CHAPTER 3

DIVISIBILITY AND DISTRIBUTION MODULO p IN GENERALIZED PASCAL TRIANGLES, AND FIBONACCI, LUCAS, AND OTHER SEQUENCES

In this chapter we consider divisibility of generalized binomial coefficients $\binom{n}{m}_s$. We give the analog of Lucas's Theorem, and prove some theorems on the divisibility by a prime p of generalized binomial coefficients in a given row of the generalized Pascal triangle of order s for s=3 and p=2,3. We also discuss the distribution of these coefficients for the moduli 2 and 3, and the situation for $n \to \infty$.

Divisibility and distributions for a prime modulus are also considered for Fibonacci, Lucas, and other sequences, as well as periodicity of these sequences with respect to a prime modulus.

3.1 DIVISIBILITY AND THE DISTRIBUTION MODULO p OF GENERALIZED BINOMIAL COEFFICIENTS

In section 1.3 we discussed generalized Pascal triangles of order s, the elements of which are the generalized binomial coefficients $\binom{n}{m}_s$, and considered their recurrence and other relations analogous to those of the binomial coefficients. Not a great deal of work has appeared on questions of divisibility and distribution of these coefficients, but we first turn to the analog of Lucas's Theorem, and some related results, given by R.C. Bollinger and C.L. Burchard [81].

Theorem 3.1. Let p be a prime, and n and m nonnegative whole numbers, $0 \le m \le n(s-1)$ with p-ary representations $n=(a_r a_{r-1} \cdots a_0)_p$, $m=(b_r b_{r-1} \cdots b_0)_p$, where $a_r \ne 0$, $0 \le a_k$, $b_k < p$. Then

$$\binom{n}{m}_s \equiv \sum_{i_k} \prod_{k=0}^{r} \binom{a_k}{i_k}_s \pmod{p}, \tag{3.1}$$

where the summation is carried out over all indicies i_k for which $i_0+i_1p+i_2p^2+\cdots+i_rp^r=m$, $0 \le i_k \le (s-1)a_k$.

We note that if the latter two conditions are not satisfied, then $\binom{n}{m}_s \equiv 0 \pmod{p}$. The authors prove this theorem and give some related examples in [81]. They also discuss the question of the number $N_s(n,p)$ of generalized binomial coefficients $\binom{n}{m}_s \not\equiv 0 \pmod{p}$ for the two cases $s=p$ and $s=p^v$, where p is a prime.

Theorem 3.2. Let $(p-1)n$ have the p-ary representation $(c_r c_{r-1} \cdots c_0)_p$. Then in the generalized Pascal triangle of order p, the number of coefficients in row n which are not divisible by p is

$$N_p(n,p) = \prod_{k=0}^{r} (1+c_k). \tag{3.2}$$

From Theorem 3.2 it also follows that if $s=p^v$, then

$$N_{p^v}(n,p) = N_p[n(p^v-1)/p-1,p] = N_2[n(p^v-1),p], \tag{3.3}$$

and, further, that for $n\to\infty$ almost all coefficients $\binom{n}{m}_s$ are divisible by p.

Theorem 3.3. Let r be a natural number. Then in row $n=p^r$ of the generalized Pascal triangle of order s, we have

$$\binom{n}{m}_s \equiv 1 \pmod p, \; m = ip^r, \qquad \binom{n}{m}_s \equiv 0 \pmod p, \; m \neq ip^r. \tag{3.4}$$

The proof of this theorem is based on the analog of Lucas's Theorem in polynomial form, and the result is used in the construction of fractal generalized Pascal triangles.

