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INTERPOLATION OF FOURIER TRANSFORMS ON SUMS OF FIBONACCI NUMBERS

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Our notation throughout this paper is that of [3]. Denote by $M(\mathbf{T})$ the Banach algebra of finite Borel measures on the circle group $\mathbf{T}$ and write $M_a(\mathbf{T})$ for those $\mu \in M(\mathbf{T})$ such that μ is absolutely continuous with respect to Lebesgue measure. Also $\mu \in M_d(\mathbf{T})$ if $\mu \in M(\mathbf{T})$ and μ is concentrated on a countable subset of $\mathbf{T}$.

The Fourier-Stieltjes transform $\hat{\mu}$ of the measure $\mu \in M(\mathbf{T})$ is defined by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} d\mu(\theta) \qquad (n \in \mathbb{Z})$$

where $\mathbb{Z}$ is the additive group of integers. In this paper we prove that there is an infinite subset $\mathcal{A}$ of the set of Fibonacci numbers $\mathcal{F}$ such that

$$M_a(\mathbf{T})^\wedge|_{\mathcal{A}+\mathcal{A}} \subset M_d(\mathbf{T})^\wedge|_{\mathcal{A}+\mathcal{A}};$$

i.e., on $\mathcal{A} + \mathcal{A} = \{a + b : a,b \in \mathcal{A}\}$ any transform of an absolutely continuous measure can be interpolated by the transform of a discrete measure. To prove this, we shall need the following interesting result of S. Burr [1]:

A natural number m is said to be defective if the Fibonacci sequence $\mathcal{F} = \{f_n\}_1^\infty$ does not contain a complete system of residues modulo m.

Theorem 1: (Burr) A number m is not defective if and only if m has one of the following forms:

$$5^k,\ 2 \cdot 5^k,\ 4 \cdot 5^k,$$
$$3^j \cdot 5^k,\ 6 \cdot 5^k,$$
$$7 \cdot 5^k,\ 14 \cdot 5^k, \text{ where } k \geq 0,\ j \equiv 1.$$

Let $\mathcal{S}^a$ denote the set of all integer accumulation points of $\mathcal{S} \subset \mathbb{Z}$ where the closure of $\mathcal{S}$ is taken with respect to the Bohr compactification $\overline{\mathbb{Z}}$ (see [3]) of $\mathbb{Z}$. In the sequel, we shall also need a theorem of Pigno and Saeki [6], which we now cite.

Theorem 2: The inclusion

$$M_a(\mathbf{T})^\wedge|_{\mathcal{S}} \subset M_d(\mathbf{T})^\wedge|_{\mathcal{S}}$$

obtains if and only if there is a measure $\mu \in M(\mathbf{T})$ such that $\hat{\mu}(\mathcal{S}) = 1$ and $\hat{\mu}(\mathcal{S}^a) = 0$.

We state and prove our main result:

Theorem 3: There is an increasing sequence $\mathcal{A} = \{f_n'\}_1^\infty$ of Fibonacci numbers such that

$$M_a(\mathbf{T})|_{\mathcal{A}+\mathcal{A}} \subset M_d(\mathbf{T})|_{\mathcal{A}+\mathcal{A}}.$$

Proof: By Theorem 1, we may find an increasing sequence $\mathcal{A} = \{f_n'\}_1^\infty$ of Fibonacci numbers such that

$$f_n' \equiv 5^n \pmod{2 \cdot 5^n} \text{ for all } n. \tag{1}$$

Now it follows from (1) that in the group of 5-adics (see [3, p. 107]) the only limit points of $\mathcal{A} + \mathcal{A}$ are 0 and each f_n'. Hence, to find the integer limit points of $\mathcal{A} + \mathcal{A}$ in $\overline{\mathbf{Z}}$ we need only look at 0 and each f_n'. Fix an f_n' and consider the arithmetic progression $\{2k + f_n' : k \in \mathbf{Z}\}$. This arithmetic progression is a neighborhood of f_n' in $\mathbf{Z}$ with the relative Bohr topology, and furthermore, $2k + f_n' = f_s' + f_t'$ is impossible because each member of $\mathcal{A}$ is odd [by (1)]. Thus, the only possible integer limit point of $\mathcal{A} + \mathcal{A}$ is 0.

Clearly the Dirac measure minus the Lebesgue measure separates $\mathcal{A} + \mathcal{A}$ and $\{0\}$ in the desired fashion. Hence we are done by Theorem 2.

Comments:

(i) Examples of related interpolation problems can be found in [2], [4], and [5].

(ii) It is an open question as to whether the sum set $\mathcal{F} + \mathcal{F}$ has the interpolation property of this paper. It is a result of the authors that if $\mathcal{A} = \{a^n : n \in \mathbf{Z}^+\}$, a any fixed positive integer, then $\mathcal{A} + \mathcal{A}$ has the interpolation property.

We wish to thank Professor V. E. Hoggatt, Jr., for the reference to [1].

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TOPOLOGICAL, MEASURE THEORETIC AND ANALYTIC PROPERTIES OF THE FIBONACCI NUMBERS

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Our purpose here is to develop a connection between the arithmetic, topological, measure theoretic and analytic properties of the set, F, of Fibonacci numbers. We begin by topologizing the set, Z, of integers in a rather natural way and then showing that F has a certain closure property.

Definition: Let $\mathcal{A}$ be the topology on Z generated by the set of all arithmetic progressions. That is, for any integer b, a neighborhood base at b is given by

$$\left\{\{an + b | n \in Z\} | a \in Z,\ a \neq 0\right\}.$$

Our main result is

Theorem 1: $\{0\} \cup F \cup -F$ is $\mathcal{A}$-closed.

Proof: We use the theorem of Halton [2] which states that if f_n is divided by f_m $(n > m)$ then either the remainder r is a Fibonacci number or $f_m - r$ is a Fibonacci number. Here f_n is the nth Fibonacci number.

Thus, if $r > 0$ and $r \notin F$, choose $f \in F$ such that $f > r$ and $f - r \notin F$. It is easy to check, using Halton's theorem, that

$$(*) \qquad \left(\{0\} \cup F \cap -F\right) \cap \left\{fn + r | n \in Z\right\} = \emptyset.$$

Also, if $r < 0$ and $-r \notin F$, choose $f \in F$ such that $f > -r$ and $f + r \notin F$. Again it follows that (*) holds. This establishes the result.

We will, in what follows, omit details for the sake of brevity. However, we cite references for those readers interested in the technicalities of the subject.

Observe that $\{0\} \cup F \cup -F$ is closed in any topology for Z which is finer than $\mathcal{A}$. Thus, $\{0\} \cup F \cup -F$ is (see [3, p. 87]) closed in Z with the relative Bohr topology. This allows us to deduce (since $\{0\} \cup F \cup -F$ is a Sidon set) that $\{0\} \cup F \cup -F$ is a strong Riesz set (see [3, p. 90]). Meyer has proved [3, p. 90] that the union of a strong Riesz set and a Riesz set is a Riesz set. One implication of this fact is the following extension of the F. and M. Riesz Theorem.

Corollary: Let T be the circle group (that is, the group, under multiplication, of complex numbers of modulus 1). Let μ be a bounded Borel measure

on T. The nth Fourier-Stieltjes coefficient $\hat{\mu}(n)$ of μ is defined by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \qquad (n \in Z).$$

Suppose $\hat{\mu}(n) = 0$ for all $n > 0$ with $n \notin F$. Then μ is absolutely continuous with respect to Lebesgue measure on T. Indeed, if R is any Riesz set and $\hat{\mu}(n) = 0$ for all $n \notin R \cup F$, then μ is absolutely continuous with respect to Lebesgue measure on T.

Comments:

(i) Observe that since $f_{n+1}/f_n \to \frac{\sqrt{5}+1}{2}$, it follows that F is a Hadamard set (see [3]). The above Corollary holds for all Hadamard sets [3, p. 94], but the proof, which depends on the comparatively deep work of Strzelecki [4], is much more difficult. This difficulty is to be expected because it is easy to see that there are Hadamard sets H with H Λ-dense in Z. Thus, it is the intrinsic arithmetical properties of F which enable us to give our proof of the Theorem.

(ii) For some interesting arithmetical examples of Riesz sets the reader is referred to [1].

It is well known that the closure in the Bohr compactification of any Hadamard set has Haar measure zero. Using Halton's theorem, we can give a simple proof of this result for the Fibonacci numbers.

Theorem 2: Let $\overline{F}$ denote the closure of F in the Bohr compactification of Z. Then $\mu(\overline{F}) = 0$ where μ is the Haar measure of the Bohr compactification of Z.

Proof: We shall use the elementary fact that Haar measure of the closure of an arithmetic progression in the Bohr compactification of Z is precisely the natural density of the progression. Thus, it suffices, given $\varepsilon > 0$, to imbed F in a union of residue classes, modulo some fixed modulus, such that the density of the union is less than ε.

Choose m so large that $\frac{2m}{f_m} < \varepsilon$, which can be done because F has density zero. For any n, by Halton's theorem, $f_n \in f_m Z + r$ where $0 \le r < f_m$ and $r \in F$ or $f_m - r \in F$.

But, there are clearly at most $2m$ integers, r, with $0 \le r < f_m$ and $r \in F$ or $f_m - r \in F$. Thus, F can be imbedded in a union of residue classes (mod f_m) whose density is $\frac{2m}{f_m} < \varepsilon$. This completes the proof.

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IDENTITIES FROM PARTITION INVOLUTIONS

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To George Pólya on the 2^{15}th day after his birth: August 31, 1977.

ABSTRACT

Subbarao and Andrews have observed that the combinatorial technique used by F. Franklin to prove Euler's famous partition identity

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots$$

can be applied to prove the more general formula

$$1-x-x^2y(1-xy)-x^3y^2(1-xy)(1-x^2y)-x^4y^3(1-xy)(1-x^2y)(1-x^3y)-\cdots$$
$$=1-x-x^2y+x^5y^3+x^7y^4-x^{12}y^6-x^{15}y^7+\cdots$$

which reduces to Euler's when $y = 1$. This note shows that several finite versions of Euler's identity can also be demonstrated using this elementary technique; for example,

$$\begin{aligned}1-x-x^2+x^5+x^7-x^{12}-x^{15}&\\ &=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\\ &\quad -x^7(1-x^2)(1-x^3)(1-x^4)(1-x^5)+x^{7+6}(1-x^3)(1-x^4)-x^{7+6+5}\\ &=(1-x)(1-x^2)(1-x^3)-x^4(1-x^2)(1-x^3)+x^{4+5}(1-x^3)-x^{4+5+6}.\end{aligned}$$

By using Sylvester's modification of Franklin's construction, it is also possible to generalize Jacobi's triple product identity.

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0. INTRODUCTION

Nearly a century ago [7], [14, §12], a young man named Fabian Franklin published what was to become one of the first noteworthy American contributions to mathematics, an elementary combinatorial proof of Euler's well-known identity

$$\prod_{j\geq 1}(1-x^j) = 1-x-x^2+x^5+x^7-\cdots = \sum_{-\infty<k<\infty}(-1)^k x^{(3k^2+k)/2}. \tag{0.1}$$

His approach was to find a nearly one-to-one correspondence between partitions with an even number of distinct parts and those with an odd number of distinct parts, thereby showing that most of the terms on the left-hand side of (0.1) cancel in pairs. Such combinatorial proofs of identities often yield further information, and in the first part of this note we shall demonstrate that Franklin's construction can be used to prove somewhat more than (0.1).

