CHAPTER 4

FRACTAL PASCAL TRIANGLES AND OTHER ARITHMETIC TRIANGLES

In this chapter we use some results from Chapter 2 to form fractal Pascal, and generalized Pascal, triangles, as well as fractal arithmetic triangles whose elements are, e.g., Gaussian binomial coefficients, and Stirling and Euler numbers. We also give some interesting geometric figures containing elements of the Pascal triangle, these elements being related to one another by various arithmetic properties.

4.1 FRACTALS AND THEIR DIMENSIONS

The objects which today we call fractals, or describe as being fractal, were first studied in the early part of the present century, although the term "fractal" has only become established in the last decade. The term was introduced by the French mathematician B.B. Mandelbrot, and comes from the Latin adjective "fractus", connoting something fractional or cut up. The most complete descriptions of various classes of fractals in nature are in the books of B.B. Mandelbrot [270-272], H.-O. Peitgen and co-authors [304, 305], and in the collection of articles [51], and also in [27, 30]. The fractal property is possessed by many geographic features, among them coastlines, mountains, and valleys. There also exist many physical and chemical processes out of which arise complicated fractal structures.

Of considerable interest is the representation of fractal constructions arising from the Pascal triangle in the works of S. Wolfram [397,398], and O. Martin, A.M. Odlyzko, S. Wolfram [276], and their application to the study of cellular automata. A. Lakhtakia, et al., [213, 249-254], constructed a new class of fractals, the Pascal-Sierpinski gaskets, investigated their properties, and gave applications to various physical problems. An interesting class of fractals, connected with the Gaussian binomial coefficients, and the Stirling and Euler numbers, was constructed by M. Sved [368]. In [12, 16] are constructed fractal Pascal triangles, pyramids, generalized triangles, and Fibonacci and Lucas triangles. Fractal Pascal triangles also appear in the book of T.M. Green and C.L. Hamberg [162], as well as in [1, 2, 94, 147, 247, 344, 403].

Fractals as geometric objects possess a variety of properties, but fundamental among these are their fractional (non-integral) dimension and their self-similarity. Roughly speaking, a self-similar geometric figure is one which may be represented in the form of a finite number of figures similar among themselves. With such figures we may associate equilateral triangles and squares, the self-similarity of which is defined in a more complicated way. As examples, we mention the well-known self-similar geometric constructions in Figure 38, known as (a) the Sierpinski triangle curve, and (b) the Sierpinski carpet, after the Polish mathematician W. Sierpinski. The methods of construction are easily explained in terms of the figures: the triangle curve is obtained by repeatedly connecting the midpoints of the sides of the successively smaller equilateral triangles; the carpet is constructed by iterating the process of discarding the middle square from among the nine squares of the preceding stage. Figure 39 shows the so-called Koch triadic (snowflake) curve, whose construction begins with an equilateral triangle, each side of which is divided into three parts, with the middle part then replaced by two line segments of length equal to a third of the original side.

As we know, there exist various definitions of dimension, corresponding to quite different points of view. One of these ideas of dimension is related to the minimal number of coordinates necessary to unambiguously define the location of a point on a line, in a plane, and in space. Another, the concept of the topological dimension, is that the dimension of any set should be one greater than the dimension of the cut which separates it into two disconnected parts. We note here that these dimensions may only be integers; both definitions imply that a line is one-dimensional, a plane is two-dimensional, the usual geometric space is three-dimensional, and so on.

But there exist other concepts, as well, and one of these is that of the dimension of self-similarity. Let n be the number of identical parts into which a given self-similar object is decomposed when the size of the original parts has been reduced by a factor of m. Then the self-similar dimension is defined by the formula

$$D = \ln n / \ln m. \tag{4.1}$$

Using this introduced concept, we find that the dimension of the self-similar square obtained by successive division into four equal squares is ln 4/ln 2=2, that of the self-similar cube is ln 8/ln 2=3, and so on. But when we use (4.1) to calculate the dimensions of the objects in Figures 38 and 39, we find for the Sierpinski triangle curve that D_1=ln 3/ln 2=1.5849, for the Sierpinski carpet that D_2=ln 8/ln 3=1.8727, and for the Koch curve that D_3=ln 4/ln 3=1.2618. That is, the self-similar dimensions of these objects are non-integral. Non-integral dimensions are usually referred to as fractional dimensions, and in Figure 40 are some additional self-similar figures: fractals, having fractional dimensions [270].