Theorem 3.4. Let p be a prime. In the multinomial coefficient $(n; m_1, m_2, \ldots, m_{s-1})$ let n and m_k be written as

$$n = (a_r a_{r-1} \ldots a_1 a_0)_p, \quad m_k = \left(b_r^{(k)} b_{r-1}^{(k)} \ldots b_1^{(k)} b_0^{(k)}\right)_p,$$

where $a_r \neq 0$, $0 \leq a_i < p$, $0 \leq b_i^{(k)} \leq p$. Then

$$\binom{n}{m}_s \equiv \sum \prod_{i=0}^{r} \left(a_i; b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(s-1)}\right) \pmod p,$$

where the summation is over all $b_i^{(k)}$ satisfying

$$b_i^{(1)} + b_i^{(2)} + \cdots + b_i^{(s-1)} = m.$$

Unlike the Pascal triangle, in which the rule for forming the binomial coefficients mod p, and their distribution, depends only on p, the distribution of the generalized binomial coefficients depends on both p and s. Thus, we will only consider here the distribution of these coefficients for s=3, p=2,3; the method itself may be used for other values of s and p.

Let p=2. We introduce the following definition.

Definition 3.1. Let the natural number n be written in binary form. We will say that n contains a block of type k - denoted by $<1>_k$ - if its binary form contains a string of k consecutive ones which has at least one zero on both the left and the right.

Clearly, any natural number n written in binary form consists of $q_k \geq 0$ blocks of type k for k=1,2,...,t, where t=t(n). For example, the binary form of n=315837 is 1011001000110111101, which contains $q_1=3$ blocks of type 1, $q_2=2$ of type 2, $q_3=0$ of type 3, and $q_4=1$ of type 4. We note also that the binary forms of distinct natural numbers may contain identical numbers of the same kinds of blocks.

Theorem 3.5. In the generalized Pascal triangle of order 3, let the row number n be written in binary form, in which there are $q_k \geq 0$ blocks of type k, $1 \leq k \leq t$. Then the number of odd trinomial coefficients in row n is given by

$$P_1(n) = U_1^{q_1} U_2^{q_2} \dots U_t^{q_t}, \quad U_k = \frac{1}{3}[2^{k+2}-(-1)^k]. \tag{3.5}$$

The proof of this theorem follows from Theorem 3.4 and the solution of the recurrence relation $U_k=U_{k-1}+2U_{k-2}$ with the initial conditions $U_0=1$, $U_1=3$; it is not difficult to show that the solution is given by the expression for U_k in (3.5).

The total number of coefficients in row n is 2n+1, so that the number of even coefficients is $P_2(n)=(2n+1)-P_1(n)$. And, if there are N rows in the generalized Pascal triangle, there will be a total of $(N+1)^2$ coefficients, and the total number of even coefficients will be given by

$$Q_2(n) = (N+1)^2 - \sum_{n=0}^{N} P_1(n).$$

If we apply the elementary rule defining evenness/oddness to the sums of three terms occurring in the recurrence relation for the trinomial coefficients, and write out the triangle, we will have the distribution of even and odd coefficients in the Pascal triangle of order 3. We show this in Figure 26 for $N=2^4+1=17$ rows, where the odd coefficients are denoted by ones and the even coefficients by dots.

