A PRIMER ON THE FIBONACCI SEQUENCE: PART I

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SIMPLE PROPERTIES OF THE FIBONACCI SEQUENCE AND MATHEMATICAL INDUCTION

The proofs of Fibonacci identities serve as very suitable examples of certain techniques encountered in a first course in college algebra. With this in mind, it is the intention of this series of articles to introduce the beginner to the Fibonacci sequence and a few techniques in proving some number theoretic identities as well as furnishing examples of well-known methods of proof such as mathematical induction. The collection of proofs that will be given in this series may serve as a source of elementary examples for classroom use.

1. SOME SIMPLE PROPERTIES OF THE FIBONACCI SEQUENCE

By observation of the sequence 1, 1, 2, 3, 5, 8, ..., it is easily seen that each term is the sum of the two preceding terms. In mathematical language, we define this sequence by

(A) $$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

for all integers n. The first few Fibonacci numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

The Lucas numbers L_n satisfy the same recurrence relation but have different starting values, namely,

(B) $$L_1 = 1, \quad L_2 = 3, \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n$$

for all integers n. The first few Lucas numbers are:

1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, ...

The following are some simple formulas which are called Fibonacci Number Identities or Lucas Number Identities for $n \geq 1$.

(1) $$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$$

(2) $$L_1 + L_2 + L_3 + \ldots + L_n = L_{n+2} - 3$$

(3) $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

(4) $$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}$$

(5) $$L_n = F_{n+1} + F_{n-1}$$

(6) $$5 \cdot F_n = L_{n+1} + L_{n-1}$$

(7) $$F_{2n+1} = F_{n+1}^2 + F_n^2$$

(8) $$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$

(9) $$F_{2n} = F_n L_n$$

(10) $$F_{n+p+1} = F_{n+1}F_{p+1} + F_n F_p$$

(11) $$F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n F_{n+1}$$

(12) $$L_n^2 - 5 \cdot F_n^2 = 4(-1)^n$$

(13) $$F_{-n} = (-1)^{n+1} F_n$$

(14) $$L_{-n} = (-1)^n L_n$$

2. MATHEMATICAL INDUCTION

Any proofs of the foregoing identities ultimately depend upon the postulate of complete mathematical induction.

First one has a formula involving an integer n. For some values of n the formula has been seen to be true. This may be one, two, or, say, twenty times. Now the excitement sets in. Is it true for all positive n? One may prove this by appealing to mathematical induction, whose three phases are:

A. Statement P(1) is true by trial. (If you can't find a first true case...why do you think it's true for any n, let alone all n? Here you need some true cases to start with.) An example of a statement P(n) is $1 + 2 + 3 + \ldots + n = n(n + 1)/2$. It is simple to see that P(1) is true; that is, that $1 = 1(1 + 1)/2$.

B. The truth of statement P(k) logically implies the truth of P(k+1). In other words, <u>if</u> P(k) is true, <u>then</u> P(k+1) is true. This step is commonly referred to as the inductive transition. The actual method used to prove this implication may vary from simple algebra to very profound theory.

C. The statement that 'The proof is complete by mathematical induction'.

As an example, let us prove identity (1). Recall from (A) that $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Statement P(n) is

P(n): $$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1.$$

A. P(1) is true, since $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, so that $F_1 = 2 - 1 = F_3 - 1$.

B. Assume that P(k) is true; that is,

P(k): $$F_1 + F_2 + F_3 + \ldots + F_k = F_{k+2} - 1.$$

From this we will show that the truth of P(k) <u>demands</u> the truth of P(k+1), which is

P(k+1): $$(F_1 + F_2 + F_3 + \ldots + F_k) + F_{k+1} = F_{k+3} - 1.$$

Since we assume P(k) is true, we may therefore assume that, in P(k+1), we may replace $(F_1 + F_2 + F_3 + \ldots + F_k)$ by $(F_{k+2} - 1)$. That is, P(k+1) may be rewritten as

$$(F_{k+2} - 1) + F_{k+1} = (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1.$$

This is now clearly true from (A), which for n = k + 1 becomes $F_{k+3} = F_{k+2} + F_{k+1}$.