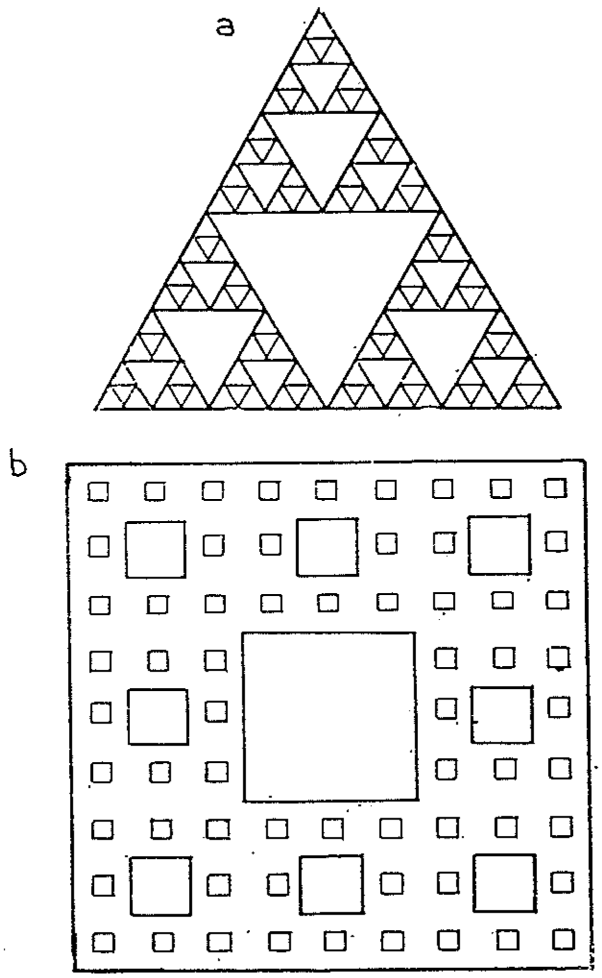

Figure 38

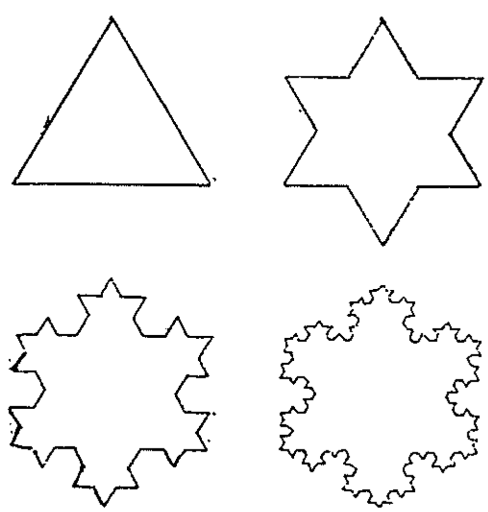

Figure 39

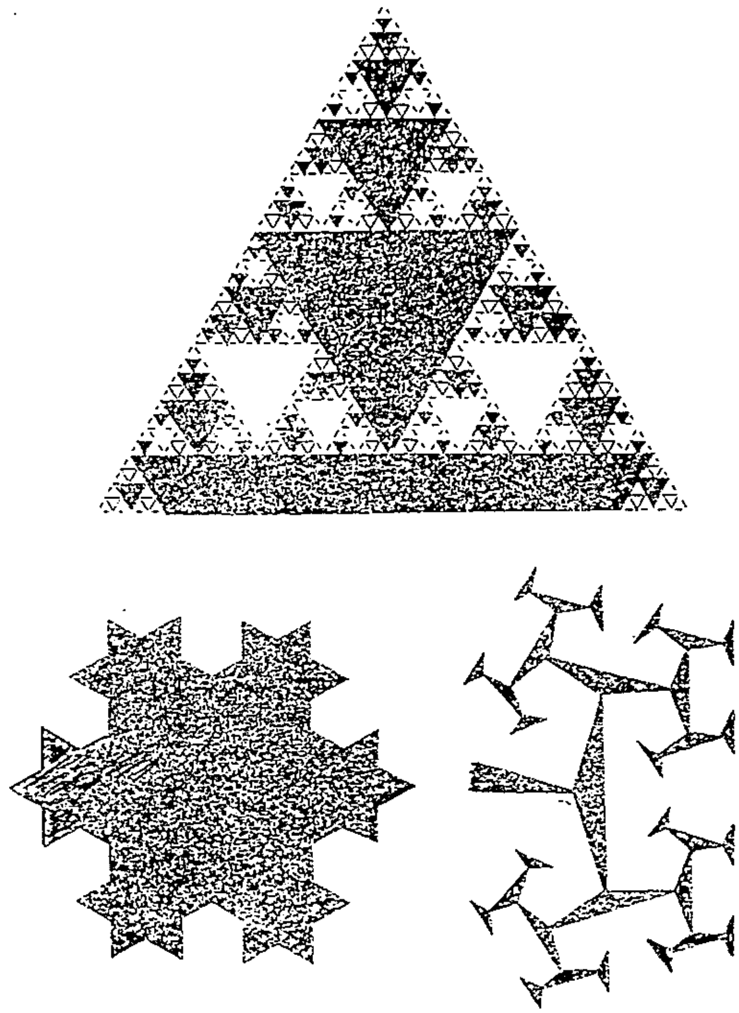

Figure 40

4.2 FRACTAL PASCAL TRIANGLES MODULO p

The elements of the Pascal triangle, reduced with respect to some modulus, in particular a prime modulus p, form various triangular geometric lattices - in fact, fractals, which played an early part in the analysis of these structures and processes. (In order to discover the properties of self-similar Pascal triangles mod p, we need to use a sufficiently large number of rows of the triangle.) A fundamental characteristic of these fractals formed from the Pascal triangle mod p is their fractional dimension D_p. Taking into account that for a prime p, the corresponding m and n (of 4.1) are $n=p(p+1)/2$ and $m=p$, it follows from (4.1) that

$$D_p = \ln \frac{p(p+1)}{2} \,/\, \ln p = 1 + \ln \frac{(p+1)}{2} \,/\, \ln p. \tag{4.2}$$

Then from (4.2) we find, for example,

$$D_2 \cong 1.585,\ D_3 \cong 1.631,\ D_5 \cong 1.683,\ \ldots.$$

And for $p \to \infty$,

$$\lim_{p\to\infty} D_p = 1 + \lim_{p\to\infty} \ln(p+1)/\ln p - \lim_{p\to\infty} \ln 2/\ln p = 2. \tag{4.3}$$

In Figures 41-43 we show the fractal Pascal triangles formed using the respective prime moduli p=2,3,5, and in Figures 44,45 using the composite moduli d=4,6 (in these latter two cases the self-similarity is of a more complicated kind, and (4.3) cannot be used to calculate the dimension). The dark ovals indicate points of the geometric lattice corresponding to the elements of the Pascal triangle not divisible by p=2,3,5 and d=4,6; the blanks indicate coefficients divisible by these moduli.

Applications of fractal Pascal triangles, mentioned earlier, are discussed in [12, 213, 249-254, 276, 395, 396].