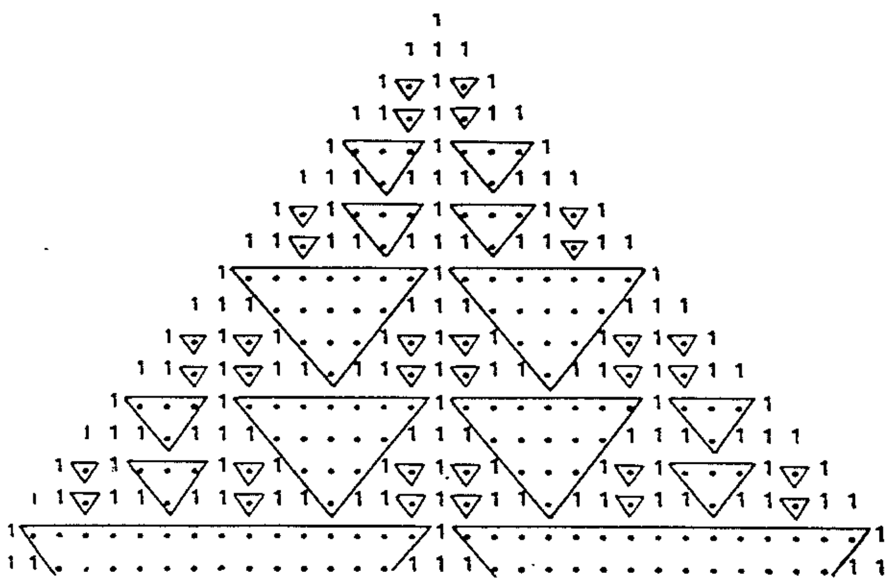

Figure 26

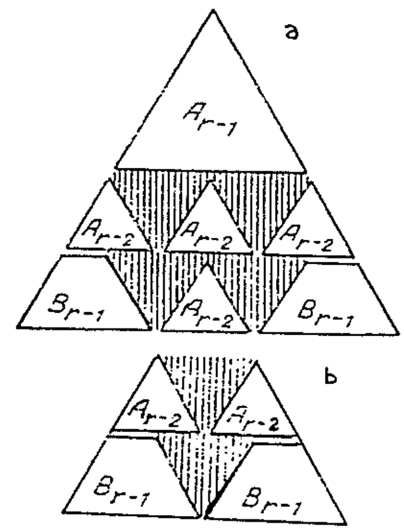

Figure 27

Let us denote by A_r the isosceles triangle (Figure 27a) whose altitude, measured by the number of rows from base to vertex inclusive, is $h_r=2^r$; the length of whose base is $d_r=2^{r+1}-1$; and the number of whose base row is $n=h_r-1=2^r-1$, $r\geq 0$.

Also, denote by B_r the isosceles trapezoid (Figure 27b) whose altitude is $h_r=2^{r-1}$; whose upper and lower bases have lengths $d_U=2^{r-1}+1$ and $d_L=2^r+2^{r-1}-1$; and the number of whose base row is $n=2^{r-1}-1$, $r\geq 1$.

In Figure 26, it is not difficult to see the triangles $A_0,\ldots,A_4$, and the trapezoids $B_1,\ldots,B_3$.

For the following theorem, we will need the Fibonacci numbers, which may be calculated by the known formula of Binet,

$$F_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right],$$

or expressed by means of the binomial coefficients

$$F_m = \sum_{i=0}^{\left[\frac{m}{2}\right]} \binom{m-i}{i}, \quad F_1 = F_2 = 1. \tag{3.6}$$

We note that the Binet formula [392] may be extended to the case of the sequence $\{G_m\}$, where

$$G_1 = G_2 = 1, \quad G_{m+2} = G_{m+1} + \frac{p-1}{4}G_m \quad (m \geq 1),$$

and

$$G_m = \frac{1}{\sqrt{p}}\left[\left(\frac{1+\sqrt{p}}{2}\right)^m - \left(\frac{1-\sqrt{p}}{2}\right)^m\right].$$

Theorem 3.6. Let the row number of the base of the generalized Pascal triangle of order 3 be $n=2^r-1$. Then for any natural number r, the number of odd trinomial coefficients in this triangle is given by

$$Q_1(2^r-1) = 2^r F_{r+2}. \tag{3.7}$$

<u>Proof</u>: It follows from Theorem 3.3 that in row $n=2^{r-1}$, which lies inside A_r, there are three odd coefficients: $\binom{n}{m}_3$ for $m=0,2^{r-1},2^r$. Each of these gives rise to a triangle A_{r-2}, the base of which has length $2^{r-2}-1$ and lies on the row $n=2^{r-1}+2^{r-2}-1$. The following row, $2^{r-1}+2^{r-2}$, according to Theorem 3.5, has five odd coefficients: for $m=0$, 2^{r-2}, $2^{r-1}+2^{r-2}$, 2^r+2^{r-2}, 2^r+2^{r-1}. From this, we can establish that the coefficient for $m=2^{r-1}+2^{r-2}$ gives rise to a triangle A_{r-2}, and each pair of coefficients for $m=0$, 2^{r-2} and $m=2^r+2^{r-2}$, 2^r+2^{r-1}, gives rise to a trapezoid B_{r-1}. Thus, the triangle A_r can be written as a "geometric" sum of the triangle A_{r-1} with base row $2^{r-1}-1$, four triangles A_{r-2}, and two trapezoids B_{r-1} (Figure 27a). Likewise, B_r is the geometric sum of two trapezoids B_{r-1} and two triangles A_{r-2} (Figure 27b).