In the second part of this note, we show that Sylvester's modification of Franklin's construction can be applied in a similar way to obtain generalizations of Jacobi's triple product identity

$$\begin{aligned}\prod_{j\geq 1}(1-q^{2j-1}z)(1-q^{2j-1}z^{-1})(1-q^{2j}) &= 1-q(z+z^{-1})+q^4(z^2+z^{-2})-\cdots \\ &= \sum_{-\infty<k<\infty}(-1)^k q^{k^2}z^k. \end{aligned}\tag{0.2}$$

1. THE BASIC INVOLUTION

First let us recall the details of Franklin's construction. Let π be a partition of n into m distinct parts, so that $\pi = \{a_1, \ldots, a_m\}$ for some integers $a_1 > \cdots > a_m > 0$, where $a_1 + \cdots + a_m = n$. We shall write

$$\Sigma(\pi) = n, \quad \nu(\pi) = m, \quad \lambda(\pi) = a_1, \tag{1.1}$$

for the sum, number of parts, and largest part of π, respectively; if π is the empty set, we let $\Sigma(\pi) = \nu(\pi) = \lambda(\pi) = 0$. Following Hardy and Wright [8], we also define the "base" $b(\pi)$ and "slope" $s(\pi)$ as follows:

$$\beta(\pi) = \min\{j \mid j \in \pi\}, \quad \sigma(\pi) = \min\{j \mid \lambda(\pi) - j \notin \pi\}. \tag{1.2}$$

Note that if π is nonempty we have

$$\lambda(\pi) \geq \beta(\pi) + \nu(\pi) - 1 \quad \text{and} \quad \nu(\pi) \geq \sigma(\pi). \tag{1.3}$$

The partition $F(\pi)$ corresponding to π under Franklin's transformation is obtained as follows:

(i) If $\beta(\pi) \leq \sigma(\pi)$ and $\beta(\pi) < \nu(\pi)$, remove the smallest part, $\beta(\pi)$, and increase each of the largest $\beta(\pi)$ parts by one.

(ii) If $\beta(\pi) > \sigma(\pi)$ and $\sigma(\pi) < \nu(\pi)$ or $\sigma(\pi) \neq \beta(\pi) - 1$, decrease each of the largest $\sigma(\pi)$ parts by one and append a new smallest part, $\sigma(\pi)$.

(iii) Otherwise $F(\pi) = \pi$. [This case holds if and only if π is empty or $\sigma(\pi) = \nu(\pi) \leq \beta(\pi) \leq \sigma(\pi) + 1$.]

These definitions are easily understood in terms of the "Ferrers graph" [14, p. 253] for the partition π, as shown in Figure 1. It is not difficult to verify that F is an involution, i.e., that

$$F(F(\pi)) = \pi \tag{1.4}$$

for all π.

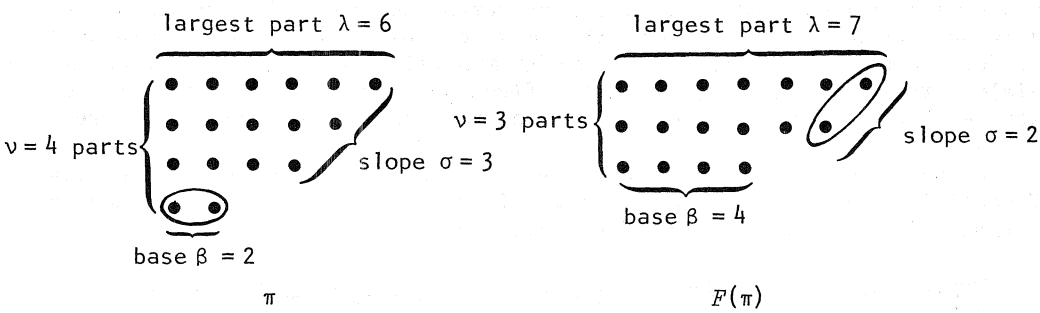

Fig. 1 Two partitions of 17 into distinct parts, obtained from each other by moving the two circled elements.

For each $\ell \geq 0$ there is exactly one partition π such that $\lambda(\pi) = \ell$ and $F(\pi) = \pi$. We shall denote this fixed point of the mapping by f_ℓ; it has $\lceil \ell/2 \rceil$ consecutive parts,

$$f_\ell = \{\ell, \ell - 1, \ldots, \lfloor \ell/2 \rfloor + 1\}. \tag{1.5}$$

(See Figure 2.) Let

$$\Phi = \{f_0, f_1, f_2, \ldots\} \tag{1.6}$$

be the set of all such partitions. Note that the somewhat similar partitions $\{2k + 1, 2k, \ldots, k + 2\}$ and $\{2k, 2k - 1, \ldots, k\}$ are not fixed under F, although their bases and slopes do intersect.

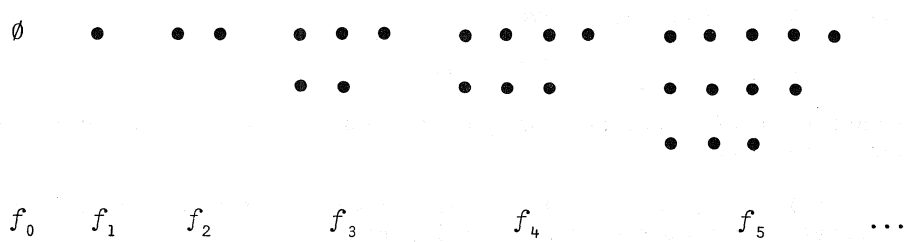

Fig. 2 The partitions which remain fixed under F.

2. EXTENDED GENERATING FUNCTIONS

If S is any set of partitions, we define the *generating function* of S by the formula

$$G_S(x, y, z) = \sum_{\pi \in S} x^{\Sigma(\pi)} y^{\lambda(\pi)} z^{\nu(\pi)}. \tag{2.1}$$

The identities we shall derive from Franklin's construction are special cases of the following elementary result:

Theorem 1: If S is any set of partitions,

$$G_S(x, y, -y) = G_{S \cap \Phi}(x, y, -y) + G_{S \setminus F(S)}(x, y, -y). \tag{2.2}$$

Proof: Let π be a partition with $\pi' = F(\pi) \neq \pi$. Then $\Sigma(\pi') = \Sigma(\pi)$, $\lambda(\pi') = \lambda(\pi) \pm 1$, and $\nu(\pi') = \nu(\pi) \mp 1$, hence

$$x^{\Sigma(\pi)} y^{\lambda(\pi)} (-y)^{\nu(\pi)} + x^{\Sigma(\pi')} y^{\lambda(\pi')} (-y)^{\nu(\pi')} = 0. \tag{2.3}$$

This equation means that π and π' do not contribute to $G_S(x, y, -y)$ if they are both members of S. The only terms which fail to cancel out are from partitions $\pi \in S$ with $F(\pi) = \pi$, namely the elements of $S \cap \Phi$, and those from partitions $\pi \in S$ with $F(\pi) \notin S$, namely the elements of $S \setminus F(S)$. ∎

3. THREE IDENTITIES

In order to get interesting corollaries of Theorem 1, we must find sets S for which the corresponding generating functions are reasonably simple.

First, let S be the set P of all partitions. Theorem 1 implies that

$$G_P(x, y, -y) = G_\Phi(x, y, -y). \tag{3.1}$$

Now

$$G_P(x, y, z) = 1 + \sum_{\ell \geq 1} x^\ell y^\ell z \prod_{1 < j < \ell} (1 + x^j z) \tag{3.2}$$

and

$$G_\Phi(x, y, z) = 1 + \sum_{\ell \geq 1} x^{\ell(\ell+1)/2 - \lfloor \ell/2 \rfloor (\lfloor \ell/2 \rfloor + 1)/2} y^\ell z^{\lceil \ell/2 \rceil} \tag{3.3}$$

$$= 1 + \sum_{k \geq 1} \left(x^{(3k^2-k)/2} y^{2k-1} z^k + x^{(3k^2+k)/2} y^{2k} z^k \right). \tag{3.4}$$

Thus we have

Corollary 1.1:

$$\sum_{\ell \geq 1} x^\ell y^{\ell+1} \prod_{1 < j < \ell} (1 - x^j y) = \sum_{k \geq 1} (-1)^{k-1} \left(x^{(3k^2-k)/2} y^{3k-1} + x^{(3k^2-k)/2} y^{3k} \right). \tag{3.5}$$

Franklin essentially considered the special case $y = 1$ of this identity, when the left-hand side reduces to $1 - \prod_{j \geq 1}(1 - x^j)$. Equation (3.5) was originally discovered by L. J. Rogers [10, §10(4)], who gave an analytic proof. The fact that Franklin's correspondence could be used to obtain (3.5) was first noticed by M. V. Subbarao [12] and G. E. Andrews [2].

Although the power series identity of Corollary 1.1 is formally true, it does not converge for all x and y; for example, if we set $y = x^{-1}$ we get the anomalous formula $x^{-1} = x^{-1} + x^{-1} - 1 - x + x^4 + x^6 - \ldots$. To better understand the rate of convergence, we can obtain an exact truncated version of the sum by restricting S to the set

$$P_n = \{\lambda(\pi) \leq n\}. \tag{3.6}$$

Since

$$\begin{aligned} P_n \backslash F(P_n) &= \{\pi \mid \lambda(\pi) = n \text{ and } \beta(\pi) \leq \sigma(\pi) \text{ and } \beta(\pi) < \nu(\pi)\} \\ &= \{\pi \mid \lambda(\pi) = n \text{ and } \beta(\pi) \leq \sigma(\pi) \text{ and } \beta(\pi) \leq n/2\}, \end{aligned} \tag{3.7}$$

we have

$$G_{P_n \backslash F(P_n)}(x, y, z) = \sum_{1 \leq b \leq n/2} (x^b y^n z)\left(\prod_{b < j \leq n-b} (1 + x^j z)\right)\left(\prod_{n-b < j \leq n} x^j z\right). \tag{3.8}$$

Thus Theorem 1 yields

Corollary 1.2:

$$\begin{aligned} &\sum_{1 \leq \ell \leq n} x^\ell y^{\ell+1} \prod_{1 \leq j < \ell} (1 - x^j y) \\ &\quad = \sum_{1 \leq k \leq (n+1)/2} (-1)^{k-1} x^{(3k^2-k)/2} y^{3k-1} + \sum_{1 \leq k \leq n/2} (-1)^{k-1} x^{(3k^2+k)/2} y^{3k} \\ &\qquad + \sum_{1 \leq b \leq n/2} (-1)^b y^{n+b+1}\left(\prod_{b < j \leq n-b} (1 - x^j y)\right)\left(\prod_{n-b < j \leq n} x^{j+1}\right). \end{aligned}$$

For example, the cases $n = 4$ and $n = 5$ of this identity are

$$\begin{aligned} &xy^2 + x^2y^3(1 - xy) + x^3y^4(1 - xy)(1 - x^2y) + x^4y^5(1 - xy)(1 - x^2y)(1 - x^3y) \\ &\quad = xy^2 + x^2y^3 - x^5y^5 - x^7y^6 - x^5y^6(1 - x^2y)(1 - x^3y) + x^{5+4}y^7; \end{aligned} \tag{3.9}$$

$$\begin{aligned} &xy^2 + x^2y^3(1 - xy) + x^3y^4(1 - xy)(1 - x^2y) + x^4y^5(1 - xy)(1 - x^2y)(1 - x^3y) \\ &\quad + x^5y^6(1 - xy)(1 - x^2y)(1 - x^3y)(1 - x^4y) \\ &\quad = xy^2 + x^2y^3 - x^5y^5 - x^7y^6 + x^{12}y^8 - x^6y^7(1 - x^2y)(1 - x^3y)(1 - x^4y) \\ &\qquad + x^{6+5}y^8(1 - x^3y). \end{aligned} \tag{3.10}$$

Setting $y = 1$ and subtracting both sides from 1 yields truncated versions of Euler's formula which appear to be new; e.g.,

$$\begin{aligned}
&1 - x - x^2 + x^5 + x^7 \\
&\quad = (1-x)(1-x^2)(1-x^3)(1-x^4) - x^5(1-x^2)(1-x^3) + x^{5+4}; && (3.11)\\
&1 - x - x^2 + x^5 + x^7 - x^{12} \\
&\quad = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \\
&\qquad - x^6(1-x^2)(1-x^3)(1-x^4) + x^{6+5}(1-x^3); && (3.12)\\
&1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} \\
&\quad = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6) \\
&\qquad - x^7(1-x^2)(1-x^3)(1-x^4)(1-x^5) + x^{7+6}(1-x^3)(1-x^4) - x^{7+6+5}. && (3.13)
\end{aligned}$$

Essentially the same formulas, but with n decreased by 2, would have been obtained if we had set $y = x^{-1}$ in the identity of Corollary 1.2.