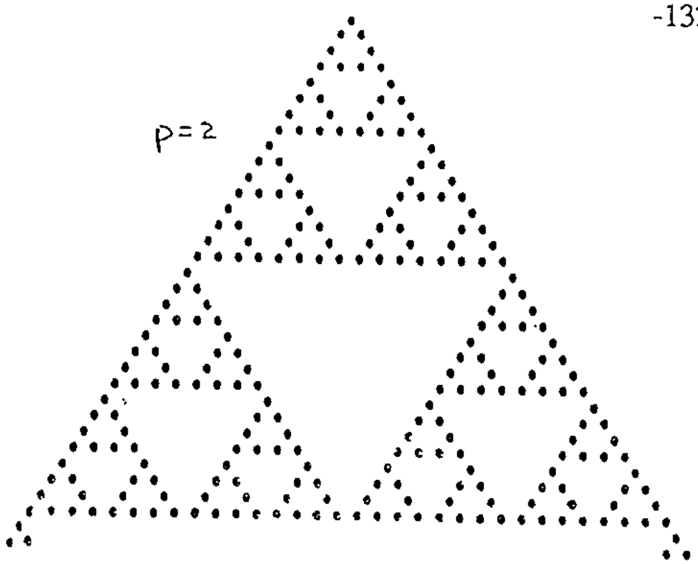

Figure 41

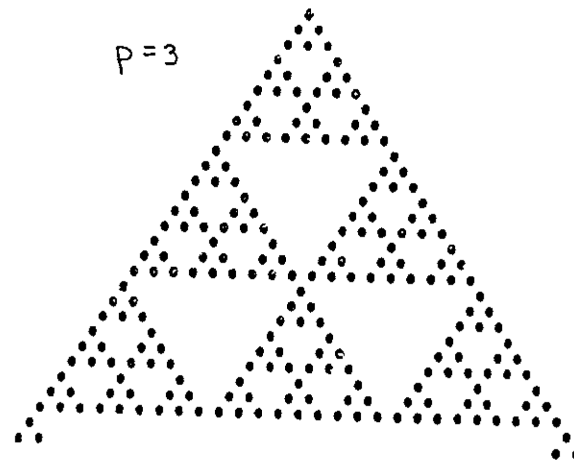

Figure 42

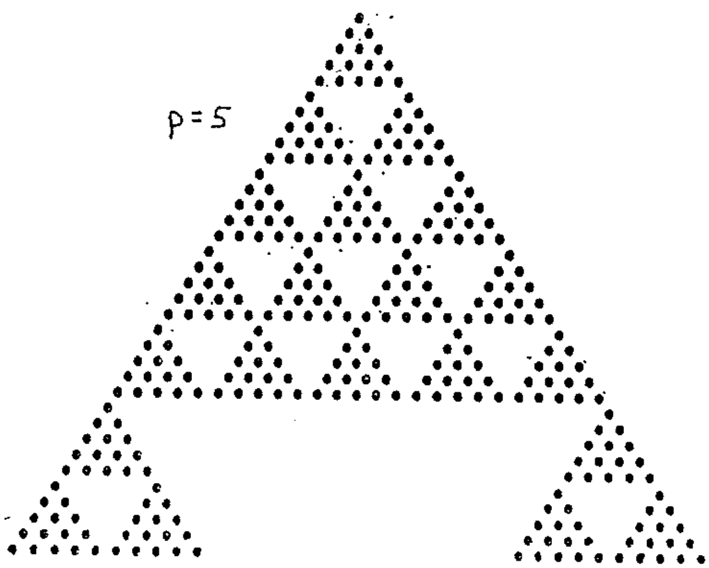

Figure 43

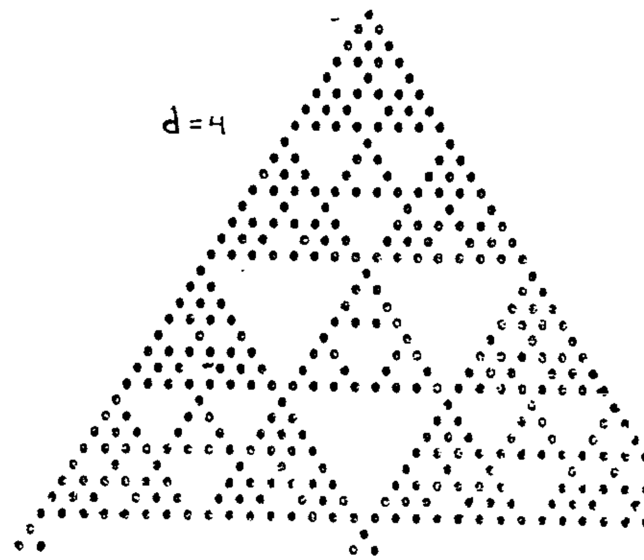

Figure 44

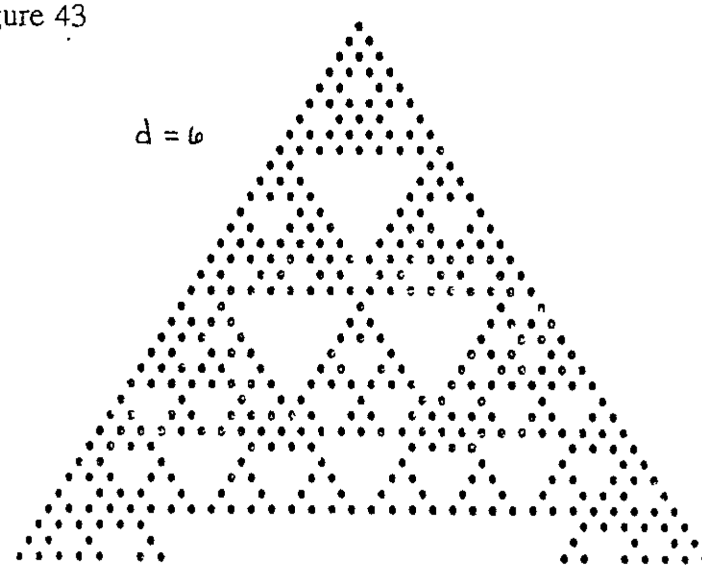

Figure 45

4.3 FRACTAL GENERALIZED PASCAL TRIANGLES AND OTHER ARITHMETIC TRIANGLES MODULO p

Fractal generalized Pascal triangles, with the coefficients $\binom{n}{m}_s$ reduced with respect to the prime moduli p=2,3 are given in [16]. M. Sved in [368] gives the constructions for fractal arithmetic triangles composed of Gaussian binomial coefficients, Stirling numbers of the first and second kind, and Euler numbers.

