\frac{\partial}{\partial y} H^{n,p}_{r;s_1,s_2}(x,y,z) = nH^{n-1,p}_{r;s_1,s_2-1}(x,y,z), \quad s_2 \neq 0,$$

$$\frac{\partial}{\partial y} H^{n,p}_{r;s_1,0}(x,y,z) = -nH^{n-1,p}_{r;s_1,r-1}(x,y,z) - nH^{n-1,p-1}_{r;s_1,r-1}(x,y,z),$$

$$\frac{\partial}{\partial z} H^{n,p}_{r;s_1,s_2}(x,y,z) = nH^{n-1,p}_{r;s_1,s_2}(x,y,z).$$

The indefinite integral formulas for the polynomials $G^{n,p}_{r,s_1,s_2}(x,y,z)$, up to an arbitrary function not dependent on the variable of integration, take the form

$$\int G^{n,p}_{r;s_1,s_2}(x,y,z)\,dx = \frac{1}{n+1} G^{n+1,p}_{r;s_1+1,s_2}(x,y,z),$$

$$\int G^{n,p}_{r;s_1,s_2}(x,y,z)\,dy = \frac{1}{n+1} G^{n+1,p}_{r;s_1,s_2+1}(x,y,z),$$

$$\int G^{n,p}_{r;s_1,s_2}(x,y,z)\,dz = \frac{1}{n+1} G^{n+1,p}_{r;s_1,s_2}(x,y,z).$$

Using the differentiation formulas, and Euler's formula as before, recurrence relations for the polynomials (7.34) take the form

$$G^{n,p}_{r;s_1,s_2}(x,y,z) = xG^{n-1,p}_{r;s_1-1,s_2}(x,y,z) + yG^{n-1,p}_{r;s_1,s_2-1}(x,y,z)$$

$$+ zG^{n-1,p}_{r;s_1,s_2}(x,y,z), \quad s_1 \neq 0, \quad s_2 \neq 0,$$

$$G^{n,p}_{r;0,s_2}(x,y,z) = xG^{n-1,p}_{r;r-1,s_2}(x,y,z) + xG^{n-1,p-1}_{r;r-1,s_2}(x,y,z)$$

$$+ yG^{n-1,p}_{r;0,s_2-1}(x,y,z) + zG^{n-1,p}_{r;0,s_2}(x,y,z), \quad s_2 \neq 0,$$

$$G^{n,p}_{r;s_1,0}(x,y,z) = xG^{n-1,p}_{r;s_1-1,0}(x,y,z) + yG^{n-1,p}_{r;s_1,r-1}(x,y,z)$$

$$+ yG^{n-1,p-1}_{r;s_1,r-1}(x,y,z) + zG^{n-1,p}_{r;s_1,0}(x,y,z), \quad s_1 \neq 0,$$

$$G^{n,p}_{r;0,0}(x,y,z) = xG^{n-1,p}_{r;r-1,0}(x,y,z) + xG^{n-1,p-1}_{r;r-1,0}(x,y,z)$$

$$+ yG^{n-1,p}_{r;0,r-1}(x,y,z) + yG^{n-1,p-1}_{r;0,r-1}(x,y,z) + zG^{n-1,p}_{r;0,0}(x,y,z).$$

Corresponding integration and recurrence formulas for the polynomials $H^{n,p}_{r,s_1,s_2}(x,y,z)$ may be derived in similar fashion.

Various methods of constructing polynomial solutions for partial differential equations, including the Laplace, polywave, polyvibration, and polyharmonic equations, may be found in [82-87, 215, 290, 317, 374, 400, 406].

7.5 OTHER CLASSES OF NONORTHOGONAL POLYNOMIALS

As is well known, orthogonal polynomials in one variable have been studied in great detail, and widely applied in problems in mathematics, mechanics, and physics; two excellent summaries are the books [48, 370]. An extensive bibliography, containing a survey of over 500 works, is given in the detailed book of P. Nevai [293], dedicated to the memory of Geza Freud.

We consider here nonorthogonal polynomials in one independent variable, whose coefficients in expansions in powers of x are binomial coefficients, Fibonacci and Lucas numbers, and other special quantities. We first discuss two classes of polynomials suggested in [10], and of interest from the point of view of constructing solutions to differential equations of Sobolev type, which arise in various mechanics contexts.

We define the polynomials by

$$P_{r,s}^{m,n}(x) = \sum_{k=0}^{n} \binom{n+m}{k+m} x^{rk+s,1}, \tag{7.36}$$

$$Q_{r,s}^{m,n}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+m}{k+m} x^{rk+s,1}, \tag{7.37}$$

where m,n are any nonnegative integers, and r=1,2,..., s=0,1,2,....