Let us also consider another family of partition sets with a reasonably simple generating function,

$$S_n = \left\{\pi \,\middle|\, \beta(\pi) > \lambda(\pi) - n \quad \text{and} \quad \sigma(\pi) \geq \lambda(\pi) - n\right\}. \tag{3.14}$$

These sets are closed under F, for if $\pi' = F(\pi) \neq \pi$ we have either

(i) $\lambda(\pi') = \lambda(\pi) + 1$, $\beta(\pi') \geq \beta(\pi) + 1$, and $\sigma(\pi') = \beta(\pi)$, or

(ii) $\lambda(\pi') = \lambda(\pi) - 1$, $\beta(\pi') \geq \sigma(\pi)$, and $\sigma(\pi') \geq \sigma(\pi)$.

Note that S_n is finite, since $\pi \in S_n$ implies that

$$2\lambda(\pi) - 2n \leq \beta(\pi) + \sigma(\pi) - 1 \leq \lambda(\pi),$$

hence $\lambda(\pi) \leq 2n$. The set of fixed points $S_n \cap \Phi$ is $\{f_0, f_1, \ldots, f_{2n}\}$, and

$$G_{S_n}(x, y, z) = G_{P_n}(x, y, z) + \sum_{n < \ell \leq 2n} x^\ell y^\ell z \left(\prod_{\ell - n < j \leq n} (1 + x^j z)\right)\left(\prod_{n < j < \ell} x^j z\right), \tag{3.15}$$

so Theorem 1 yields a companion to Corollary 1.2:

Corollary 1.3:

$$\sum_{1 \leq \ell \leq n} x^\ell y^{\ell+1} \prod_{1 \leq j < \ell} (1 - x^j y) = \sum_{1 \leq k \leq n} (-1)^{k-1}\left(x^{(3k^2-k)/2} y^{3k-1} + x^{(3k^2+k)/2} y^{3k}\right)$$
$$+ \sum_{1 \leq b \leq n} (-1)^b y^{2b+n} \left(\prod_{b < j \leq n} (1 - x^j y)\right)\left(\prod_{n < j \leq n+b} x^j\right).$$

For example, the cases $n = 2, 3$ of this identity are

$$xy^2 + x^2y^3(1-xy) = xy^2 + x^2y^3 - x^5y^5 - x^7y^6 - x^3y^4(1-x^2y) + x^{3+4}y^6;$$
$$xy^2 + x^2y^3(1-xy) + x^3y^4(1-xy)(1-x^2y) = xy^2 + x^2y^3 - x^5y^5 - x^7y^6 + x^{12}y^8$$
$$+ x^{15}y^9 - x^4y^5(1-x^2y)(1-x^3y) + x^{4+5}y^7(1-x^3y) - x^{4+5+6}y^9.$$

Setting $y = 1$ and subtracting from 1 leads to formulas somewhat analogous to (3.11) and (3.13):

$$1 - x - x^2 + x^5 + x^7 = (1-x)(1-x^2) - x^3(1-x^2) + x^{3+4}; \tag{3.16}$$
$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} = (1-x)(1-x^2)(1-x^3) - x^4(1-x^2)(1-x^3)$$
$$+ x^{4+5}(1-x^3) - x^{4+5+6}. \tag{3.17}$$

Let us restate the identities arising from Corollaries 1.2 and 1.3 when $y = 1$, where n is even in Corollary 1.2:

$$1 + \sum_{1\le k\le n} (-1)^k\left(x^{(3k^2-k)/2} + x^{(3k^2+k)/2}\right)$$
$$= \sum_{0\le k\le n} (-1)^k x^{(2n+2)k-k(k+1)/2} \prod_{k<j\le 2n-k} (1 - x^j) \tag{3.18}$$
$$= \sum_{0\le k\le n} (-1)^k x^{nk+k(k+1)/2} \prod_{k<j\le n} (1 - x^j). \tag{3.19}$$

The latter formula was discovered by D. Shanks [11] in the course of some experiments on nonlinear transformations of series; he observed that it can be proved by induction on n without great difficulty. There is also a short proof of (3.18): Let

$$A(k, n) = (1-x^k) + x^k(1-x^k)(1-x^{k+1}) + \cdots + x^{kn}(1-x^k)\cdots(1-x^{k+n}), \tag{3.20}$$

$$R(k, n) = x^{(n+1)k}(1-x^{k+1})\cdots(1-x^{k+n}). \tag{3.21}$$

Then $A(0, n) = 0$, $A(k, 0) = 1 - x^k$, $A(k, -1) = 0$, and it is not difficult to show that

$$A(k, n) = 1 - x^{2k+1} - R(k, n) - x^{3k+2}A(k+1, n-2) \text{ if } n > 0. \tag{3.22}$$

Iteration of this recurrence yields identity (3.18). The use of this recurrence is actually only a slight extension of Euler's original technique [6] for proving (0.1).

It is interesting to compare (3.18) and (3.19) to "classical" formulas on terminating basic hypergeometric series, as suggested in a note to the authors by G. E. Andrews. If we set $a = 1$, $b = c = d = \infty$, and $q = x$ in a highly general identity given by R. P. Agarwal [1, Eq. (4.2)], we obtain

$$1 + \sum_{1\le k\le n} (-1)^k\left(x^{(3k^2-k)/2} + x^{(3k^2+k)/2}\right)$$
$$= \sum_{0\le k\le n} (-1)^k x^{k(k+1)/2}\left(\prod_{k<j\le 2n-k} (1 - x^j)\right)\Big/\prod_{1\le j\le n-k} (1 - x^j). \tag{3.23}$$

In particular, when $n = 3$ this formula gives the following analog of (3.13) and (3.17):

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} = \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)}{(1-x)(1-x^2)(1-x^3)}$$

$$- x^1 \frac{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}{(1-x)(1-x^2)} + x^{1+2} \frac{(1-x^3)(1-x^4)}{(1-x)} - x^{1+2+3}. \tag{3.24}$$

4. SYLVESTER'S INVOLUTION

Let us now turn to Jacobi's identity (0.2), which is formally equivalent under the substitution $q^2 = uv$ and $z^2 = uv^{-1}$ to

$$\prod_{j \geq 1} (1 - u^j v^{j-1})(1 - u^j v^j)(1 - u^{j-1} v^j)$$
$$= 1 + \sum_{k \geq 1} (-1)^k \left(u^{(k^2+k)/2} v^{(k^2-k)/2} + u^{(k^2-k)/2} v^{(k^2+k)/2} \right). \tag{4.1}$$

The left-hand side of this equation can be interpreted as involving partitions of Gaussian integers $m + ni$ into distinct parts of the form $p + qi$, where $\max(p, q) > 0$ and $|p - q| \leq 1$; the coefficient of $u^m v^m$ will be the excess of the number of such partitions with an even number of parts over those with an odd number of parts. The right-hand side says that there exists a nearly one-to-one correspondence between such even and odd partitions, the only unmatched partitions being of the form

$$\{1, 2+i, \ldots, k+(k-1)i\} \quad \text{or} \quad \{i, 1+2i, \ldots, k-1+ki\}. \tag{4.2}$$

An explicit correspondence of this sort was discovered by J. J. Sylvester [14, §§57-61, 64-68] shortly after he had learned of Franklin's construction; at that time Sylvester was a professor at Johns Hopkins University in Baltimore.*

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*The literature contains several incorrect references to the history of Sylvester's construction. Sudler [13] says that the approach taken by Wright [15] is essentially that of Sylvester; but in fact it is essentially the same as another construction due to Arthur S. Hathway, quoted by Sylvester in [14, §62]. Zolnowsky [16] independently rediscovered Sylvester's rules (i)-(iv), and observed that these were sufficient to prove Jacobi's identity since they will handle all cases $m + ni$ with $m \geq n$.

Sylvester's original treatment has apparently never been cited by anyone else, possibly because it comes at the end of a very long paper; furthermore his notation was rather obscure, and he made numerous careless errors that a puzzled reader must rectify. Indeed, the present authors may never have been able to understand what Sylvester was talking about if Zolnowsky's clear presentation had not been available.

We shall represent complex partitions π by three real partitions, π_+, π_0, π_-, containing respectively $\max(p, q)$ for those parts $p+qi$ in which $p-q=+1$, 0, or -1. For example, the complex partition

$$\pi = \{3+2i,\ 2+i,\ 1,\ 3+3i,\ 1+i,\ 3+4i\}$$

of $13+11i$ will be represented by

$$\pi_+ = \{3,\ 2,\ 1\},\quad \pi_0 = \{3,\ 1\},\quad \pi_- = \{4\}.$$

Sylvester noted that if i is artificially set equal to 2, we obtain a one-to-one correspondence between the complex partitions of $m+ni$ and a subset of the real partitions of $m+2n$ into distinct parts; π_+, π_0, and π_- map into the parts congruent respectively to +1, 0, and -1 modulo 3, hence Jacobi's identity implies Euler's.

In order to present Sylvester's construction, we recall the definitions of $\Sigma(\pi)$, $\nu(\pi)$, $\lambda(\pi)$, $\beta(\pi)$, and $\sigma(\pi)$ for real partitions in Section 1 above; we also add two more attributes,

$$\tau[\pi] = \min\{k \mid k+1 \notin \pi\}, \tag{4.3}$$

and

$$\alpha[\ell] = \min\{k \mid k \in \pi \text{ and } k > \tau(\pi)\}. \tag{4.4}$$

By convention, the minimum over an empty set is ∞; thus, $\beta[\pi] = \infty$ if and only if π is empty, and $\alpha[\pi] = \infty$ if and only if π has the form $\{1, 2, \ldots, k\}$ for some $k \geq 0$. Sylvester defined an involution $F(\pi)$ on complex partitions π by what amounts to the following seven rules:

(i) If $\beta(\pi_0) \leq \sigma(\pi_+)$, remove the smallest part, $\beta(\pi_0)$, from π_0 and increase each of the largest $\beta(\pi_0)$ parts of π_+ by one.

(ii) If $\beta(\pi_0) > \sigma(\pi_+) > 0$ and $\sigma(\pi_+) \neq \lambda(\pi_+)$, decrease each of the largest $\sigma(\pi_+)$ parts of π_+ by one and append a new smallest part, $\sigma(\pi_+)$, to π_0.

(iii) If $\beta(\pi_0) > \sigma(\pi_+) = \lambda(\pi_+)$ and $\beta(\pi_0) < \sigma(\pi_+) + \beta(\pi_-)$, remove the smallest part, $\beta(\pi_0)$, from π_0 and append a new largest part, $\sigma(\pi\ ) + 1$, to π and a new smallest part, $\beta(\pi_0) - \sigma(\pi_+)$, to π_-.

(iv) If $\beta(\pi_0) > \sigma(\pi_+) = \lambda(\pi_+) > 0$ and $\beta(\pi_0) + 1 > \sigma(\pi_+) + \beta(\pi_-)$, remove the largest part, $\sigma(\pi_+)$, from π_+ and the smallest part, $\beta(\pi_-)$, from π_- and append a new smallest part, $\sigma(\pi_+) + \beta(\pi_-) - 1$, to π_0.

(v) If $\lambda(\pi_+) = 0$ and $\alpha(\pi_-) > \beta(\pi_0) + \tau(\pi_-)$ and $\tau(\pi_-) > 0$, remove the smallest part, $\beta(\pi_0)$, from π_0 and replace the part $\tau(\pi_-)$ in π_- by $\tau(\pi_-) + \beta(\pi_0)$.

(vi) If $\lambda(\pi_+) = 0$ and $\alpha(\pi_-) < \beta(\pi_0) + \tau(\pi_-) + 1$, replace the part $\alpha(\pi_-)$ in π_- by $\tau(\pi_-) + 1$, and append a new smallest part, $\alpha(\pi_-) - \tau(\pi_-) - 1$, to π_0.

(vii) Otherwise $F(\pi) = \pi$. [This happens if and only if π has the form (4.2).]

