When we speak of a Fibonacci matrix, we shall have in mind matrices which contain members of the Fibonacci sequence as elements. An example of a Fibonacci matrix is the $Q$ matrix as defined by King in [1], where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The determinant of $Q$ is $-1$, written $\det Q = -1$. From a theorem in matrix theory, $\det Q^n = (\det Q)^n = (-1)^n$. By mathematical induction, it can be shown that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

so that we have the familiar Fibonacci identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ by finding $\det Q^n$.

The lambda function of a matrix was studied extensively in [2] by Fenton S. Stancliff, who was a professional musician. Stancliff defined the lambda function $\Lambda(M)$ of a matrix $M$ as the change in the value of the determinant of $M$ when the number one is added to each element of $M$. If we define $(M + k)$ to be that matrix formed from $M$ by adding any given number $k$ to each element of $M$, we have the identity

$$(1) \quad \det (M + k) = \det M + k\Lambda(M).$$

For an example, the determinant $\Lambda(Q^n)$ is given by

$$\Lambda(Q^n) = \begin{vmatrix} F_{n+1} + 1 & F_n + 1 \\ F_n + 1 & F_{n-1} + 1 \end{vmatrix} - \det Q^n$$

$$= (F_{n+1}F_{n-1} - F_n^2) + (F_{n-1} + F_{n+1} - 2F_n) - \det Q^n$$

$$= F_{n-3}$$

which follows by use of Fibonacci identities. Now if we add $k$ to each element of $Q^n$, the resulting determinant is
\[
\begin{vmatrix}
F_{n+1} + k & F_n + k \\
F_n + k & F_{n-1} + k
\end{vmatrix}
= \det Q^n + k \cdot F_{n-3}
\]

However, there are more convenient ways to evaluate the lambda function.

For simplicity, we consider only $3 \times 3$ matrices.

**THEOREM.** For the given general $3 \times 3$ matrix $M$, $\lambda(M)$ is expressed by either of the expressions (2) or (3). For

\[
M = \begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & j
\end{pmatrix},
\quad
\lambda(M) = \begin{vmatrix}
a + e - b - d & b + f - c - e \\
d + h - g - e & e + j - h - f
\end{vmatrix},
\]

(2)

\[
\lambda(M) = \begin{vmatrix}
l & b & c \\
l & e & f \\
l & h & j
\end{vmatrix} + \begin{vmatrix}
a & l & c \\
d & 1 & f \\
g & 1 & j
\end{vmatrix} + \begin{vmatrix}
a & b & l \\
d & e & 1 \\
g & h & 1
\end{vmatrix},
\]

(3)

Proof: This is made by direct evaluation and a simple exercise in algebra.

An application of the lambda function is in the evaluation of determinants.

Whenever there is an obvious value of $k$ such that $\det (M + k)$ is easy to find, we can use equation (1) advantageously. To illustrate this fact, consider

\[
M = \begin{pmatrix}
1000 & 998 & 554 \\
990 & 988 & 554 \\
675 & 553 & 554
\end{pmatrix}.
\]

We notice that, if we add $k = -554$ to each element of $M$, then $\det (M + k) = 0$ since every element in the third column will be zero. From (2) we compute

\[
\lambda(M) = \begin{vmatrix}
0 & 10 \\
-120 & 435
\end{vmatrix} = 1200;
\]

and from (1) we find that $0 = \det M + (-554)(1200)$, so that $\det M = (554)(1200)$.

Readers who enjoy mathematical curiosities can create determinants which are not changed in value when any given number $k$ is added to each element, by writing any matrix $D$ such that $\lambda(D) = 0$.

**LEMMA:** If two rows (or columns) of a matrix $D$ have a constant difference between corresponding elements, then $\lambda(D) = 0$.

Proof: Evaluate $\lambda(D)$ directly, by (2) or (3).