We will give differentiation, indefinite integration, and recurrence formulas for the polynomials (7.36); those for the polynomials (7.37) are similar.

It is easy to establish the formulas

$$\left.\begin{aligned} \frac{d}{dx}P_{r,s}^{m,n}(x) &= P_{r,s-1}^{m,n}(x), \quad s \neq 0, \\ \frac{d}{dx}P_{r,0}^{m,n}(x) &= P_{r,r-1}^{m+1,n-1}(x), \end{aligned}\right\} \tag{7.38}$$

$$\int P_{r,s}^{m,n}(x)\,dx = P_{r,s+1}^{m,n}(x), \tag{7.39}$$

$$P_{r,s}^{m+1,n}(x) = P_{r,s}^{m+1,n-1}(x) + P_{r,s}^{m,n}(x). \tag{7.40}$$

From (7.38)-(7.40) we may derive other relations, including integrals of products of two polynomials for values of m,n,s the same or differing. We calculate, for example, the r^{th} derivative, which is needed in applications; we have

$$\frac{d^r}{dx^r} P_{r,s}^{m,n}(x) = \frac{d^{r-s}}{dx^{r-s}} \frac{d^s}{dx^s} P_{r,s}^{m,n}(x) = \frac{d^{r-s}}{dx^{r-s}} P_{r,0}^{m,n}(x)$$

$$= \frac{d^{r-s-1}}{dx^{r-s-1}} \frac{d}{dx} P_{r,0}^{m,n}(x) = \frac{d^{r-s-1}}{dx^{r-s-1}} P_{r,r-1}^{m+1,n-1}(x) = P_{r,s}^{m+1,n-1}(x),$$

that is,

$$\frac{d^r}{dx^r} P_{r,s}^{m,n}(x) = P_{r,s}^{m+1,n-1}(x).$$

In a similar way, we could consider the extension of these polynomials, as in

$$P_{r,s}^{m,n;p,q}(x) = \sum_{k=p}^{n} \binom{k}{p} \binom{n+m}{k+m} x^{rk+rq+s,1},$$

$$Q_{r,s}^{m,n;p,q}(x) = \sum_{k=p}^{n} (-1)^k \binom{k}{p} \binom{n+m}{k+m} x^{rk+rq+s,1},$$

which also have convenient analytic and computational properties.

We include now a short review of references dealing with the construction and study of Fibonacci, Lucas, and other generalized polynomials.

V.E. Hoggatt and M. Bicknell [202] considered Fibonacci, tribonacci, and more general r-bonacci polynomials, defined by the relations

$$R_{-(r-2)}(x) = R_{-(r-1)}(x) = \cdots = R_{-1}(x) = R_0(x) = 0,$$
$$R_1(x) = 1,\ R_2(x) = x^{r-1},$$
$$R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \cdots + R_n(x).$$

For r=2, the polynomial $R_n(x)$ reduces to the Fibonacci polynomial $F_n(x)$, and for r=3 to the tribonacci polynomial $T_n(x)$. They showed that the r-bonacci polynomial $R_n(x)$, written with descending powers of x, has as coefficients the coefficients in the n^{th} ascending diagonal of the generalized Pascal triangle of order r, i.e., the coefficients in the expansion

$$(1 + x + x^2 + \cdots + x^{r-1})^n,\ n = 0, 1, 2, \ldots.$$

The general representation of $R_n(x)$ has the form

$$R_n(x) = \sum_{j=0}^{[(r-1)(n-1)/r]} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj},$$

where $\binom{n}{j}_r = 0$ for $j > n$. For r=2,3, this gives

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1},$$
$$T_n(x) = \sum_{j=0}^{[2(n-1)/3]} \binom{n-j-1}{j}_3 x^{2n-3j-2}.$$

They also studied Q-matrices, generating r-bonacci polynomials.

V.E. Hoggatt, M. Bicknell, and E.L. King [207] defined four sequences of polynomials: the Fibonacci polynomials $f_n(x)$, the Lucas polynomials $l_n(x)$, and the polynomials $g_n(x)$, $h_n(x)$ satisfying

$$g_0(x) = 0,\ g_1(x) = 1,\ g_{n+2}(x) = xg_{n+1}(x) - g_n(x),$$
$$h_0(x) = 2,\ h_1(x) = x,\ h_{n+2}(x) = xh_{n+1}(x) - h_n(x),$$

and established various identities and relations among these.