C. The proof is complete by mathematical induction.

3. A BIT OF THEORY (Cramer's Rule)

Next we need a bit of determinant theory. Given a second order determinant, by definition

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The following theorem can be proved using the definition and simple algebra.

<u>THEOREM</u>: For any real numbers x and y,

$$\begin{vmatrix} ax + by & b \\ cx + dy & d \end{vmatrix} = \begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = xD$$

Suppose a system of two simultaneous equations possesses a unique solution (x_0, y_0); that is,

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

is satisfied if and only if $x = x_0$ and $y = y_0$. This is specified by saying that

$$\text{(C)} \qquad \begin{cases} ax_0 + by_0 = e \\ cx_0 + dy_0 = f \end{cases}$$

are true statements. From our definition of determinant and the theorem, for $x = x_0$ and $y = y_0$, we may write

$$x_0 D = \begin{vmatrix} ax_0 + by_0 & b \\ cx_0 + dy_0 & d \end{vmatrix} = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

where we used (C) to rewrite the determinant. Thus,

$$x_0 = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y_0 = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad D \neq 0,$$

which is Cramer's Rule.

Algebraically, we see that we must take $D \neq 0$. Geometrically, $D = 0$ if the graphs of the two linear equations are distinct parallel lines (inconsistent equations) or if the graphs are the same line (redundant equations). If the graphs (lines) are not parallel or coincident, then the common point of intersection is (x_0, y_0).

4. A CLEVER DEVICE IN ACHIEVING AN INDUCTIVE TRANSITION

Now, to apply our theory in a proof by mathematical induction, suppose we write two examples of definition (A), for $n = k$ and for $n = k - 1$, obtaining $F_{k+2} = F_{k+1} + F_k$ and $F_{k+1} = F_k + F_{k-1}$, and then let us try to solve the pair of simultaneous linear equations,

(D)
$$\begin{cases} F_{k+2} = xF_{k+1} + yF_k \\ F_{k+1} = xF_k + yF_{k-1}. \end{cases}$$

This is silly because we know that the answer is $x_0 = 1$ and $y_0 = 1$, but using Cramer's Rule we note:

(E)
$$y_0 = 1 = \frac{\begin{vmatrix} F_{k+1} & F_{k+2} \\ F_k & F_{k+1} \end{vmatrix}}{\begin{vmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{vmatrix}} = \frac{F_{k+1}^2 - F_kF_{k+2}}{F_{k+1}F_{k-1} - F_k^2}$$

Let us now use Mathematical Induction to prove Identity (3) which is

P(n): $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

If we note that $F_0 = 0$ is valid, then P(1): $F_2F_0 - F_1^2 = (-1)^1 = -1$ is true, and part A is done.

Suppose P(k) is true. From (E) and the induction hypothesis,

$$1 = \frac{F_{k+1}^2 - F_kF_{k+2}}{F_{k+1}F_{k-1} - F_k^2} = \frac{F_{k+1}^2 - F_kF_{k+2}}{(-1)^k}$$

so that P(k+1): $F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}$ is indeed true! Thus part B is done. The proof is complete by mathematical induction, and part C is done.

All of the other identities given in this article can be proved by mathematical induction. To test your understanding, you should prove several of them.

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Problem B-1 (Proposed by I. D. Ruggles) Show that the sum of twenty consecutive Fibonacci numbers is divisible by F_{10}.

B-2 (Proposed by Verner E. Hoggatt, Jr.) Show that

$$u_{n+1} + u_{n+2} + \cdots + u_{n+10} = 11u_{n+7}$$

holds for generalized Fibonacci numbers such that $u_{n+2} = u_{n+1} + u_n$, where $u_1 = p$ and $u_2 = q$.

B-3 (Proposed by J. E. Householder) Show that F_{n+24} is congruent to F_n (modulo 9), where F_n is the nth Fibonacci number.