The fractal triangles for $\binom{n}{m}_3$ mod 2 and mod 3 are shown in Figures 46,47; for the Gaussian binomial coefficients,q=2,mod 3 in Figure 48; for the Stirling numbers of the second kind mod 2 in Figure 49; and for the Euler numbers mod 3 in Figure 50.

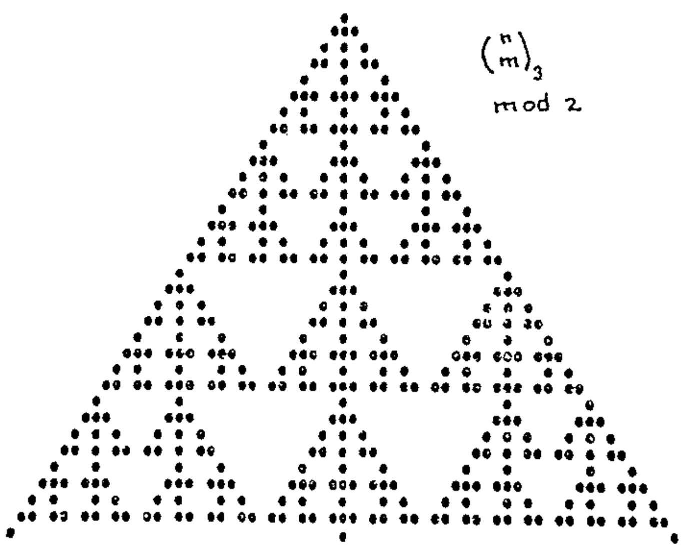

Figure 46

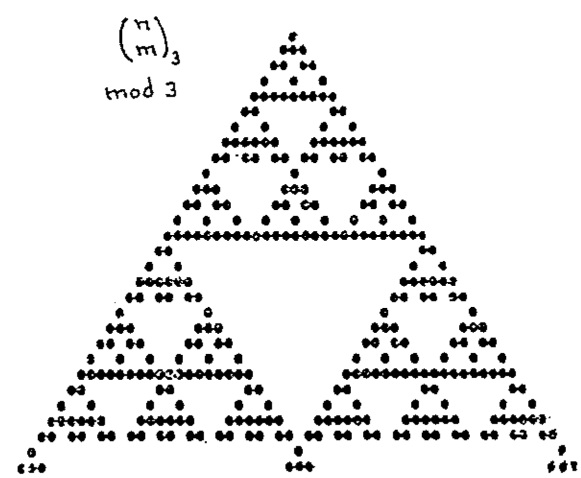

Figure 47

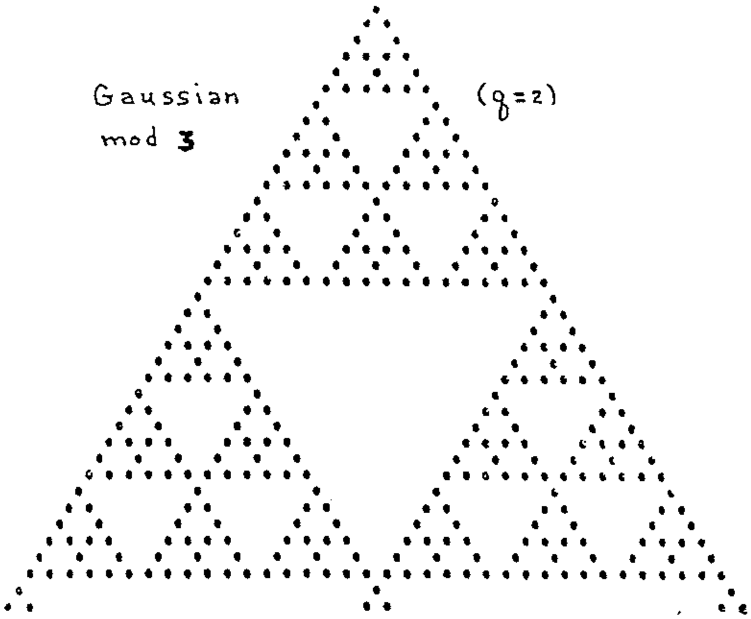

Figure 48

Figure 49

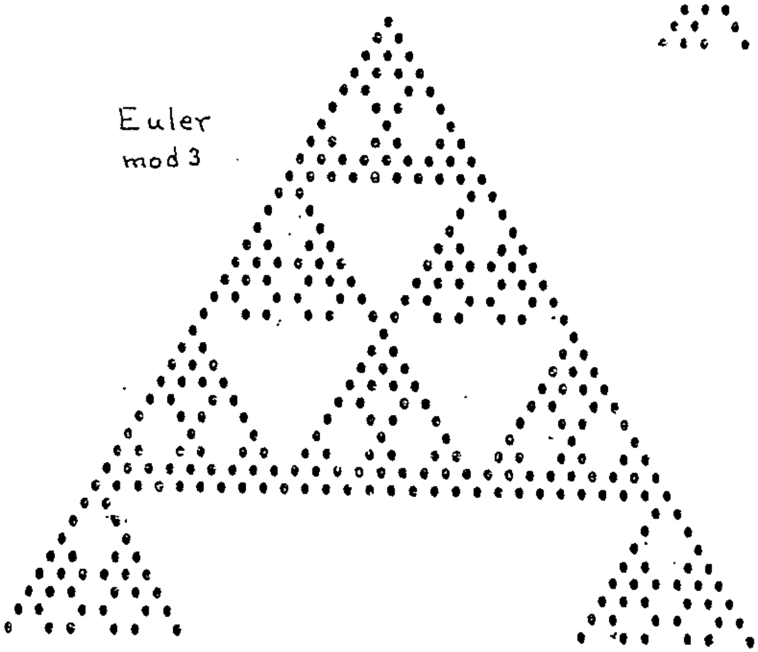

Figure 50

4.4 GEOMETRIC ARRANGEMENTS OF BINOMIAL COEFFICIENTS WHOSE PRODUCTS YIELD PERFECT POWERS

V.E. Hoggatt and W. Hansell [208] discovered an interesting property which can be described as follows. Let $\binom{n}{r}$ be an interior element of the Pascal triangle, and let this element be located at the center of a regular hexagon whose vertices consists of the neighboring binomial coefficients

$$\binom{n-1}{m-1}, \binom{n}{m+1}, \binom{n+1}{m}, \binom{n-1}{m}, \binom{n}{m-1}, \binom{n+1}{m+1}.$$

They showed that the product of these six binomial coefficients is a perfect square (of some number), and further that the product of the first three equals the product of the last three. Using a notation introduced in [384], denote the first three elements by O's and the last three by X's. Then the coefficients listed above form the figure shown in Figure 51a, and a specific example is shown in Figure 51b for the case n=8, m=2, where $7 \cdot 56 \cdot 36 = 8 \cdot 21 \cdot 84$.

V.E. Hoggatt and G.L. Alexanderson [196] extended this property to the case of multinomial coefficients. And H.W. Gould [157] showed there are arrangements of eight, and of ten, binomial coefficients in the Pascal triangle which also have this property; two of these are shown in Figure 51c, d.