In [212] V.E. Hoggatt and J.W. Phillips used the generalized binomial coefficients $C_n(p,r)$, $r \geq 2$, $p \geq 0$, and the Fibonacci and Lucas polynomials

$$f_1(x) = 1,\ f_2(x) = x,\ f_n(x) = xf_{n-1} + f_{n-2}(x),$$
$$l_1(x) = x,\ l_2(x) = x^2+2,\ l_n(x) = xl_{n-1}(x) + l_{n-2}(x),$$

to define new classes of polynomials in the form of the sums

$$\sum_{n=0}^{p(r-1)} (\pm 1)^n C_n(p,r) f_{bn+j}^m(x), \quad \sum_{n=0}^{p(r-1)} (\pm 1)^n C_n(p,r)\, l_{bn+j}^m(x), \quad C_n(p,r) = \binom{p}{n}_r .$$

V.E. Hoggatt and C.T. Long [211] introduced the so-called generalized Fibonacci polynomials

$$u_0(x,y) = 0,\ u_1(x,y) = 1,\ u_{n+2}(x,y) = xu_{n+1}(x,y) + yu_n(x,y),\ n = 0,1,2,\ldots .$$

They showed that these polynomials have a number of properties analogous to the Fibonacci sequence, and studied some of their divisibility properties.

A. Krishnaswami [244] discussed a class of functions which he called Pascal functions, the coefficients of which are formed from the Pascal triangle diagonals; he showed that the Fibonacci polynomials are a special case of the Pascal functions.

H. Hosoya [217, 219] studied some interesting analytic, combinatorial, and graph properties of orthogonal polynomials, including those of Chebyshev, Hermite, and Laguerre.

In a cycle of papers: A.N. Phillipou [310]; A.N. Phillipou, C. Georghiou, G.N. Phillipou [311, 312]; A.N. Phillipou, F.S. Makri [313]; and G.N. Phillipou and C. Georghiou [314] considered generalized Fibonacci polynomials, and Fibonacci polynomials of order k. A review of these works and a bibliography is given in the detailed article of A.N. Phillipou [310]. A sequence of polynomials $\left\{f_n^{(k)}(x)\right\}_{n=0}^{\infty}$ is said to be a sequence of Fibonacci polynomials of order k if $f_0^{(k)}(x)=0$, $f_1^{(k)}(x)=1$, and

$$f_n^{(k)}(x) = \begin{cases} \sum_{i=1}^{n} x^{k-i} f_{n-i}^{(k)}(x), & 2 \le n \le k+1, \\ \sum_{i=1}^{k} x^{k-i} f_{n-i}^{(k)}(x), & n \ge k+2. \end{cases}$$

If we set $f_n^{(r)}(x)=0$ for $-(r-2) \le n \le -1$, then $f_n^{(r)}(x)=R_n(x)$, $n \ge -(r-2)$, i.e., reduces to the r-bonacci polynomial mentioned earlier. In [311], the explicit representation of these polynomials is obtained in a form involving multinomial coefficients,

$$f_{n+1}^{(k)}(x) = \sum_{n_1,\dots,n_k} \binom{n_1+\dots+n_k}{n_1,\dots,n_k} x^{k(n_1+\dots+n_k)-n}, \quad n \ge 0.$$

In [312, 313] other properties are studied, and an application to probability theory is given, and [314] is a further consideration of their properties, including their connection with the generalized Pascal triangle of order k, whose coefficients are defined by

$$A_{0,0}^{(k)} = 1,\ A_{0,m}^{(k)} = 0, \quad m \geq 1,$$

$$A_{n,m}^{(k)} = \begin{cases} \sum_{i=0}^{m} A_{n-1,m-i}^{(k)}, & 0 \leq m < k, \\ \sum_{i=0}^{k-1} A_{n-1,m-i}^{(k)}, & m \geq k, \end{cases}$$

where n≥1. They showed that

$$F_{n+1}^{k}(x) = \sum_{i=0}^{[n-n/k]} A_{n-i,i}^{(k)} x^{n-i}, \quad n \geq 0,$$

and that

$$A_{n,m}^{(k)} = \sum_{i=0}^{\left[\frac{m}{k}\right]} (-1)^{i} \binom{n}{i} \binom{m+n-1-ki}{n-1}.$$

V.E. Hoggatt and D.A. Lind [210] discussed the question of the so-called height of the Fibonacci polynomial

$$f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1},$$

and its connection with these functions, where the height is taken to be the largest coefficient of the polynomial. They proved that the heights of two successive polynomials lie either in the same column, or in neighboring columns, of the Pascal triangle. In the latter case, $m(n)/m(n+1)=k/h(k)$, where $m(n)$ is the height of $f_n(x)$, and $h(k)=\frac{1}{2}\left(k+1+\sqrt{5k^2-2k+1}\right)$ if $m(n+1)$ lies in the k^{th} column. They also showed that F_n/F_{n+1} for $n\geq 2$, and L_n/L_{n+1} for $n\geq 4$, may be represented as the quotient of the heights of two successive polynomials whose heights do not lie in the same column.