If we denote by a_r the number of odd coefficients in A_r, and by b_r the number in B_r, we can, on the basis of the arguments given above, write the system of recurrence relations

$$\left.\begin{aligned} a_r &= a_{r-1} + 4a_{r-2} + 2b_{r-1} \\ b_r &= 2b_{r-1} + 2a_{r-2} \end{aligned}\right\} \tag{3.8}$$

where $r\geq 2$ and the initial data is $a_0=1$, $a_1=4$, $b_1=2$, determined by the number of odd coefficients in A_0, A_1, and B_1. The solution of (3.8) may be expressed in terms of the Fibonacci numbers as

$$a_r = 2^r F_{r+2}, \quad b_r = 2^{r-1} F_{r+2}. \tag{3.9}$$

Substituting (3.9) in (3.8), and using the fact that $F_{r+2}=F_{r+1}+F_r$, it is easy to show that (3.9) is correct. Thus, $Q_1=a_r=2^rF_{r+2}$, and the proof is complete.

If $n=2^r-1$, then the total number of trinomial coefficients in triangle A_r is 2^{2r}, and thus the number of even coefficients, using (3.7), is given by

$$Q_2(2^r-1) = 2^{2r} - 2^r F_{r+2}. \qquad (3.10)$$

Then, using (3.7) and (3.10) we can show that from some r onward $Q_2(2^r-1) >> Q_1(2^r-1)$. Thus, $Q_2(2^4-1)=128$, $Q_1(2^4-1)=128$; $Q_2(2^7-1)=12032$, $Q_1(2^7-1)=4352$; $Q_2(2^{10}-1)=1048576$, $Q_1(2^{10}-1)=147456$.

Theorem 3.7. For $n\to\infty$, $\lim [Q_1(n)/Q_2(n)]=0$.

Proof: Since $Q_1(n)$ and $Q_2(n)$ are nondecreasing functions of n, then for $2^r-1 \leq n < 2^{r+1}-1$,

$$Q_1(n)/Q_2(n) < Q_1(2^r-1)/Q_2(2^r-1).$$

Consequently,

$$\lim_{n\to\infty} Q_1(n)/Q_2(n) < \lim_{r\to\infty} Q_1(2^{r+1}-1)/Q_2(2^r-1).$$

Using (3.7) and (3.10) we find that

$$\begin{aligned} Q_1(2^{r+1}-1)/Q_2(2^r-1) &= 2^{r+1}F_{r+3}/(2^{2r}-2^rF_{r+2}) \\ &= 2F_{r+3}/(2^r-F_{r+2}) \\ &< F_{r+3}/(2^r-F_{r+3}) \\ &= 1/((2^r/F_{r+3})-1). \end{aligned}$$

But for $r\to\infty$, $\lim 2^r/F_{r+3}=\infty$, and so

$$\lim_{n\to\infty} Q_1(n)/Q_2(n) < \lim_{r\to\infty} Q_1(2^{r+1}-1)/Q_2(2^r-1) = 0,$$

which proves the theorem.

We consider now the distribution of the trinomial coefficients in the generalized Pascal triangle of order 3 with respect to the modulus $p=3$.

Definition 3.2. Let the natural number n be written in ternary form. We will say that n contains a 1-block of type k - denoted by $<1>_k$ - if its ternary form contains a string of k consecutive ones which is bounded on the left by at least one zero or one two, and on the right by at least one zero.

Definition 3.3. With n in ternary form, as above, we will say that n contains a 2-block of type i - denoted by $<2>_i$ - if it contains a string of i consecutive twos, ignoring imbedded ones, which is bounded on the left and right by at least one zero or by ones.