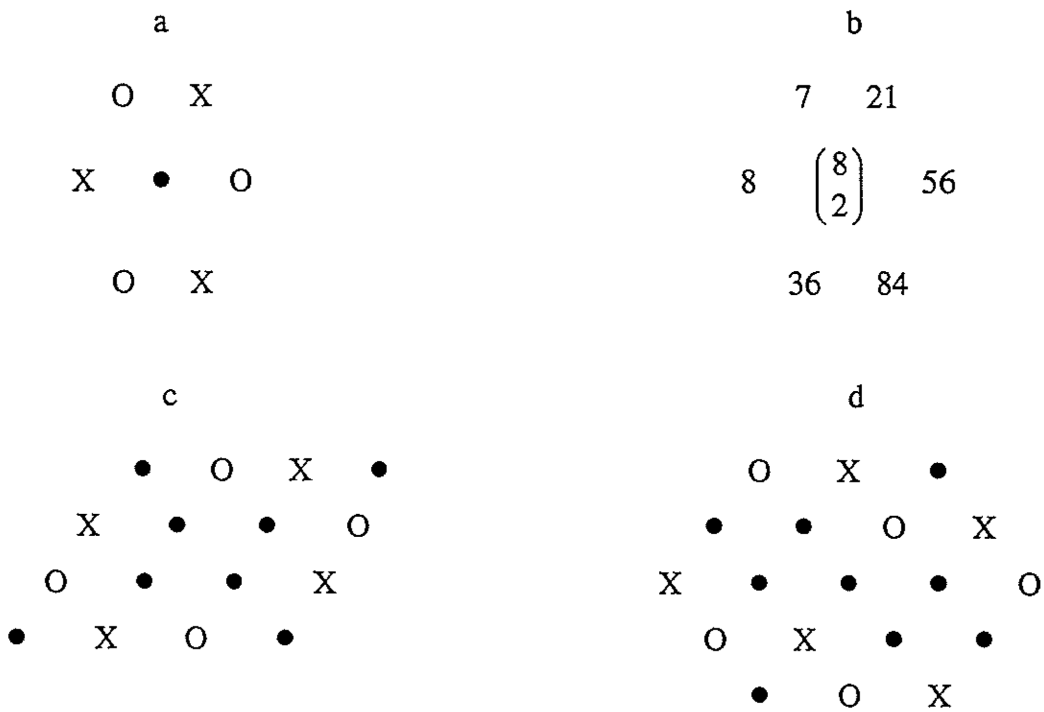

Figure 51

Further, if this property, for a given arrangement of binomial coefficients, does not depend on the choice of the values of m and n, then such a figure is said to be a perfect square pattern (PSP). Z. Usiskin [384] discussed the problem of conditions for the existence of PSP's, and showed that if the figure contains an even number of elements in each row and in its main diagonals then it belongs to the class of PSP's. As a result of this theorem, he constructed new PSP figures, shown in Figure 52a-d. He also discussed arrangements of binomial coefficients, the product of which is a third power. Thus, in Figure 52e, composed of three rhomboids, the products of the elements at the nodes denoted by O, X, Y are equal among themselves, and the product of all elements is a cube. This property extends also to the case of n^{th} powers, if the figure is composed of rhomboids whose sides consist of n elements.

