SOME NEW FIBONACCI IDENTITIES

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In this paper, some new Fibonacci and Lucas identities are generated by matrix methods.

The matrix

\[
R = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix}
\]

satisfies the matrix equation \( R^3 - 2R^2 - 2R + I = 0 \). Multiplying by \( R^n \) yields

\[
R^{n+3} - 2R^{n+2} - 2R^{n+1} + R^n = 0.
\]

(1)

It has been shown by Brennan [1] and appears in an earlier article [2] that

\[
R^n = \begin{pmatrix}
F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\
2F_nF_{n-1} & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\
F_n^2 & F_nF_{n+1} & F_{n+1}^2
\end{pmatrix},
\]

(2)

where \( F_n \) is the \( n \)th Fibonacci number.

By the definition of matrix addition, corresponding elements of \( R^{n+3} \), \( R^{n+2} \), \( R^{n+1} \), and \( R^n \) must satisfy the recursion formula given in Equation (1). That is, for example,

\[
F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0,
\]

\[
F_{n+3}F_{n+4} - 2F_{n+2}F_{n+3} - 2F_{n+1}F_{n+2} + F_nF_{n+1} = 0.
\]

Returning again to \( R^3 - 2R^2 - 2R + I = 0 \), this equation can be rewritten as

\[
(R + I)^3 = R^3 + 3R^2 + 3R + I = 5R(R + I).
\]

In general, by induction, it can be shown that

\[
R^p(R + I)^{2n+1} = 5^nR^n + p(R + I).
\]

(3)
Equating the elements in the first row and third column of the above matrices, by means of Equation (2), we obtain

\[
\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i+p}^2 = 5^n F_{2(n+p)+1} .
\]

(4)

It is not difficult to show that the Lucas numbers and members of the Fibonacci sequence have the relationship

\[
L_n^2 - 5F_n^2 = (-1)^R 4 .
\]

Since also

\[
\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+p} = 0,
\]

we can derive the following sum of squares of Lucas numbers,

\[
\sum_{i=0}^{2n+1} \binom{2n+1}{i} L_{i+p}^2 = 5^n L_{2(n+p)+1} .
\]

by substitution of the preceding two identities in Equation (4).

Upon multiplying Equation (3) on the right by \((R+I)\), we obtain

\[
R^P(R+I)^{2n+2} = 5^n R^{n+p}(R+I)^2 .
\]

Then, using the expression for \(R^n\) given in Equation (2) and the identity

\[
L_k = F_{k-1} + F_{k+1} ,
\]

we find that

\[
(R^{n+1} + R^n)(R+I) = \begin{pmatrix}
F_{2n-1} & F_{2n} & F_{2n+1} \\
2F_{2n} & 2F_{2n+1} & 2F_{2n+2} \\
F_{2n+1} & F_{2n+2} & F_{2n+3}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 \\
0 & 2 & 2 \\
1 & 1 & 2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
L_{2n} & L_{2n+1} & L_{2n+2} \\
2L_{2n+1} & 2L_{2n+2} & 2L_{2n+3} \\
L_{2n+2} & L_{2n+3} & L_{2n+4}
\end{pmatrix} .
\]
Finally, by equating the elements in the first row and third column of the matrices of Equation (5), we derive the two identities

\[ \sum_{i=0}^{2n+2} \binom{2n+2}{i} R_{i}^{2} = 5^{n}L_{2(n+p)+2}, \]

\[ \sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i+p}^{2} = 5^{n+1}L_{2(n+p)+2}. \]

By similar steps, by equating the elements appearing in the first row and second column of the matrices of Equations (3) and (5), we can write the additional identities,

\[ \sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i-1+p}F_{i+p} = 5^{n}F_{2(n+p)} \]

\[ \sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i-1+p}F_{i+p} = 5^{n}L_{2(n+p)+1}. \]

REFERENCES

1. From the unpublished notes of Terry Brennan.

Editorial Comment

Form the \((n+1) \times (n+1)\) matrix \(P_n\) with Pascal's triangle appearing on and below its secondary diagonal, e.g.,

\[
P_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 1 & 3 & 6 \\
0 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Surely the reader will see \(R = P_2\) and matrix \(P_1\) very like \(Q\) in the lower left.

The element occurring in the lower left corner of \(P_n^k\) is always \(P_n^0\), and the characteristic equation of \(P_n\) has the Binomial coefficients appearing, leading to identities such as described in the next article.