It can be shown that $F(F(\pi)) = \pi$, and that in fact rules (i)-(ii), (iii)-(iv), (v)-(vi) undo each other.*

For example, Sylvester's correspondence pairs up the complex partitions in the following way, if we denote partitions by listing the respective elements of π_+, π_0, π_- separated by vertical bars:†

$3\|1\| \leftrightarrow 4\|\|$		rules (i) and (ii)
$21\|1\|1 \leftrightarrow 31\|\|1$		rules (i) and (ii)
$1\|21\| \leftrightarrow 2\|2\|$		rules (i) and (ii)
$1\|3\| \leftrightarrow 21\|\|2$		rules (iii) and (iv)
$\|2\|21 \leftrightarrow \|\|41$		rules (v) and (vi)
$\|1\|31 \leftrightarrow \|\|32$		rules (v) and (vi)

5. GENERATING FUNCTIONS REVISITED

If S is a set of complex partitions, we let

$$G_S(u, v, y, z) = \sum_{\pi \in S} u^{\mathcal{R}\Sigma(\pi)} v^{\mathcal{S}\Sigma(\pi)} y^{\lambda(\pi)} z^{\nu(\pi_0)}, \tag{5.1}$$

where

$$\begin{aligned} \mathcal{R}\Sigma(\pi) &= \Sigma(\pi_+) + \Sigma(\pi_0) + \Sigma(\pi_-) - \nu(\pi_-); \\ \mathcal{S}\Sigma(\pi) &= \Sigma(\pi_+) - \nu(\pi_+) + \Sigma(\pi_0) + \Sigma(\pi_-); \\ \lambda(\pi) &= \begin{cases} \lambda(\pi_+) & \text{if } \lambda(\pi_+) > 0; \\ -\tau(\pi_-) & \text{if } \lambda(\pi_+) = 0. \end{cases} \end{aligned} \tag{5.2}$$

These definitions have the property we want, as shown in the following theorem.

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*At this point one cannot resist quoting Sylvester, who stated that these rules possess what he called Catholicity, Homoeogenesis, Mutuality, Inertia, and Enantiotropy: "I need hardly say that so highly organized a scheme . . . has not issued from the mind of its composer in a single gush, but is the result of an analytical process of continued residuation or successive heaping of exception upon exception in a manner dictated at each point in its development by the nature of the process and the resistance, so to say, of its subject-matter" [14, p. 314].

†These are the complex partitions whose sums have the form $k + (11 - 2k)i$. Sylvester gave an incorrect table corresponding to these 12 partitions at the bottom of [14, p. 315]; in his notation, he should have written

"1st Species. 11 3.8; 6.3.2 6.5; 8.2.1 3.5.2.1.
2d Species. 9.2 5.2.4.
3d Species. 10.1 6.4.1; 7.4 3.7.1."

Theorem 2: Let S be any set of complex partitions, and let Φ be the set of all complex partitions of the form (4.2). Then

$$G_S(u, v, y, -y) = G_{S\cap\Phi}(u, v, y, -y) + G_{S\setminus F(S)}(u, v, y, -y). \tag{5.3}$$

Proof: As in Theorem 1, we need only verify that if $\pi' = F(\pi) \neq \pi$ we have $\Sigma(\pi') = \Sigma(\pi)$, $\lambda(\pi') = \lambda(\pi) \pm 1$, and $\nu(\pi_0') = \nu(\pi_0) \mp 1$. Rules (i), (iii), (v) all leave Σ unchanged, decrease $\nu(\pi_0)$, and increase $\lambda(\pi)$; rules (ii), (iv), (vi) are the inverses. There is one slightly subtle case worth discussing: Rule (iii) applies when $\lambda(\pi_+) = 0$ and it changes $\lambda(\pi_+)$ to 1; in that case the hypothesis $\beta(\pi_0) < \beta(\pi\ )$ implies that $\tau(\pi_-) = 0$, hence $\lambda(\pi) = 0$. ■

6. JACOBI-LIKE IDENTITIES

We shall apply Theorem 2 only to two infinite sets of partitions, leaving it to the reader to discover interesting finite versions of Jacobi's identity analogous to Corollaries 1.2 and 1.3.

If P is the set of all complex partitions, we have

$$\begin{aligned} G_P(u, v, y, z) = \Bigg(&\sum_{\ell\geq 1} u^\ell v^{\ell-1} y^\ell \Bigg(\prod_{1\leq j<\ell}(1+u^j v^{j-1})\Bigg)\Bigg(\prod_{j\geq 1}(1+u^{j-1}v^j)\Bigg) \\ &+ \sum_{\ell\geq 0} y^{-\ell}\Bigg(\prod_{1\leq j\leq \ell} u^{j-1}v^j\Bigg)\Bigg(\prod_{j>\ell+1}(1+u^{j-1}v^j)\Bigg)\Bigg) \prod_{j\geq 1}(1+u^j v^j z); \end{aligned} \tag{6.1}$$

furthermore

$$G_\Phi(u, v, y, z) = 1 + \sum_{k\geq 1}\left(u^{(k^2+k)/2}v^{(k^2-k)/2}y^k + u^{(k^2-k)/2}v^{(k^2+k)/2}y^{-k}\right). \tag{6.2}$$

Setting $z = -y$ in (6.1) gives the identity $G_P(u, v, y, -y) = G_\Phi(u, v, y, -y)$, which can be rewritten as

Corollary 2.1:

$$\begin{aligned} &\sum_{-\infty<\ell<\infty} \frac{y^\ell u^\ell v^{\ell-1}}{\prod_{j\leq 0}(1+u^{j+\ell}v^{j+\ell-1})}\left(\prod_{j\leq 1}(1+u^{j-1}v^j)(1+u^j v^{j-1})(1-u^j v^j y)\right) \\ &\qquad = \sum_{-\infty<k<\infty} u^{(k^2+k)/2}v^{(k^2-k)/2}y^k. \end{aligned}$$

Our derivation makes it clear that this formula reduces to (4.1) if we set $y = 1$ and replace (u, v) by $(-u, -v)$; it is therefore a three-parameter generalization of Jacobi's identity.

The right-hand side of Corollary 2.1 can be expressed as

$$\sum_{-\infty<k<\infty}(uy)^{(k^2+k)/2}(vy^{-1})^{(k^2-k)/2} = \prod_{j\leq 1}(1+u^{j-1}v^j y^{-1})(1+u^j v^{j-1}y)(1-u^j v^j)$$

by Jacobi's identity (4.1), hence Corollary 2.1 implies that

$$\sum_{-\infty<\ell<\infty} \frac{y^\ell u^\ell v^{\ell-1}}{\prod_{j\geq 0}(1+u^{j+\ell}v^{j+\ell-1})} = \prod_{j\geq 1} \frac{(1+u^{j-1}v^j y^{-1})}{(1+u^j v^{j-1})} \frac{(1+u^j v^{j-1} y)}{(1+u^j v^{j-1})} \frac{(1-u^j v^j)}{(1-u^j v^j y)}.$$

Let us set $a = -v^{-1}$, $q = uv$, and $x = uvy$, to make the structure of this formula slightly more clear; we obtain

$$\sum_{-\infty<n<\infty} \frac{x^n}{\prod_{j\geq 0}(1-aq^{j+n})} = \prod_{k\geq 0} \frac{(1-a^{-1}x^{-1}q^{j+1})(1-axq^j)(1-q^{j+1})}{(1-a^{-1}q^{j+1})(1-aq^j)(1-xq^j)}. \tag{6.3}$$

This three-parameter identity turns out to be merely the special case $b = 0$ of a "remarkable formula with many parameters" discovered by S. Ramanujan (see [8, Eq. (12.12.2)]); Ramanujan's formula, for which a surprisingly simple analytic proof has recently been found [5], can be written

$$\sum_{-\infty<n<\infty} x^n \prod_{j\geq 0}\left(\frac{1-bq^{j+n}}{1-aq^{j+n}}\right)$$

$$= \prod_{j\geq 0} \frac{(1-ba^{-1}q^j)(1-a^{-1}x^{-1}q^{j+1})(1-axq^j)(1-q^{j+1})}{(1-ba^{-1}x^{-1}q^j)(1-a^{-1}q^{j+1})(1-aq^j)(1-xq^j)}. \tag{6.4}$$

If we let S be the set of all complex partitions with π_+ nonempty, $G_S(u, v, y, z)$ and $G_{S\cap\Phi}(u, v, y, z)$ are given by the terms in (6.1) and (6.2) involving y^ℓ for $\ell \geq 1$. The set $S\backslash F(S)$ consists of those partitions with $\pi_+ = \{1\}$ and $\beta(\pi_-) < \beta(\pi_0)$, hence

$$G_{S\backslash F(S)}(u, v, y, z) = uy\sum_{b\geq 1} u^{b-1}v^b \prod_{j>b}(1+u^j v^j z)(1+u^{j-1}v^j).$$

By Theorem 2, we obtain

Corollary 2.2:

$$\left(\sum_{\ell\geq 1} u^\ell v^{\ell-1} y^\ell \prod_{1\leq j<\ell}(1+u^j v^{j-1})\right)\left(\prod_{j\geq 1}(1-u^j v^j y)(1+u^{j-1}v^j)\right)$$

$$= \sum_{k\geq 1} u^{(k^2+k)/2} v^{(k^2-k)/2} y^k + y\sum_{b\geq 1} u^b v^b \prod_{j>b}(1-u^j v^j y)(1+u^{j-1}v^j).$$

If we subtract this identity from that of Corollary 2.1, we get the formula for the complement of S, namely

$$\sum_{\ell\geq 0} y^{-\ell}\left(\prod_{1\leq j\leq \ell} u^{j-1}v^j\right)\left(\prod_{j>\ell+1}(1+u^{j-1}v^j)\right)\left(\prod_{j\geq 1}(1-u^j v^j y)\right)$$

$$= \sum_{k \geq 0} u^{(k^2-k)/2} v^{(k^2+k)/2} y^{-k} - y \sum_{b \geq 1} u\ v \prod_{j > b} (1 - u^j v^j y)(1 + u^{j-1} v^j). \tag{6.5}$$

Putting $y = 1$ reduces the left-hand side to $\prod_{j>0}(1 - u^j v^j)(1 + u^{j-1} v^j)$; hence we obtain

$$\sum_{b \geq 0} u^b v^b \prod_{j > b} (1 - u^j v^j)(1 + u^{j-1} v^j) = \sum_{k \geq 0} u^{(k^2-k)/2} v^{(k^2+k)/2}. \tag{6.6}$$

Let $q = uv$ and $x = -u^{-1}$; this formula is equivalent to the identity

$$\sum_{b \geq 0} q^b \prod_{j > b} (1 - q^j)(1 - q^j x) = \sum_{k \geq 0} (-x)^k q^{(k^2+k)/2}. \tag{6.7}$$

Equation (6.7) can be derived readily from known identities on basic hypergeometric functions. Let us first divide both sides by $\prod_{j \geq 1}(1 - q^j)(1 - q^j x)$, obtaining

$$\sum_{n \geq 0} \frac{q^n}{\prod_{0 \leq j < n} (1 - xq^{j+1})(1 - q^{j+1})} = \left(\frac{1}{\prod_{j \geq 0} (1 - xq^{j+1})(1 - q^{j+1})} \right) \sum_{k \geq 0} (-x)^k q^{(k^2+k)/2}.$$

Now we use E. Heine's important transformation of such series, a five-parameter identity [9, Eq. 79] which essentially states that

$$f(u, v; a, b; q) = f(v, u; b, a; q)$$

if

$$f(u, v; a, b; q) = \left(\sum_{n \geq 0} u^n \prod_{0 \leq j < n} \frac{(1 - aq^j)(1 - vq^j)}{(1 - bvq^j)(1 - q^{j+1})} \right) \left(\prod_{j \geq 0} \left(\frac{1 - uq^j}{1 - auq^j} \right) \right). \tag{6.8}$$

In our case we let $u = q$, $v = x/b$, $a = 0$, and $b \to \infty$, obtaining the desired result:

$$\left(\sum_{n \geq 0} \frac{q^n}{\prod_{0 \leq j < n} (1 - xq^{j+1})(1 - q^{j+1})} \right) \left(\prod_{j \geq 0} (1 - q^{j+1}) \right)$$
$$= \left(\sum_{n \geq 0} x^k \prod_{0 \leq j < n} (-q^j) \right) \left(\prod_{j \geq 0} \frac{1}{(1 - xq^{j+1})} \right).$$

It is not clear whether or not the more general equation (6.5) is related to known formulas in an equally simple way.