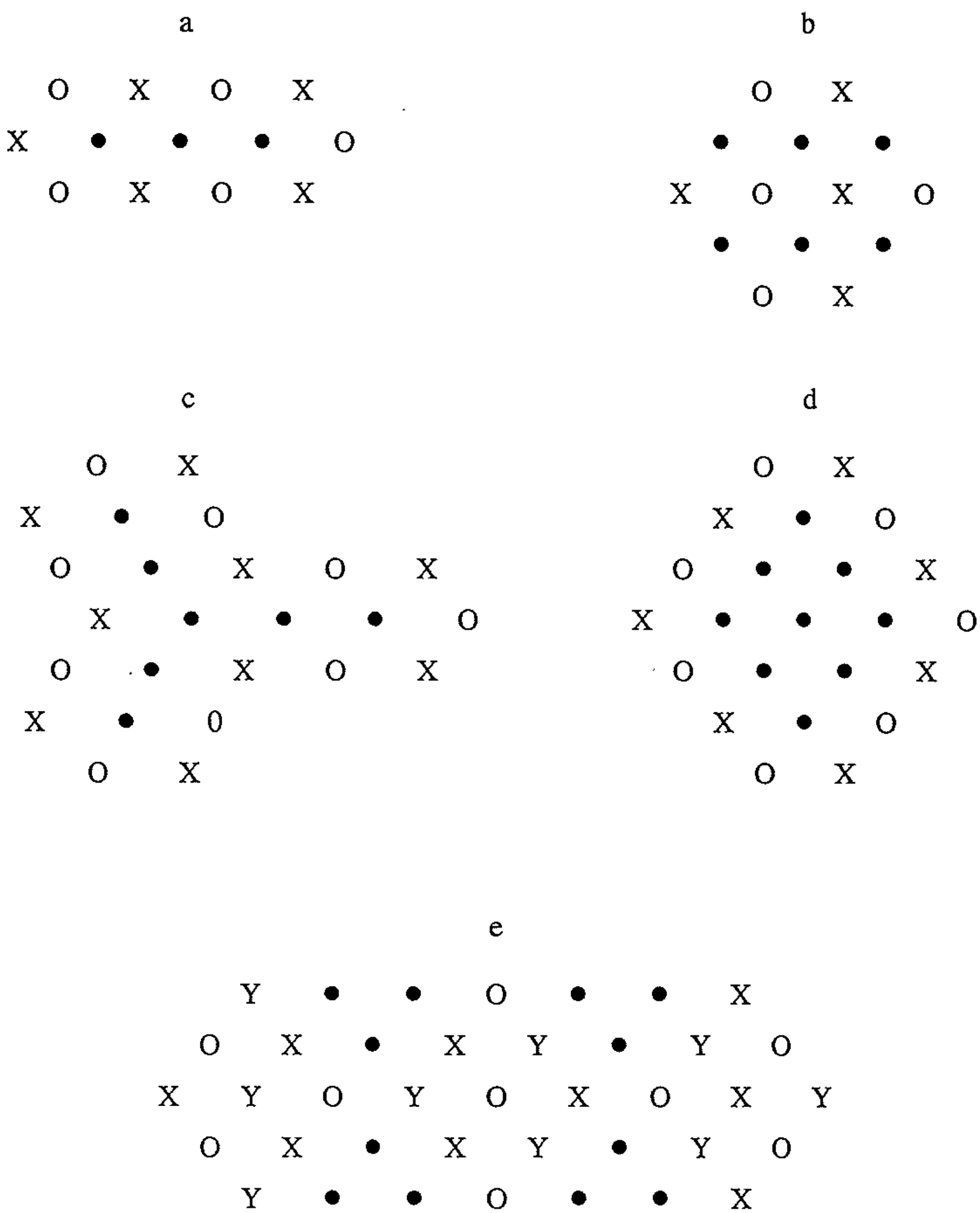

Figure 52

C.L. Moore [284] extended the results of [208] and established that for the binomial coefficients forming a regular hexagon whose sides lie along the horizontal rows and main diagonals of the Pascal triangle, and which contain $j+1$ elements, the product yields a perfect square if j is odd. Similar results are discussed by A.K. Gupta in [165].

C.T. Long [260], and C.T. Long and V.E. Hoggatt [264], using a lemma they proved, generalized known results and presented new geometric figures in the lattice of points forming the Pascal triangle (denoted by * in Figures 53, 54). They proved that the products of the binomial coefficients at the points of these figures form perfect squares. The assertion of their lemma is as follows: the product of the binomial coefficients at the vertices of the pair of parallelograms oriented as shown in Figure 53, is a perfect square. (If the parallelograms partially