THE GOLDEN RATIO: COMPUTATIONAL CONSIDERATIONS

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1. INTRODUCTION

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel"--so wrote Kepler (1571-1630)[1].

The famous golden section involves the division of a given line segment into mean and extreme ratio, i.e., into two parts a and b, such that $a/b = b/(a + b)$, $a < b$. Setting $x = b/a$ we have $x^2 - x - 1 = 0$. Let us designate the positive root of this equation by $\phi$ (the golden ratio). Thus

$$\phi^2 - \phi - 1 = 0 .$$

Since the roots of (1) are $\phi = (1 + \sqrt{5})/2$ and $-1/\phi = (1 - \sqrt{5})/2$ we may write Binet's formula [2] for the nth Fibonacci number in the form

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

2. POWERS OF THE GOLDEN RATIO

Returning to (1), let us "solve for $\phi^2$" by writing

$$\phi^2 = \phi + 1 .$$

Multiplying both members by $\phi$, we get $\phi^3 = \phi^2 + \phi = (\phi + 1) + \phi = 2\phi + 1$. Proceeding in a similar fashion we can write all of

$$\phi^3 = 2\phi + 1 ;$$
$$\phi^4 = 3\phi + 2 ;$$
$$\phi^5 = 5\phi + 3 .$$

This pattern suggests
(4) \[ \varphi^n = F_n \varphi + F_{n-1}, \quad n = 1, 2, 3, \ldots \]

To prove (4) by mathematical induction, we note that it is true for \( n = 1 \) and \( n = 2 \) (since \( F_0 = 0 \) by definition). Assume that \( \varphi^k = F_k \varphi + F_{k-1} \). Then

\[ \varphi^{k+1} = F_k \varphi^2 + F_{k-1} \varphi = F_k (\varphi + 1) + F_{k-1} \varphi = (F_k + F_{k-1}) \varphi + F_k = F_{k+1} \varphi + F_k, \]

which completes the proof.

The computational advantage of (4) over expansion of \( \left( \frac{1 + \sqrt{5}}{2} \right)^n \) by the binomial theorem is striking.

Dividing both members of (3) by \( \varphi \), we obtain

(5) \[ \frac{1}{\varphi} = \varphi - 1 \]

Thus \( 1/\varphi^2 = 1 - 1/\varphi = 1 - (\varphi - 1) = - (\varphi - 2) \). Using this result and (5),

\[ 1/\varphi^3 = 2/\varphi - 1 = 2(\varphi - 1) - 1 = 2\varphi - 3 \]

Proceeding in a similar fashion, one may write all of the following:

\[ \frac{1}{\varphi^2} = - \varphi - 2 \]

\[ \frac{1}{\varphi^3} = 2\varphi - 3 \]

\[ \frac{1}{\varphi^4} = - (3\varphi - 5) \]

Via induction, the reader may provide a painless proof of

(6) \[ \varphi^{-n} = (-1)^{n+1}(F_n \varphi - F_{n+1}), \quad n = 1, 2, 3, \ldots \]

3. A LIMIT OF FIBONACCI RATIOS

If we "solve" \( x^2 - x - 1 = 0 \) for \( x \) by writing \( x = 1 + 1/x \) and then consider the related recursion relation

(7) \[ x_1 = 1, \quad x_{n+1} = 1 + \frac{1}{x_n}, \]
Fibonacci numbers start popping out! We immediately deduce \( x_2 = 1 + 1/x_1 \)
\[ = 1 + 1/1 = 2/1, \quad x_3 = 3/2, \quad x_4 = 5/3, \quad x_5 = 8/5, \text{ etc.} \]
This suggests that \( x_n = F_{n+1}/F_n \).

Now suppose the sequence \( x_1, x_2, x_3, \ldots \) has a limit, say \( L \), as \( n \)
tends toward infinity. Then

\[
\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = L
\]

whence (7) yields \( L = 1 + 1/L \) or \( L = \emptyset \) since the \( x_i \) are positive. Indeed,
there are many ways of proving Kepler's observation that

\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \emptyset.
\]

For example, from (2)

\[
\frac{F_{n+1}}{F_n} = \frac{\phi^{n+1} - (-\phi)^{-n-1}}{\phi^n - (-\phi)^{-n}} = \frac{\phi - \frac{1}{(-\phi)^{n+1}\phi^n}}{1 - \frac{1}{(-\phi)^{n}\phi^n}} \to \emptyset
\]

as \( n \to \infty \) since \( \phi = (1 + \sqrt{5})/2 > 1 \) implies that the fractions involving \( \phi^n \)
approach 0 as \( n \to \infty \).

4. AN APPROXIMATE ERROR ANALYSIS

Just how accurate are the above approximations to the golden ratio?
Let us denote the exact error at the \( n \)th iteration by

\[
(9) \quad e_n = x_n - \emptyset
\]

The trick is to express \( e_{n+1} \) in terms of \( e_n \) using (7) and then to make
use of the identity

\[
(10) \quad \frac{1}{1 + w} = 1 - w + w^2 - w^3 + w^4 - \ldots, \quad w < 1
\]

(The latter may be discovered by dividing \( 1 \) by \( 1 + w \)).
Thus
\[ e_{n+1} = x_{n+1} - \phi \]
\[ = 1 + \frac{1}{x_n} - \phi \]
\[ = 1 - \phi + \frac{1}{e_n + \phi} \]
\[ = 1 - \phi + \frac{1}{\phi} \frac{1}{1 + (e_n/\phi)} \]
\[ = 1 - \phi + \frac{1}{\phi} [1 - (e_n/\phi) + (e_n/\phi)^2 - (e_n/\phi)^3 + \ldots] \]
\[ = -\frac{e_n}{\phi^2} + \frac{e_n}{\phi^3} - \frac{e_n}{\phi^4} + \ldots \]

since $1/\phi = \phi - 1$ by (5). However, the terms involving the higher powers of $e_n$ are quite small in comparison with the first term. Thus, following the customary practice of neglecting high order terms, we will approximate the error at the $(n + 1)st$ step by $e_{n+1} = -e_n \phi^{-2}$. Finally, we may note that

\[ e_2 = -e_1 \phi^{-2}, \quad e_3 = -e_2 \phi^{-2} = -e_1 \phi^{-4}, \quad e_4 = -e_3 \phi^{-4}, \quad \text{and, in general,} \]

\[ e_n = (-1)^{n+1} e_1 \phi^{-2(n-1)} \quad \ldots \quad (11) \]

If $x_1 = 1$, then $e_1 = 1 - \phi = -1/\phi$ by (9) and (5), making (11) become

\[ e_n = (-1)^n \phi^{-2(n-1)} \quad \ldots \quad (12) \]

(Sections 5 and 6 of the original paper are omitted here.)

REFERENCES
