FIBONACCI NUMBERS AND GENERALIZED BINOMIAL COEFFICIENTS

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1. INTRODUCTION

The first time most students meet the binomial coefficients is in the expansion

\[(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j, \quad n \geq 0,\]

where

\[\binom{n}{m} = 0 \quad \text{for} \quad m > n, \quad \binom{n}{n} = \binom{n}{0} = 1, \quad \text{and} \]

\[\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \quad 0 < m < n.\]  

Consistent with the above definition is

\[\binom{n}{m} = \frac{n(n-1) \cdots 2 \cdot 1}{m(m-1) \cdots 2 \cdot 1} \left(\frac{n-m}{n-m-1} \cdots 2 \cdot 1\right) = \frac{n!}{m!(n-m)!}, \]

where

\[n! = n(n-1)(n-2) \cdots 2 \cdot 1 \quad \text{and} \quad 0! = 1.\]

Given the first lines of Pascal's arithmetic triangle one can extend the table to the next line using directly definition (2) or the recurrence relation (1).

We now can see just how the ordinary binomial coefficients \(\binom{n}{m}\) are related to the sequence of integers 1, 2, 3, ..., k, .... Let us generalize this observation using the Fibonacci sequence.

2. THE FIBONOMIAL COEFFICIENTS

Let the Fibonacci coefficients (which are a special case of the generalized binomial coefficients) be defined as

\[\binom{n}{m} = \frac{F_n F_{n-1} \cdots F_{n-m}}{(F_m F_{m-1} \cdots F_{m-n})(F_{n-m} F_{n-m-1} \cdots F_{n-1})}, \quad 0 < m < n,\]

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and \([ n \choose 0 ] = [ n \choose n ] = 1\), where \(F_n\) is the \(n\)th Fibonacci number, defined by
\[
F_n = F_{n-1} + F_{n-2}, \quad F_1 = F_2 = 1.
\]

We next seek a convenient recurrence relation, like (1) for the ordinary binomial coefficients, to get the next line from the first few lines of the Fibonomial triangle.

To find two such recurrence relations we recall the \(Q\)-matrix,
\[
Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},
\]
for which it is easily established by mathematical induction that
\[
Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad n \geq 0.
\]

The laws of exponents hold for the \(Q\)-matrix so that \(Q^n = Q^{m}\cdot Q^{n-m}\). Thus
\[
\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n-m+1} & F_{n-m} \\ F_{n-m} & F_{n-m-1} \end{pmatrix}
\]
\[
= \begin{pmatrix} F_{m+1}F_{n-m+1} + F_mF_{n-m} & F_{m+1}F_{n-m} + F_mF_{n-m-1} \\ F_mF_{n-m+1} + F_{m-1}F_{n-m} & F_mF_{n-m} + F_{m-1}F_{n-m-1} \end{pmatrix}
\]
yielding, upon equating corresponding elements,

(A) \(F_n = F_{m+1}F_{n-m} + F_mF_{n-m-1}\) \hspace{1cm} \text{(upper right)},

(B) \(F_n = F_mF_{n-m+1} + F_{m-1}F_{n-m}\) \hspace{1cm} \text{(lower left)}.

Define \(C\) so that
\[
\begin{pmatrix} n \\ m \end{pmatrix} = \frac{F_nF_{n-1}\cdots F_{2}F_1}{(F_mF_{m-1}\cdots F_2F_1)(F_{n-m}F_{n-m-1}\cdots F_{2}F_1)} = F_n C^n.
\]

With \(C\) defined above, then
\[
\begin{pmatrix} n-1 \\ m \end{pmatrix} = F_{n-m} C \quad \text{and} \quad \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} = F_m C.
\]
Returning now to identity (A), we may write for $C \neq 0$,

$$F_nC = F_{m+1}(F_{n-m}C) + F_{n-m-1}(F_mC)$$

but by the definition of $C$, we have derived

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$  \hspace{1cm} (D)

Similarly, using identity (B), one can establish

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m-1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$  \hspace{1cm} (E)

It is thus now easy to establish by mathematical induction that if the Fibonomial coefficients are integers for an integer $n$ ($m = 0, 1, \ldots, n$), then they are integers for an integer $n+1$ ($m = 0, 1, 2, \ldots, n+1$).