An amusing special case of (6.7) can be obtained by setting $q = x^2$ and multiplying both sides by x:

$$\sum_{k \text{ odd}} x^k \prod_{j>k} (1 - x^j) = x - x^4 + x^9 - \cdots = \sum_{k \geq 0} (-1)^k x^{(k+1)^2}. \tag{6.9}$$

"The partitions of n into an odd number of distinct parts in which the least part is odd are equinumerous with its partitions into an even number of distinct parts in which the least part is odd, unless n is a perfect square." An equivalent statement was posed as a problem by G. E. Andrews several years ago [3], and he has sketched a combinatorial proof in [4, pp. 156-157]. However, there must be an involution on partitions which proves this formula! If the reader can find one, it might well lead to a number of interesting new identities.

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EXPANSION OF THE FIBONACCI NUMBER F_{nm} IN nTH POWERS OF FIBONACCI OR LUCAS NUMBERS

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Fibonacci numbers are defined by the recurrence relation $F_m + F_{m+1} = F_{m+2}$ and the initial values $F_0 = 0$, $F_1 = 1$. Lucas numbers are defined by $L_m = F_{m-1} + F_{m+1}$. The well-known identities $F_{2m} = F_{m+1}^2 - F_{m-1}^2$ and $F_{3m} = F_{m+1}^3 + F_m^3 - F_{m-1}^3$ are shown to be the first members of two families of identities of a more general nature. Similar identities for L_{2m} and L_{3m} can be generalized in similar ways.

1. Let $n = 2p$ be an even positive integer, m be any integer, and k be any integer except zero. Then

$$F_{nm} = \sum_{r=-p}^{p} a_r F_{m+rk}^n = 5^{-p} \sum_{r=-p}^{p} a_r L_{m+rk}^n$$

where $a_0 = 0$, $a_{-r} = -a_r$ and $a_1, a_2, \ldots, a_p$ are the solution of the p simultaneous equations

$$(1) \quad \sum_{r=1}^{p} a_r (-1)^{rks} F_{rk(n-2s)} = \begin{cases} 5^{p-1} & \text{for } s = 0 \\ 0 & \text{for } s = 1, 2, \ldots, p-1 \end{cases}$$

2. Let n, p, m, and k be as in 1. Then

$$L_{nm} = \sum_{r=-p}^{p} b_r F_{m+rk}^n = 5^{-p} \sum_{r=-p}^{p} b_r L_{m+rk}^n$$

where $b_{-r} = b_r$ and $b_0, b_1, \ldots, b_p$ are the solution of the $p+1$ simultaneous equations

$$(2) \quad b_0 + \sum_{r=1}^{p} b_r (-1)^{rks} L_{rk(n-2s)} = \begin{cases} 5^p & \text{for } s = 0 \\ 0 & \text{for } s = 1, 2, \ldots, p. \end{cases}$$

3. Let $n = 2p+1$ be an odd positive integer, and let m and k be as in 1. Then

$$F_{nm} = \sum_{r=-p}^{p} c_r F_{m+rk}^n \quad \text{and} \quad L_{nm} = 5^{-p} \sum_{r=-p}^{p} c_r L_{m+rk}^n$$

where $c_{-r} = (-1)^{rk} c_r$ and $c_r = b_r$ for $r \geq 0$.

4. Since the proofs are similar in all cases, only that for the first identity need be given. The Fibonacci numbers are first written in the Binet form

$$F_u = 5^{-\frac{1}{2}}\left\{g^u - (-g)^{-u}\right\}$$

where $g = \frac{1}{2}\left(5^{\frac{1}{2}} + 1\right)$. Then for n even:

$$F_{nm} = 5^{-\frac{1}{2}}(g^{nm} - g^{-nm}) = \sum_{r=-p}^{p} a_r 5^{-\frac{1}{2}n}\left\{g^{m+rk} - (-g)^{-m-rk}\right\}^n$$

$$= 5^{-p}\sum_{s=0}^{n}\binom{n}{s}(-1)^{ms+s} g^{m(n-2s)} \sum_{r=-p}^{p} a_r(-1)^{rks} g^{rk(n-2s)}$$

Equating coefficients of like powers of g for each value of s gives:

$$(3) \quad 5^{p-\frac{1}{2}} = \sum_{r=-p}^{p} a_r g^{rkn} \quad \text{for } s = 0$$

$$(4) \quad 5^{p-\frac{1}{2}} = -\sum_{r=-p}^{p} a_r g^{-rkn} \quad \text{for } s = n$$

$$(5) \quad 0 = \sum_{r=-p}^{p} a_r(-1)^{rks} g^{rk(n-2s)} \quad \text{for } s = 1, 2, \ldots, n-1$$

These $n+1$ equations can be rewritten in terms of Fibonacci numbers as follows: Equating the coefficients of like powers of g in (3) and (4) gives $a_{-r} = -a_r$ and $a_0 = 0$. Equations (3) and (4) are thus equivalent and can be rewritten in a common form

$$(6) \quad 5^{p-1} = 5^{-\frac{1}{2}}\left\{\sum_{r=1}^{p} a_r g^{rkn} + \sum_{r=-1}^{-p} a_r g^{rkn}\right\} = 5^{-\frac{1}{2}}\sum_{r=1}^{p} a_r(g^{rkn} - g^{-rkn})$$

$$= \sum_{r=1}^{p} a_r F_{rkn}$$

Similarly, (5) can be rewritten as

$$(7) \quad 0 = \sum_{r=1}^{p} a_r(-1)^{rks} F_{rk(n-2s)} \quad \text{for } s = 1, 2, \ldots, n-1$$

However, since $F_{-u} = -(-1)^u F_u$, this summation is unchanged when s is replaced with $n-s$, and since each term is zero when $s = p$, only $\frac{1}{2}(n-2)$ $p-1$ values of s give independent equations. These values can be taken as $s = 1, 2, \ldots, p-1$. Thus (6) and (7) together give p equations for the coefficients $a_1, a_2, \ldots, a_p$, and it is obvious that the conditions for the existence and uniqueness of the solution are satisfied.

5. For small values of n, the explicit expressions for a_r and b_r obtained by solving (1) and (2) can be reduced to simple forms by repeated use of the identities $L_{2u} = L_u^2 - 2(-1)^u = 5F_u^2 + 2(-1)^u$. The results for $n = 2, 3, 4$ are:

$n = 2$ $\quad 1/a_1 = F_2 \qquad 1/b_0 = -\frac{1}{2}(-1)^k F_k^2 \qquad 1/b_1 = F_k^2$

$n = 3$ $\quad 1/b_0 = -(-1)^k F_k^2 \qquad 1/b_1 = L_k F_k^2$

$n = 4$ $\quad 1/a_1 = F_{2k}\left\{F_k^2 - (-1)^k F_{2k}^2\right\} \qquad 1/a_2 = -(-1)^k L_{2k}/a_1$

$1/b_0 = \frac{1}{2}(-1)^k F_k^4 L_k^2 \qquad 1/b_1 = F_k/L_k a_1$

$1/b_2 = -(-1)^k L_k^2/b_1$

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SOME POLYNOMIALS RELATED TO FIBONACCI AND EULERIAN NUMBERS

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1. INTRODUCTION

Put

$$\frac{1}{1 - kx - x^2} = \sum_{n=0}^{\infty} c_{kn} x^n \tag{1.1}$$

and

$$C_n(y) = \sum_{k=0}^{\infty} c_{kn} y^k \qquad (n = 0, 1, 2, \ldots). \tag{1.2}$$

By (1.1),

$$\sum_{n=0}^{\infty} c_{kn} x^n = \sum_{j=0}^{\infty} x^j (k + x)^j = \sum_{j=0}^{\infty} \sum_{s=0} \binom{j}{s} k^{j-s} x^{j+s} = \sum_{n=0}^{\infty} x^n \sum_{2s \le n} \binom{n-s}{s} k^{n-2s},$$

so that

$$c_{kn} = \sum_{2s \le n} \binom{n-s}{s} k^{n-2s}. \tag{1.3}$$

Since c_{kn} is a polynomial in k of degree n, it follows that

$$C_n(y) = \frac{r_n(y)}{(1 - y)^{n+1}} \qquad (n = 0, 1, 2, \ldots), \tag{1.4}$$

where $r_n(y)$ is a polynomial in y of degree n. Moreover, since

$$c_{k,n+1} = kc_{k,n} + c_{k,n-1},$$

it follows from (1.2) that

$$C_{n+1}(x) = \sum_{k=0}^{\infty} (kc_{k,n} + c_{k,n-1}) x^k.$$

This gives

$$C_{n+1}(x) = C_n'(x) + C_{n-1}(x) \qquad (n \ge 1). \tag{1.5}$$

Hence, by (1.4),

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$$r_{n+1}(x) = (n+1)xr_n(x) + x(1-x)r_n'(x) + (1-x)^2 r_{n-1}(x) \qquad (n \geq 1) \tag{1.6}$$

with $r_0(x) = r_1(x) = 1$.

If we put

$$r_n(x) = \sum_{k=0}^{n} R_{n,k}\, x^k, \tag{1.7}$$

then, by (1.6), we get the recurrence

$$(n - k + 2)R_{n,k-1} + kR_{n,k} + R_{n-1,k} - 2R_{n-1,k-1} + R_{n-1,k-2}. \tag{1.8}$$

By means of (1.8) the following table is easily computed.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	•	1						
2	1	-1	2					
3	•	3	•	3				
4	1	•	14	4	5			
5	•	8	22	60	22	8		
6	1	6	99	244	279	78	13	
7	•	21	240	1251	2016	1251	240	21

It follows from (1.6) that

$$R_{n+1,n+1} = R_{n,n} + R_{n-1,n-1}.$$

Hence, since $R_{0,0} = R_{1,1} = 1$,

$$R_{n,n} = F_{n+1} \qquad (n = 0, 1, 2, \ldots). \tag{1.9}$$

Hoggatt and Bicknell [2] have conjectured that

$$R_{2n+1,k} = R_{2n+1,2n-k+2} \qquad (1 \leq k \leq 2n+1). \tag{1.10}$$

We shall prove that this is indeed true and that

$$R_{2n,2n-k+1} + (-1)^k \binom{2n+1}{k} = R_{2n,k} \qquad (1 \leq k \leq 2n). \tag{1.11}$$

The proof of (1.10) and (1.11) makes use of the relationship of $r_n(x)$ to the polynomial $A_n(x)$ defined by [1], [3, Ch. 2]

$$\frac{1-x}{1-xe^{(1-x)z}} = 1 + \sum_{n=1}^{\infty} A_n(x)\frac{x^n}{n!} \tag{1.12}$$

The relation in question is

$$r(x) = \sum_{2k \le n} \binom{n-k}{k} (1-x)^{2k} A_{n-2k}(x) \tag{1.13}$$

with $A_0(x) = 1$. The polynomial $A_n(x)$ is of degree n:

$$A_n(x) = \sum_{k=1}^{n} A_{n,k} x^k \qquad (n \ge 1), \tag{1.14}$$

where the $A_{n,k}$ are the Eulerian numbers. Since

$$A_n(x) = x^{n+1} A_n\left(\frac{1}{x}\right), \tag{1.15}$$

it is easily seen that (1.10) and (1.11) are implied by (1.13).

It seems difficult to find a simple explicit formula for $R_{n,k}$ or a simple generating function for $r_n(x)$. An explicit formula for $R_{n,k}$ is given in (2.11). As for a generating function, we show that

$$\sum_{n=0}^{\infty} r_n(x) \frac{z^n}{n!} = \sum_{n=0}^{\infty} A_n(x) f_n\left((1-x)z\right)(1-x)^{-n} z^n, \tag{1.16}$$

where

$$f_n(z) = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!\,(2k+n)!} z^{2k+n}. \tag{1.17}$$

Moreover

$$f_n(z) = P_n(z) \cosh z + Q_n(z) \sinh z, \tag{1.18}$$

where $P_n(z)$, $Q_n(z)$ are polynomials of degree n, $n-1$, respectively, that are given explicitly below.

While (1.16) is not a very satisfactory generating function, the explicit result (1.18) for $f_n(z)$ seems of some interest. It is reminiscent of the like result concerning Bessel functions of order half an integer [4, p. 52].