In connection with definition 3.2, note that strings of consecutive ones which precede twos are not considered. Thus, in $n=(211122)_3$ we ignore the three ones, and count what remains as a block of type $<2>_3$.

Example: Suppose n in ternary form is $n=2012210211202221101221$. To count blocks in n we first exclude ones which precede twos. As a result, we find the block form of n to be

$$<n> = <202210220222110221>,$$

and say that n contains two $<1>_1$ blocks, one $<1>_2$ block, one $<2>_1$ block, three $<2>_2$ blocks, and one $<2>_3$ block.

Theorem 3.8. In the generalized Pascal triangle of order 3, let the row number n when written in ternary form consist of $p_k \geq 0$ blocks $<1>_k$, $1 \leq k \leq s$, and q_i blocks $<2>_i$, $1 \leq i \leq t$. Then in row n the number of trinomial coefficients not divisible by three is

$$N_{1,2}(n) = \prod_{k=1}^{s} V_k^{p_k} \prod_{i=1}^{t} W_i^{q_i}, \quad V_k = 3^k, \quad W_i = 3^i+3^{i-1}. \tag{3.11}$$

To prove Theorem 3.8, as in Theorem 3.5 we use the three-dimensional analog of Lucas's Theorem, and find the expressions for V_k and W_i as the solutions of the corresponding recurrence relations.

In Figure 28 we have written out, using the modulus p=3, the rows of the triangle up through row N=15, in which the coefficients not divisible by three appear as 1's and 2's, and those divisible by three are represented by dots.

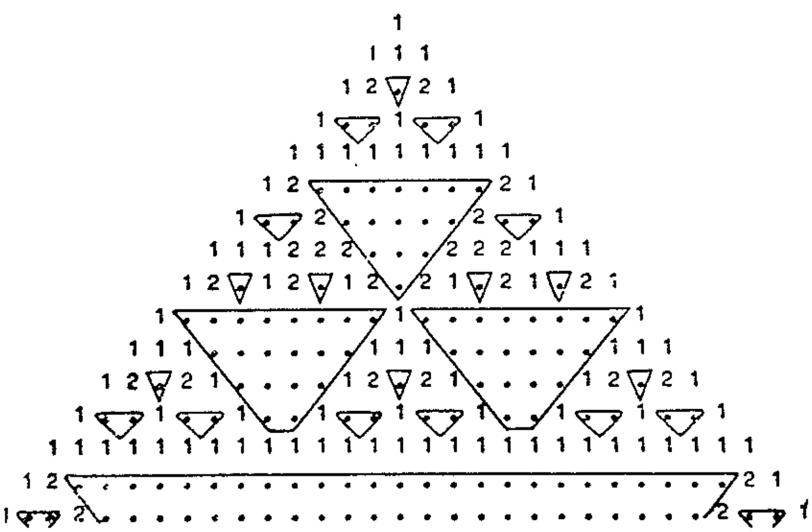

Figure 28

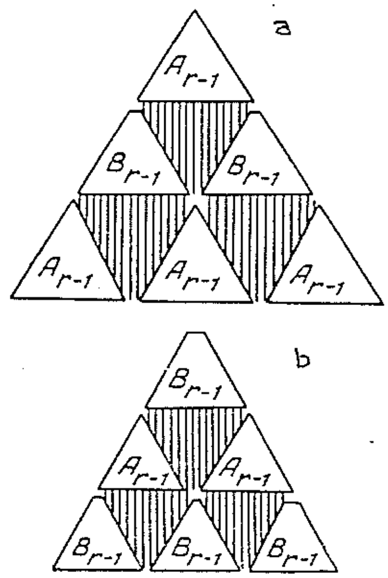

Figure 29

Denote by A_r the isosceles triangle (Figure 29a) whose height, width of base, and row number of the base are, respectively,

$$h_r = \frac{1}{2}(3^r+1), \quad d = 3^r, \quad n = \frac{1}{2}(3^r-1), \quad r \geq 0.$$