For example, we write the matrix $D$, where corresponding elements in the first and second rows differ by 4, such that

\[
\det D = \begin{vmatrix}
1 & 2 & 3 \\
5 & 6 & 7 \\
4 & 9 & 8
\end{vmatrix}.
\]

\[
\det D = \begin{vmatrix}
1 + k & 2 + k & 3 + k \\
5 + k & 6 + k & 7 + k \\
4 + k & 9 + k & 8 + k
\end{vmatrix} = 24.
\]
Now, we consider other Fibonacci matrices. Suppose that we want to write a Fibonacci matrix $U$ such that $\det U = F_n$. We can write $F_n = F_1 F_2 F_n$ for any $n$, and for some $n$ we will also have other Fibonacci factorizations. Hence, for

$$ U = \begin{pmatrix} F_1 & F_0 & F_0 \\ F_m & F_2 & F_0 \\ F_k & F_p & F_n \end{pmatrix}, $$

$\det U = F_n$ where $F_0 = 0$. If we choose $m = k = 3$ and $p = 2$, we find that $\lambda(U) = 0$. If we choose $m = 1$ or $2$, $k = 1$ or $2$, and let $p$ be an arbitrary integer, then $\lambda(U) = F_n$.

A more elegant way to write such a matrix was suggested by Ginsburg in [3], who wrote a matrix with the same first two columns as $U$ below but with all elements in the third column equal to $n$ and thus with determinant value $n$. We can write $F_m = \det U$, where

$$ U = \begin{pmatrix} F_{2p} & F_{2p+1} & F_m \\ F_{2p+1} & F_{2p+2} & F_m \\ F_{2p+2} & F_{2p+3} & F_m \end{pmatrix} $$

We have, using equation (3),

$$ \lambda(U) = \begin{vmatrix} 1 & F_{2p+1} & F_m \\ 1 & F_{2p+2} & F_m \\ 1 & F_{2p+3} & F_m \end{vmatrix} + \begin{vmatrix} F_{2p} & 1 & F_m \\ F_{2p+1} & 1 & F_m \\ F_{2p+2} & 1 & F_m \end{vmatrix} + \begin{vmatrix} F_{2p} & F_{2p+1} & 1 \\ F_{2p+1} & F_{2p+2} & 1 \\ F_{2p+2} & F_{2p+3} & 1 \end{vmatrix} $$

$$ = 0 + 0 + 1 = 1 $$

If we let $k = F_{m-1}$, from (1) we see that $\det (U + F_{m-1}) = F_m + (F_{m-1}) (1) = F_{m+1}$.

Notice the possibilities for finding Fibonacci identities using the lambda function and evaluation of determinants. As a brief example, we let $k = F_n$ and consider $\det (q^n + F_n)$, which gives us

$$ \begin{vmatrix} F_{n+1} + F_n & F_n + F_n \\ F_n + F_n & F_{n-1} + F_n \end{vmatrix} = \det q^n + F_n \lambda(q^n) $$

or
\[
\begin{vmatrix}
F_{n+2} & 2F_n \\
2F_n & F_{n+1}
\end{vmatrix} = (-1)^n + F_n F_{n-3}
\]
so that

\[
4F_n^2 = F_{n+2}F_{n+1} - F_n F_{n-3} + (-1)^{n+1}.
\]

As a final example of a Fibonacci matrix, we take the matrix \( R \), given by

\[
R = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix},
\]

which has been considered by Brennan [4]. It can be shown that

\[
R^n = \begin{pmatrix}
F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\
2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\
F_n^2 & F_nF_{n+1} & F_{n+1}^2
\end{pmatrix}
\]

by mathematical induction. The reader may verify that by equation (2) and by Fibonacci identities,

\[
\lambda(R^n) = (-1)^n(F_{n-1}^2 - F_{n-3}F_{n-2}),
\]

the center element of \( R^{n-2} \) multiplied by \((-1)^n\).

REFERENCES

2. From the unpublished notes of Fenton S. Stancliff.
4. From the unpublished notes of Terry Brennan.

Problem B-24 (Proposed by Brother Alfred Brousseau): It is evident that the determinant below has a value of zero. Prove that if the same quantity \( k \) is added to each element, the value becomes \((-1)^{n-1}k\).