2. PROOF OF (1.10) AND (1.11)

By (1.2) and (1.3) we have

$$C_n(x) = \sum_{k=0}^{\infty} x^k \sum_{2s \le n} \binom{n-s}{s} k^{n-2s} = \sum_{2s \le n} \binom{n-s}{s} \sum_{k=0}^{\infty} k^{n-2s} x^k.$$

Since [3, p. 39]

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}},$$

it follows that

$$C_n(x) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{-n+2s} \quad A_{n-2s}(x)$$

and therefore

$$r_n(x) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{2s} A_{n-2s}(x). \tag{2.1}$$

Thus we have proved (1.13).

Note that by (1.7) and (1.14), (2.1) yields

$$R_{n,k} = \sum_{2s \le n} \sum_{j=0}^{k} (-1) \binom{2j}{j} \binom{n-s}{s} A_{n-2s,k-j}. \tag{2.2}$$

In the next place, since

$$A_n(x) = x^{n+1} A_n\left(\frac{1}{x}\right) \qquad (n > 0),$$

(2.1) gives

$$x^{n+1} r\left(\frac{1}{x}\right) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{2s} x^{n-2s+1} A_{n-2s}\left(\frac{1}{x}\right). \tag{2.3}$$

We now consider separately the cases n odd and n even.

Replacing n by $2n+1$, (2.3) becomes

$$x^{2n+2} r_{2n+1}\left(\frac{1}{x}\right) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{2s} A_{n-2s}(x),$$

so that

$$r_{2n+1}(x) = x^{2n+2} r_{2n+1}\left(\frac{1}{x}\right). \tag{2.4}$$

On the other hand

$$\begin{aligned} r^{2n+1} r_{2n}\left(\frac{1}{x}\right) &= \sum_{s=0}^{n-1} \binom{2n-s}{s} (1-x)^{2s} x^{2n-2s+1} A_{2n-2s}\left(\frac{1}{x}\right) + x(1-x)^{2n} \\ &= \sum_{s=0}^{n} \binom{2n-s}{s} (1-x)^{2s} A_{2n-2s}(x) - (1-x)^{2n+1}, \end{aligned}$$

so that

$$r_{2n}(x) = r^{2n+1} r_{2n}\left(\frac{1}{x}\right) + (1-x)^{2n+1}. \tag{2.5}$$

By (2.4) and (1.7) it follows at once that

$$R_{2n+1,k} = R_{2n+1,2n-k+2} \qquad (1 \le k \le 2n+1). \tag{2.6}$$

Similarly, by (2.5),

$$\sum_{k=0}^{2n} R_{2n,k} x^k = \sum_{k=0}^{2n} R_{2n,k} x^{2n-k+1} + \sum_{k=0}^{2n+1} (-1) \binom{2n+1}{k} x^k,$$

which gives

$$R_{2n,k} = R_{2n,2n-k+1} + (-1)^k\binom{2n+1}{k} \qquad (1 \le k \le 2n), \tag{2.7}$$

as well as

$$R_{2n,0} = 1 \qquad (n = 0, 1, 2, \ldots). \tag{2.8}$$

The companion formula

$$R_{2n+1,0} = 0 \qquad (n = 0, 1, 2, \ldots) \tag{2.9}$$

is implied by (2.4).

Clearly, by (1.9) and (2.6),

$$R_{2n+1,1} = F_{2n+2} \qquad (n = 0, 1, 2, \ldots) \tag{2.10}$$

while, by (2.7),

$$R_{2n,1} = F_{2n+1} - (2n+1) \qquad (n = 0, 1, 2, \ldots). \tag{2.11}$$

Since

$$A_n(y) = y\sum_{j=0}^{n}(y-1)^{n-j}\Delta^j 0^n,$$

where, as usual,

$$\Delta^j 0^n = \sum_{s=0}^{j}(-1)^{j-s}\binom{j}{s}s^n = j!S(n, j),$$

where $S(n, j)$ is a Stirling number of the second kind, (2.1) implies

$$\begin{aligned} r_n(x) &= x\sum_{2k\le n}\binom{n-s}{s}(1-x)^{2s}\sum_{j=0}^{n-2s}(x-1)^{n-2s-j}\Delta^j 0^{n-2s} \\ &= x\sum_{2s\le n}\binom{n-s}{s}\sum_{j=0}^{n-2s}(x-1)^{n-j}\Delta^j 0^{n-2s}. \end{aligned}$$

Hence

$$R_{n,k} = \sum_{2s\le n}\binom{n-s}{s}\sum_{j=0}^{n-2s}(-1)^{n\ \ j-k+1}\binom{n-j}{k-1}\Delta^j 0^{n-2s}. \tag{2.12}$$

For example, for $k = n$, (2.12) reduces to

$$R_{n,n} = \sum_{2s\le n}\binom{n-s}{s} = F_{n+1}.$$

3. GENERATING FUNCTIONS

To obtain a generating function for $r_n(x)$, we again make use of (2.1). Thus

$$\sum_{n=0}^{\infty} r_n(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{2k\le n}\binom{n-k}{k}(1-x)^{2k}A_{n-2k}(x)$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+k}{k}\frac{z^{n+2k}}{(n+2k)!}(1-x)^{2k}A_n(x).$$

If we put

$$f_n(z) = \sum_{k=0}^{\infty}\frac{(k+n)!}{k!(2k+n)!}z^{2k+n} = \sum_{k=0}^{\infty}\frac{(k+1)_n}{(2k+n)!}z^{2k+n}, \tag{3.1}$$

we get

$$\sum_{n=0}^{\infty} r_n(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty}A_n(x)f_n\big((1-x)z\big)(1-x)^{-n}z^n. \tag{3.2}$$

Clearly

$$f_0(z) = \sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!},\quad 2f_1(z) = \sum_{k=0}^{\infty}\frac{2k+2}{(2k+1)!}z^{2k+1} = z\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!},$$

so that

$$f_0(z) = \cosh z,\quad 2f_1(z) = z\cosh z + \sinh z. \tag{3.3}$$

For $n = 2$ we get

$$4f_2(z) = \sum_{k=0}^{\infty}\frac{4(k+1)(k+2)}{(2k+2)!}z^{2k+2} = \sum_{k=0}^{\infty}\frac{(2k+1)(2k+2)+3(2k+2)}{(2k+2)!}z^{2k+2},$$

which reduces to

$$4f_2(z) = z^2\cosh z + 3z\sinh z. \tag{3.4}$$

With a little more computation we find that

$$8f_3(z) = (z^3+3z^2)\cosh z + (6z^2-3)\sinh z. \tag{3.5}$$

These special results suggest that generally

$$2^n f_n(z) = P_n(z)\cosh z + Q_n(z)\sinh z, \tag{3.6}$$

where $P_n(z)$, $Q_n(z)$ are polynomials in z of degree n, $n-1$, respectively. We shall show that this is indeed the case and evaluate $P_n(z)$, $Q_n(z)$.

If we put

$$S_n(z) = P_n(z) + Q_n(z),\quad T_n(z) = P_n(z) - Q_n(z), \tag{3.7}$$

then (3.6) becomes

$$2^n f_n(z) = \frac{1}{2}\left(S_n(z)e^z + T_n(z)e^{-z}\right). \tag{3.8}$$

By (3.1) we have

$$2^n f_n(z) = \sum_{k=0}^{\infty} \frac{2^n (k+1)_n}{(2k+n)!} z^{2k+n}. \tag{3.9}$$

This suggests that we put

$$2^n (x+1)_n = \sum_{j=0}^{n} a_{nj} (2x + j + 1)_{n-j}, \tag{3.10}$$

where the a_{nj} are independent of x. Clearly the a_{nj} are uniquely determined by (3.10). Indeed, rewriting (3.10) in the form

$$2^n\left(\frac{1}{2}(x-n)+1\right)_n = \sum_{j=0}^{n} (n-j)!\, a_{nj} \binom{x}{n-j},$$

it is evident, by finite differences, that

$$\begin{aligned} a_{n,n-j} &= \frac{2^n}{j!}\sum_{s=0}^{j} (-1)^{j-s}\binom{j}{s}\left(\frac{1}{2}(s-n)+1\right)_n \\ &= \frac{1}{j!}\sum_{s=0}^{j} (-1)^{j-s}\binom{j}{s}(s+n)(s+n-2)\cdots(s-n+2). \end{aligned} \tag{3.11}$$

Substituting from (3.10) in (3.9) we get

$$\begin{aligned} 2^n f_n(z) &= \sum_{k=0}^{\infty} \frac{z^{2k+n}}{(2k+n)!} \sum_{j=0} a_{nj}(2k+j+1) \\ &= \sum_{j=0}^{n} a_{nj} z^{n-j} \sum_{k=0}^{\infty} \frac{z^{2k+j}}{(2k+j)!} \\ &= \sum_{2j \le n} a_{n,2j} z^{n-2j} \left\{ \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} - \sum_{t=0}^{j-1} \frac{z^{2t}}{(2t)!} \right\} \\ &\quad + \sum_{2j<n} a_{n,2j+1} z^{n-2j-1} \left\{ \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} - \sum_{t=0}^{j-1} \frac{z^{2t+1}}{(2k+1)!} \right\} \\ &= \sum_{2j \le n} a_{n,2j} z^{n-2j} \cosh z + \sum_{2j<n} a_{n,2j+1} z^{n-2j-1} \sinh z \\ &\quad - \sum_{2j \le n} \sum_{t<j} a_{n,2j} \frac{z^{n-2j-2t}}{(2t)!} - \sum_{2j<n} \sum_{t<j} a_{n,2j+1} \frac{z^{n-2j+2t}}{(2t+1)!}. \end{aligned} \tag{3.12}$$

Now

$$\sum_{2j\le n}\sum_{t<j} a_{n,2j}\frac{z^{n-2j+2t}}{(2t)!}+\sum_{2j<n}\sum_{t<j} a_{n,2j+1}\frac{z^{n-2j+2t}}{(2\ \ +1)!}$$

$$=\sum_{<2t<n} z^{n-2t}\left\{\sum_{2t\le j<n}\frac{a_{n,2j}}{(2j-2t)!}+\frac{a_{n,2j+1}}{(2j-2t+1)!}\right\}$$

$$=\sum_{0<2t<n} z^{n-2t}\sum_{2t\le j\le n}\frac{a_{n,j}}{(j-2t)!}.$$

By (3.11)

$$\sum_{2t\le j\le n}\frac{a_{nj}}{(j-2t)!}$$

$$=\sum_{2t\le j\le n}\frac{1}{(j-2t)!\,(n-j)!}\sum_{s=0}^{n-j}(-1)^{n-j-s}\binom{n-j}{s}(s+n)(s+n-2)\cdots(s-n+2)$$

$$=\sum_{j=0}^{n-2t}\frac{1}{j!\,(n-2t-j)!}\sum_{s=0}^{j}(-1)^{j-s}\binom{j}{s}(s+n)(s+n-2)\cdots(s-n+2)$$

$$=\frac{1}{(n-2t)!}\sum_{s=0}^{n-2t}\binom{n-2t}{s}(s+n)(s+n-2)\cdots(s-n+2)\sum_{j=s}^{n-2t}(-1)^{j-s}\binom{n-2t-s}{j-s}.$$

The inner sum vanishes unless $n = 2t + s$. Since $n > 2t$, the double sum must vanish. Therefore, (3.12) reduces to

$$2^n f_n(z) = \sum_{2j\le n} a_{n,2j}\, z^{n-2j}\cosh z + \sum_{2j<n} a_{n,2j+1}\, z^{n-2j-1}\sinh z. \tag{3.13}$$

Comparing (3.13) with (3.6), it is evident that

$$\begin{cases} P_n(z) = \displaystyle\sum_{2j\le n} a_{n,2j}\, z^{n-2j} \\[2ex] Q_n(z) = \displaystyle\sum_{2j<n} a_{n,2j+}\ z^{n-2j-1}. \end{cases} \tag{3.14}$$

Hence, as asserted above, $P_n(z)$, $Q_n(z)$ are polynomials of degree n, $n - 1$, respectively. It is in fact necessary to verify that $a_{n,0} \neq 0$, $a_{n,1} \neq 0$.