overlap then, in the total, vertices must be taken into account twice, or excluded, in the corresponding product.) Using the lemma, they proved a theorem which applies to many interesting geometric figures, some of which are shown in Figure 54. In effect, the theorem says that the products of the binomial coefficients located at the points on the contours of these figures, are perfect squares.

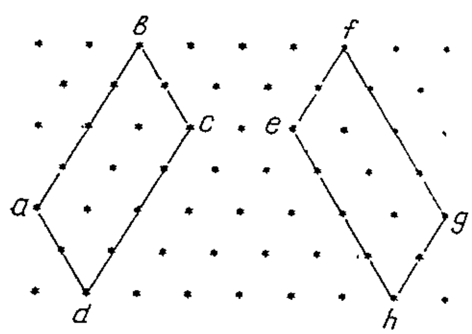

Figure 53

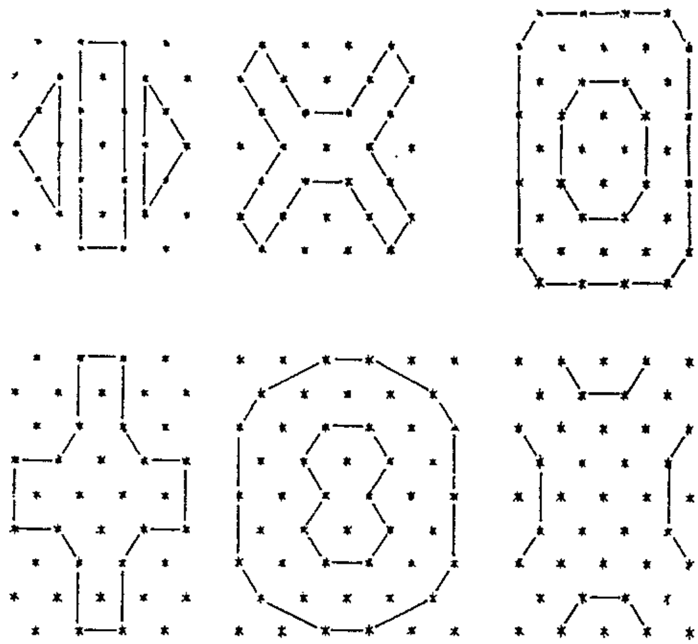

Figure 54

B. Gordon, D. Sato, and E. Straus [152] discussed P_k-sets of vertex points of the Pascal triangle lattice. They proved that the products of the binomial coefficients located at these points were k^{th} powers, and gave a description of these P_k-sets and a method for determining the minimum of f(k), where f(k) is the cardinal number of all P_k-sets. They also consider the problem of extending the results obtained to the case of the multinomial coefficients.