Other discussions related to some classes of non-orthogonal polynomials may be found in the references [124, 198, 273, 283].

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APPENDIX

ADDITIONAL REFERENCES AND A BRIEF SURVEY OF RECENT RESULTS

This appendix, added in translation, gives an additional one hundred and twenty-six references relevant to the continuing development of the properties and applications of the Pascal triangle and its generalizations. The first eighty items (Additional References I) have appeared in the literature in the last five years; preceding the list, there is also (below) a brief review of some of these and other results. The remaining items (Additional References II) were published prior to 1987 but were not originally included in the main body of references. All this material is taken from part of the author's recent paper (in Russian): "On the Pascal triangle and its generalizations. The distribution of the binomial and trinomial coefficients modulo 4 in the Pascal triangle and Pascal pyramid," Voprosy Vychisl. i Prikl. Mat., Vyp. 93. Tashkent, 1992, pp. 5-24.

The publications [1-80] from 1987-1991 discuss a variety of complicated problems connected with the further study of the Pascal triangle and its generalizations. Foremost among these are problems related to the divisibility and the distribution with respect to prime or composite moduli of the binomials and other combinatorial numbers appearing in their corresponding arithmetic triangles [2-9, 13, 14, 17, 23, 31-33, 39-51, 56-59, 62, 69-71, 73-76, 78-80]. Other main themes include the construction of fractal arithmetic structures [3, 25, 40, 42, 52-55, 60-63, 66, 67], various generalized Pascal triangles and their properties [11, 12, 15, 18, 20, 26-30, 35-38], and the construction and study of graph models, determinants, and criteria for primality [10, 16, 19, 21, 22, 34, 64, 65, 68, 72, 77]. Many of the less recent publications [81-126] also deal with these problems, as the article titles indicate.

One of the difficult problems connected with the Pascal triangle and its generalizations is that of determining the distribution of the binomial coefficients, and other combinatorial numbers occurring in arithmetic structures, with respect to a prime modulus, and especially with respect to a composite modulus. Analytic solutions of this problem are discussed in [1-8, 23, 45, 73-76, 78, 94, 95, 110].

The problem of the distribution of the binomial coefficients with respect to a prime modulus p was first discussed in 1957 by J.B. Roberts [110]. He gave, for p=3,5, exact formulas for the number of binomial coefficients congruent modulo p in the Pascal triangle whose base is the row numbered $N=p^n-1$.

In 1978 E. Hexel and H. Sachs [95] generalized the results in [110] by devising a method for determining the number of congruent binomial coefficients not just in the triangle as a whole, but in an arbitrary one of its rows. Again for p=3,5, they established exact formulas for any row of the Pascal triangle with base $N=p^n-1$.

In [1-5], the present author employed normalized p-Latin squares and their associated matrices to derive exact formulas for the distributions of binomial and trinomial coefficients, Stirling numbers of the first and second kinds, Gaussian binomial coefficients, and Euler numbers, in their corresponding arithmetic structures, with respect to the moduli p=3,5,7.

M. Sved [73-76], using a computer, implemented the construction of the triangular distributions of the binomial and Gaussian binomial coefficients, Stirling numbers of the first and second kinds, and Euler and other numbers, with respect to prime and composite moduli for a considerable number of rows of the corresponding triangles.

The references mentioned above are essentially all concerned with the distributions of coefficients and combinatorial numbers with respect to a prime modulus. Only in 1991 have the first results begun to appear for a composite modulus d: K. Davis and W. Webb [23] established an exact formula for the distribution of the binomial coefficients mod 4 in an arbitrary row of the Pascal triangle. F. Howard [45] then generalized the results of [23] and [95] to the case of Gaussian binomial and multinomial coefficients. And even more recently the present author, in the paper mentioned in the introductory paragraph, has obtained exact formulas for the distributions of the binomial and trinomial coefficients modulo 4.

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