By (3.7) and (3.14) we have

$$S_n(z) = \sum_{j=0}^{n} a_{n,j} z^{n-j}, \quad T_n(z) = \sum_{j=0}^{n} (-1)\ a_{n,j} z^{n-j}. \tag{3.15}$$

4. ANOTHER EXPLICIT FORMULA

While we have found a_{nj} explicitly in (3.11), we shall now obtain another formula that exhibits a_{nj} as a polynomial in n of degree $2j$. To begin with we have, by (3.11),

$$\begin{aligned} e^z S_n(z) &= \sum_{k=0}^{\infty} z^k \sum_j \frac{j}{(k-j)!} a_{n,n-j} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \sum_{s=0}^{j} (-1)^{j-s} \binom{j}{s} (s+n)(s+n-2) \cdots (s-n+2) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{s=0}^{k} \binom{k}{s} (s+n)(s+n-2) \cdots (s-n+2) \sum_{j=s}^{k} (-1)^{j-s} \binom{k-s}{j-s}. \end{aligned}$$

It follows that

$$U_n(z) \equiv e^z S_n(z) = \sum_{k=0}^{\infty} \frac{(k+n)(k+n-2) \cdots (k-n+2)}{k!} z^k. \tag{4.1}$$

Then

$$zU_n'(z) - nU_n(z) = \sum_{k=0}^{\infty} \frac{(k+n)(k+n-2) \cdots (k-n)}{k!} z^k,$$

$$\left(zU_n'(z) - nU_n(z)\right)' = \sum_{k=0}^{\infty} \frac{(k+n+1)(k+n-1) \cdots (k-n+1)}{k!} z^k = U_{n+1}(z).$$

Carrying out the differentiation this reduces to

$$S_{n+1}(z) = zS_n''(z) + (2z-n+1)S_n'(z) + (z-n+1)S_n(z). \tag{4.2}$$

Comparing coefficients we get

$$a_{n+1,j} = a_{nj} + (n-2j+3)a_{n,j-1} - (j-2)(n-j+2)a_{n,j-2}. \tag{4.3}$$

Hence, for $j = 0$, we get

$$a_{n,0} = 1. \tag{4.4}$$

For $j = 1$, (4.3) becomes

$$a_{n+1,1} = a_{n1} + (n+1)a_{n0},$$

which gives

$$a_{n1} = \binom{n+1}{2}. \tag{4.5}$$

For $n = 2$, (4.3) reduces to

$$a_{n+1,2} = a_{n2} + (n-1)a_{n1},$$

which gives

$$a_{n2} = 3\binom{n+1}{4}. \tag{4.6}$$

With a little more computation we get

$$a_{n3} = 15\binom{n+1}{6} - 3\binom{n+1}{4} \tag{4.7}$$

$$a_{n4} = 105\binom{n+1}{8} - 45\binom{n+1}{6} \tag{4.8}$$

$$a_{n5} = 3 \cdot 5 \cdot 7 \cdot 9\binom{n+1}{10} - 630\binom{n+1}{8} + 45\binom{n+1}{6}. \tag{4.9}$$

These special results suggest that generally

$$a_{n,j} = \sum_{2s<j} (-1)^s c_{js}\binom{n+1}{2j-2s}. \tag{4.10}$$

Indeed assuming that (4.10) holds up to j, it follows from (4.3) that

$$a_{n+1,j+1} - a_{n,j=1}$$

$$= (n-2j+1)\sum_{2s<j}(-1)^s c_{js}\binom{n+1}{2j-2s} - (j-1)(n-j+1)\sum_{2s<j-1}(-1)^s c_{j-1,s}\binom{n+1}{2j-2s-2}$$

$$= \sum_{2s<j}(-1)^s\binom{n+1}{2j-2s}\left\{(n-2j+1)c_{js} + (j-1)(n-j+1)c_{j-1,s-1}\right\}$$

$$= \sum_{2s\le j+1}(-1)^s c_{j+1,s}\binom{n+1}{2j-2s+1}$$

provided

$$(n-2j+2s+1)c_{j+1,s} = (2j-2s+1)\left((n-2j+1)c_{js} + (j-1)(n-j+1)c_{j-1,s-1}\right).$$

This gives

$$c_{j,s} = 2^{-j}\frac{(j-1)!(2j-2s)!}{s!(j-s)!(j-2s-1)!}. \tag{4.11}$$

Thus (4.10) becomes

$$a_{n,j} = \sum_{2s<j} 2^{-j} \frac{(j-1)!\,(2j-2s)!}{s!\,(j-s)!\,(j-2s-1)!} \binom{n+1}{2j-2s} \tag{4.12}$$

$a_{n,j}$:

$n \backslash j$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3				
3	1	6	3	-3		
4	1	10	15	-15		
5	1	15	45	-30	-45	+45

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A RECURRENCE SUGGESTED BY A COMBINATORIAL PROBLEM

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SECTION 1

Recurrences of the following kind occur in connection with a certain combinatorial problem (see §5 below). Let $e_1, \ldots, e_n$ be non-negative integers and q a parameter. Consider the recurrence

$$F(e_1, \ldots, e_n) = \sum_{j=1}^{n} q^{jN} F(e_1 - \delta_{1j}, \ldots, e_n - \delta_{nj}), \tag{1.1}$$

where

$$N = e_1 + \cdots + e_n, \tag{1.2}$$

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j), \end{cases} \tag{1.3}$$

$$F(0, \ldots, o) = 1 \tag{1.4}$$

and $F(e_1, \ldots, e_n) = 0$ if any $e_i < 0$.

Note that, for $q = 1$, (1.1) reduces to

$$F(e_1, \ldots, e_n) = \sum_{j=1}^{n} F(e_1 - \delta_{1j}, \ldots, e_n - \delta_{nj})$$

and $F(e_1, \ldots, e_n)$ becomes the multinomial coefficient

$$\frac{(e_1 + e_2 + \cdots + e_n)!}{e_1! e_2! \ldots e_n!}.$$

If we put

$$\boldsymbol{e} = (e_1, \ldots, e_n), \quad \delta_j = (\delta_{1j}, \ldots, \delta_{nj}), \tag{1.5}$$

then (1.1) becomes

$$F(\boldsymbol{e}) = \sum_{j=1}^{n} q^{jN} F(\boldsymbol{e} - \delta_j). \tag{1.6}$$

For $n = 1$, the recurrence (1.1) is simply

$$F(N) = q^N F(N - 1), \quad F(0) = 1. \tag{1.7}$$

The solution of (1.7) is immediate, namely

$$F(N) = q^{\frac{1}{2}N(N+1)}. \tag{1.8}$$

For $n = 2$, the situation is less simple. We take

$$F(e_1, e_2) = q^N F(e_1 - 1, e_2) + q^{2N} F(e_1, e_2)(N = e_1 + e_2). \tag{1.9}$$

Iteration of (1.9) gives

$$\begin{aligned} F(e_1, e_2) &= q^{2N-1}F(e_1-2, e_2) + q^{3N-2}(1+q)F(e_1-1, e_2-1) + q^{4N-2}F(e_1, e_2-2) \\ &= q^{3N-3}F(e\ -3, e_2) + q^{4N-5}(1+q+q^2)F(e_1-\ , e_2-1) \\ &\qquad + q^{5N-6}(1+q+q^2)F(e_1-1, e_2-2) + q^{6N-6}F(e_1, e_2-3). \end{aligned}$$

It is helpful to isolate the exponents as indicated in the following table.

$m \backslash r$	0	1	2	3	4	5
0	1					
1	N	$2N$				
2	$2N - 1$	$3N - 2$	$4N - 2$			
3	$3N - 3$	$4N - 5$	$5N - 6$	$6N - 6$		
4	$4N - 6$	$5N - 9$	$6N - 11$	$7N - 12$	$8N - 12$	
5	$5N - 10$	$6N - 14$	$7N - 17$	$8N - 19$	$9N - 20$	$10N - 20$

The special results above suggest that generally, for $m \geq 0$,

$$F(e_1, e_2) = \sum_{r+s=m} \begin{bmatrix} m \\ r \end{bmatrix} q^{(m+r)N - m(m-1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r), \tag{1.10}$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1}) \cdots (1-q^{m-r+1})}{(1-q)(1-q^2) \cdots (1-q^r)}. \tag{1.11}$$

It follows from (1.11) that

$$\begin{bmatrix} m \\ r \end{bmatrix} q^r + \begin{bmatrix} m \\ r-1 \end{bmatrix} = \begin{bmatrix} m+1 \\ r \end{bmatrix}. \tag{1.12}$$

For $m = 1$, (1.10) reduces to (1.9). Assume that (1.10) holds for all $m \leq M$. Then by (1.9)

$$
\begin{aligned}
F(e_1, e_2) &= \sum_{r+s=M} \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r)N - M(M-1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r) \\
&= \sum_{r+s=M} \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r)N - M(M-1) + \frac{1}{2}s(s-1)} \Big\{ q^{N-r-s} F(e_1 - s - 1, e_2 - r) \\
&\qquad\qquad + q^{2(N-r-s)} F(e_1 - s, e_2 - r - 1) \Big\} \\
&= \sum_{r+s=M} \Big\{ \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r+1)N - M^2 + \frac{1}{2}s(s-1)} F(e_1 - s - 1, e_2 - r) \\
&\qquad + \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r+2)N - M(M+1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r - 1) \Big\} \\
&= \sum_{r+s=M+1} \Big\{ \begin{bmatrix} M \\ r \end{bmatrix} q^{(M+r+1)N - M^2 + \frac{1}{2}(s-1)(s-2)} \\
&\qquad + \begin{bmatrix} M \\ r-1 \end{bmatrix} q^{(M+r+1)N - M(M+1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r) \Big\} \\
&= \sum_{r+s=M+1} q^{(M+r+1)N - M(M+1) + \frac{1}{2}s(s-1)} \Big\{ \begin{bmatrix} M \\ r \end{bmatrix} q^{M-s+1} \\
&\qquad\qquad + \begin{bmatrix} M \\ r-1 \end{bmatrix} \Big\} F(e_1 - s, e_2 - r) \\
&= \sum_{r+s=M+1} \begin{bmatrix} M+1 \\ r \end{bmatrix} q^{(M+r+1)N - M(M+1) + \frac{1}{2}s(s-1)} F(e_1 - s, e_2 - r),
\end{aligned}
$$

by (1.12). Thus (1.10) holds for $M + 1$ and therefore for all $m \geq 0$.