Recalling

$$L_m = F_{m+1} + F_{m-1}$$

and adding (D) and (E) yields

$$2\begin{bmatrix} n \\ m \end{bmatrix} = L_m\begin{bmatrix} n-1 \\ m \end{bmatrix} + L_{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix},$$  \hspace{1cm} (3)

where $L_m$ is the $m$th Lucas number. From (3) it is harder to show that the Fibonomial coefficients are integers.

3. THE FIBONOMIAL TRIANGLE

Pascal's arithmetic triangle

$$\begin{array}{ccccccc}
 & & & & & 1 & \\
 & & & & 1 & 1 & \\
 & & & 1 & 2 & 1 & \\
 & & 1 & 3 & 3 & 1 & \\
 & 1 & 4 & 6 & 4 & 1 & \\
\end{array}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix} \begin{bmatrix} n \\ 1 \end{bmatrix} \cdots \begin{bmatrix} n \\ m \end{bmatrix} \cdots \begin{bmatrix} n \\ n-1 \end{bmatrix} \begin{bmatrix} n \\ n \end{bmatrix}$$

has been the subject of many studies and has always generated interest. We note here to get the next line we merely use the recurrence relation (1). Here we point out two interpretations, one of which shows a direction for
generalization. The usual first meeting with Pascal's triangle lies in the
binomial theorem expansion of $(x + y)^n$. However, of much interest to us is
the difference equation interpretation. The difference equation satisfied by
$n^0$ is

$$(n + 1)^0 - n^0 = 0,$$

while the difference equation satisfied by $n$ is

$$(n + 2) - 2(n + 1) + n = 0.$$  

For $n^2$ the difference equation is

$$(n + 3)^2 - 3(n + 2)^2 + 3(n + 1)^2 - n^2 = 0.$$  

Certainly one notices the binomial coefficients with alternating signs
appearing here. In fact,

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (n + m + 1 - j)^m = 0.$$  

It is this connection with the difference equations for the powers of the
integers that leads us naturally to the Fibonomial triangle.

Similar to the difference equation coefficients array for the powers of
the positive integers which results in Pascal's arithmetic triangle with
alternating signs, there is the Fibonomial triangle made up of the Fibonomial
coefficients, with doubly alternated signs. We first write down the Fibonomial
triangle for the first six levels.

1
1 1
1 1 1
1 2 2 1
1 3 6 3 1
1 5 15 15 5 1
1 8 40 60 40 8 1

The top line is the zeroth row and the coefficients in the difference equation
satisfied by $F^k_n$ are the numbers in the $(k + 1)$st row. Of course, we can get
the next line of Fibonomial coefficients by using our recurrence relation (D),
\[
\begin{bmatrix}
n \\
m
\end{bmatrix} = F_{m+1} \begin{bmatrix}
n - 1 \\
m
\end{bmatrix} + F_{n-m-1} \begin{bmatrix}
n - 1 \\
m - 1
\end{bmatrix}, \quad 0 < m < n.
\]

We now rewrite the Fibonomial triangle with appropriate signs so that the rows are properly signed to be the coefficients in the difference equation satisfied by \( F_n^k \).

\[
\begin{array}{cccc}
F_0^0 & 1 & 1 & -1 \\
F_1^1 & 1 & -1 & -1 \\
F_1^2 & 1 & -2 & -2 & +1 \\
F_2^3 & 1 & -3 & -6 & +3 & +1 \\
F_3^4 & 1 & -5 & -15 & +15 & +5 & -1 \\
F_4^5 & 1 & -8 & -40 & +60 & +40 & -8 & -1 \\
\end{array}
\]

Thus, from the above we may write

\[
F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0
\]

and

\[
F_{n+5}^4 - 5F_{n+4}^4 - 15F_{n+3}^4 + 15F_{n+2}^4 + 5F_{n+1}^4 - F_n^4 = 0.
\]

The auxiliary polynomial for the difference equation satisfied by \( F_n^m \) is

\[
\sum_{h=0}^{m+1} \begin{bmatrix} m+1 \\ h \end{bmatrix} (-1)^h (h+1)/2 \ x^{m+1-h}
\]

which shows that the sign pattern of doubly alternating signs persists. (See [1], [2].) (Further generalizations given in the original paper are here omitted.)

REFERENCES