For $m = N$, (1.10) reduces to

$$F(s, r) = q^{(2r+s)(r+s) - (r+s)(r+s-1) + \frac{1}{2}s(s-1)} \begin{bmatrix} r+s \\ r \end{bmatrix}.$$

Simplifying and interchanging r and s, we get

$$F(r, s) = q^{\frac{1}{2}r(r-1) + (r+s)(s+1)} \begin{bmatrix} r+s \\ r \end{bmatrix}. \tag{1.13}$$

By a familiar identity, (1.13) gives

$$\sum_{r=0}^{m} q^{-m(m-r+1)} F(r, m - r) x^r = (1 + x)(1 + qx) \cdots (1 + q^{m-1}x). \tag{1.14}$$

SECTION 2

The case $n = 3$ of (1.1) is more difficult. We have

$$
\begin{aligned}
F(e_1, e_2, e_3) = q^N F(e_1 - 1, e_2, e_3) &+ q^{2N} F(e_1, e_2 - 1, e_3) \\
&+ q^{3N} F(e_1, e_2, e_3 - 1).
\end{aligned} \tag{2.1}
$$

Iteration gives

$$\begin{aligned}
F(e_1, e_2, e_3) &= q^N\Big(q^{N-1}F(e_1 - 2, e_2, e_3) + q^{2N-2}F(e_1 - 1, e_2 - 1, e_3) \\
&\quad + q^{3N-3}F(e_1 - 1, e_2, e_3 - 1)\Big) + q^{2N}\Big(q^{N-1}F(e_1 - 1, e_2 - 1, e_3) \\
&\quad + q^{2N-2}F(e_1, e_2 - 2, e_3) + q^{3N-3}F(e_1, e_2 - 1, e_3 - 1)\Big) \\
&\quad + q^{3N}\Big(q^{N-1}F(e_1 - 1, e_2, e_3 - 1) + q^{2N-2}F(e_1, e_2 - 1, e_3 - 1) \\
&\quad + q^{3N-3}F(e_1, e_2, e_3 - 2)\Big) \\
&= q^{2N-1}F(e_1 - 2, e_2, e_3) + q^{3N-2}(1 + q)F(e_1 - 1, e_2 - 1, e_3) \\
&\quad + q^{4N-3}(1 + q)F(e_1 - 1, e_2, e_3 - 1) + q^{4N-2}F(e_1, e_2 - 2, e_3) \\
&\quad + q^{5N-3}(1 + q)F(e_1, e_2 - 1, e_3 - 1) + q^{6N-3}F(e_1, e_2, e_3 - 2).
\end{aligned}$$

A second iteration gives

$$\begin{aligned}
F(e_1, e_2, e_3) &= q^{3N-3}F(e_1 - 3, e_2, e_3) + q^{6N-6}F(e_1, e_2 - 3, e_3) \\
&\quad + q^{9N-9}F(e_1, e_2, e_3 - 3) \\
&\quad + q^{6N-8}(1 + 2q + 2q^3 + q^4)F(e_1 - 1, e_2 - 1, e_3 - 1) \\
&\quad + q^{4N-5}(1 + q + q^2)F(e_1 - 2, e_2 - 1, e_3) \\
&\quad + q^{5N-7}(1 + q^2 + q^4)F(e_1 - 2, e_2, e_3 - 1) \\
&\quad + q^{5N-6}(1 + q^2)F(e_1 - 1, e_2 - 2, e_3) \\
&\quad + q^{7N-9}(1 + q^2 + q^4)F(e_1 - 1, e_2, e_3 - 2) \\
&\quad + q^{7N-8}(1 + q + q^2)F(e_1, e_2 - 2, e_3 - 1) \\
&\quad + q^{8N-9}(1 + q + q^2)F(e_1, e_2 - 1, e_3 - 2).
\end{aligned}$$

It follows from the above that

$$F(1, 0, 0) = q,\ F(0, 1, 0) = q^2,\ F(0, 0, 1) = q^3, \tag{2.2}$$

$$\begin{cases} F(2, 0, 0) = q^3,\ F(0, 2, 0) = q^6,\ F(0, 0, 2) = q^9 \\ F(1, 1, 0) = q^4(1+q),\ F(1, 0, 1) = q^5(1+q^2),\ F(0, 1, 1) = q^7(1+q), \end{cases} \tag{2.3}$$

$$\begin{cases} F(3, 0, 0) = q^6,\ F(0, 3, 0) = q^{12},\ F(0, 0, 3) = q^{18} \\ F(2, 1, 0) = q^7(1+q+q^2),\ F(2, 0, 1) = q^8(1+q^2+q^4) \\ F(0, 2, 1) = q^{13}(1+q+q^2),\ F(1, 2, 0) = q^9(1+q+q^2) \\ F(1, 0, 2) = q^{12}(1+q^2+q^4),\ F(0, 1, 2) = q^{15}(1+q+q^2) \\ F(1, 1, 1) = q^{10}(1+2q+2q^3+q^4). \end{cases} \tag{2.4}$$

It is convenient to write (2.1) in operational form. Define the operators E_1^{-1}, E_2^{-1}, E_3^{-1} by means of

$$\begin{aligned} E_1^{-1}\phi(e_1, e_2, e_3) &= \phi(e_1 - 1, e_2, e_3),\ E_2^{-1}\phi(e_1, e_2, e_3) \\ &= \phi(e_1, e_2 - 1, e_3),\ E_3^{-1}\phi(e_1, e_2, e_3) = \phi(e_1, e_2, e_3 - 1) \end{aligned} \tag{2.5}$$

and put

$$\Omega = q^N E_1^{-1} + q^{2N} E_2^{-1} + q^{3N} E_3^{-1} \qquad (N = e_1 + e_2 + e_3). \tag{2.6}$$

Then (2.1) becomes

$$F(e_1, e_2, e_3) = \Omega F(e_1, e_2, e_3) \qquad (N > 0). \tag{2.7}$$

For $m \geq 0$ we may write

$$\Omega^m = \sum_{r+s+t=m} q^{(r+2s+3t)N} C(r, s, t) E_1^{-r} E_2^{-s} E_3^{-t}, \tag{2.8}$$

where $C(r, s, t)$ is independent of N. Moreover

$$C(0, 0, 0) = 1 \tag{2.9}$$

and $C(r, s, t) = 0$ if any one of r, s, $t = 0$.

By (2.7) and (2.8),

$$\begin{aligned} F(e_1, e_2, e_3) &= \Omega^N F(e_1, e_2, e_3) \\ &= \sum_{r+s+t=N} q^{(e_1+2e_2+3e_3)N} C(r, s, t) F(e_1 - r, e_2 - s, e_3 - t), \end{aligned}$$

so that

$$F(e_1, e_2, e_3) = q^{(e_1+2e_2+3e_3)N} C(e_1, e_2, e_3). \tag{2.10}$$

Hence (2.8) becomes

$$\Omega^m = \sum_{r+s+t=m} q^{(r+2s+3t)(N-m)} F(r, s, t) E_1^{-r} E_2^{-s} E_3^{-t} \tag{2.11}$$
$$(N = e_1 + e_2 + e_3,\ 0 \leq m \leq N).$$

Since

$$F(e_1, e_2, e_3) = \Omega^m F(e_1, e_2, e_3) \qquad (0 \leq m \leq N),$$

it therefore follows from (2.11) that

$$F(e_1, e_2, e_3) = \sum_{r+s+t=m} q^{(r+2s+3t)(N-m)} F(r, s, t) F(e_1 - r, e_2 - s, e_3 - t). \tag{2.12}$$

This may be written in a more symmetrical form:

$$F(a, b, c) = \sum_{\substack{r+r'=a \\ s+s'=b \\ t+t'=c \\ r+s+t=m}} q^{(r+2s+3t)(r'+s'+t')} F(r, s, t) F(r', s', t') \tag{2.13}$$
$$(0 \leq m \leq a + b + c),$$

with a, b, c, m fixed.

For example, with $a = b = c = 1$, $m = 2$, (2.13) gives

$$\begin{aligned} F(1,1,1) &= q^5F(0,1,1)F(1,0,0) + q^4F(1,0,1)F(0,1,0) + q^3F(1,1,0)F(0,0,1) \\ &= q^{13}(1+q) + q^{11}(1+q^2) + q^{10}(1+q) \\ &= q^{10}(1+2q+2q^3+q^4). \end{aligned}$$

Note that with $a = b = c = 1$, $m = 1$, we get

$$\begin{aligned} F(1,1,1) &= q^2F(1,0,0)F(0,1,1) + q^4F(0,1,0)F(1,0,1) + q^6F(0,0,1)F(1,1,0) \\ &= q^{10}(1+q) + q^{11}(1+q^2) + q^{13}(1+q) \\ &= q^{10}(1+2q+2q^3+q^4). \end{aligned}$$

Indeed (2.13) is not completely symmetrical in appearance. If we put $m' = r' + s' + t'$, then (2.13) yields

$$F(a,b,c) = \sum_{\substack{r'+r=a \\ s'+s=b \\ t'+t=c \\ r'+s'+t'=m}} q^{(r'+2s'+3t')(r+s+t)}F(r',s',t')F(r,s,t). \tag{2.14}$$

The equivalence of (2.13) and (2.14) can be verified directly by merely interchanging the roles of the primed and unprimed letters in (2.13).

By means of (2.13) a number of special values are easily computed. For example we have

$$\begin{cases} F(a,0,a) = q^{a-1}F(1,0,0)F(a-1,0,0) \\ F(0,b,0) = q^{2(b-1)}F(0,1,0)F(0,b-1,0) \\ F(0,0,c) = q^{3(c-1)}F(0,0,1)F(0,0,c-1). \end{cases}$$

It then follows that

$$f(a,0,0) = q^{\frac{1}{2}a(a+1)}, \quad f(0,b,0) = q^{b(b+1)}, \quad f(0,0,c) = q^{\frac{3}{2}c(c+1)}. \tag{2.15}$$

As another example

$$F(a,1,0) = q^aF(1,0,0)F(-1,1,0) + q^{2a}F(0,1,0)F(a,0,0)$$

and we find that

$$F(a,1,0) = q^{\frac{1}{2}(a^2+3a+4)}(1+q+\cdots+q^a). \tag{2.16}$$

Similarly

$$F(0,1,a) = q^{2a}F(0,1,0)F(0,0,a) + q^{3a}F(0,0,1)F(0,1,a-1),$$

which gives

$$F(0,1,a) = q^{\frac{1}{2}(a+1)(3a+4)}(1+q+\cdots+q^a). \tag{2.17}$$

Also

$$\begin{cases} F(a, 0, 1) = q^{\frac{1}{2}(a^2+3a+4)}(1+q^2+\cdots+q^{2a}) \\ F(1, 0, a) = q^{\frac{1}{2}(a+1)(3a+2)}(1+q^2+\cdots+q^{2a}) \end{cases} \tag{2.18}$$

$$\begin{cases} F(1, a, 0) = q^{(a+1)^2}(1+q+\cdots+q^a) \\ F(0, a, 1) = q^{a^2+3a+3}(1+q+\cdots+q^a). \end{cases} \tag{2.19}$$

Note that it follows from (2.16), (2.17), (2.18), and (2.19) that

$$\begin{cases} F(0, 1, a) = q^{a(a+2)}F(a, 1, 0) \\ F(1, 0, a) = q^{a(a-1)}F(a, 0, 1) \\ F(0, a, 1) = q^{a(a+2)}F(1, a, 0). \end{cases} \tag{2.20}$$

SECTION 3

It is evident from (2.1) that $F(a, b, c)$ is a polynomial in q with non-negative integral coefficients. Put

$$f(a, b, c) = \deg F(a, b, c), \tag{3.1}$$

the degree of $F(a, b, c)$. To evaluate $f(a, b, c)$ we use (2.1):

$$F(a, b, c) = q^N F(a-1, b, c) + q^{2N}F(a, b-1, c) + q^{3N}F(a, b, c-1) \qquad (N = a+b+c).$$

Then

$$f(a, b, c) = \max\left\{N+f(a-1, b, c),\ 2N+f(a, b-1, c),\ 3N+f(a, b, c-1)\right\}. \tag{3.2}$$

In particular

$$f(a, b, c) \geq 3N + f(a, b, c-1) \qquad (c > 0),$$

so that

$$f(a, b, c) \geq 3N + (N-1) + \cdots + 3(N-c+1) + f(a, b, 0).$$

Since, by (3.2),

$$f(a, b, 0) \geq 2(a+b) + f(a, b-1, 0),$$

we get

$$f(a, b, c) \geq 3N + 3(N-1) + \cdots + 3(N-c+1) + 2(N-c) + 2(N-c-1) + \cdots + 2(N-c-b+1) + f(a, 0, 0).$$

Hence, by (2.15)

$$f(a, b, c) \geq \frac{1}{2}a(a+1) + b(b+1) + \frac{3}{2}c(c+1) + 2ab + 3ac + 3bc. \tag{3.3}$$

We shall prove that, in fact,

$$f(a, b, c) = \frac{1}{2}a(a+1) + b(b+1) + \frac{3}{2}c(c+1) + 2ab + 3ac + 3bc. \tag{3.4}$$

This is evidently true for $a+b+c = 0, 1, 2, 3$. Assume that (3.4) holds for $a+b+c < M$. By (3.4), with $a+b+c = M$, we have

$$f(a, b, c-1) + a + b + c - f(a, b-1, c) = c$$
$$f(a, b-1, c) + a + b + c - f(a-1, b, c) = b + c.$$

Hence (3.2) reduces to

$$f(a, b, c) = 3(a+b+c) + f(a, b, c-1)$$

and it follows that (3.4) holds for $a+b+c = M$.

This completes the proof of (3.4).

The formulas (2.20) suggest the possibility of a relation of the following kind

$$F(a, b, c) = q^{d(a, b, c)}F(c, b, a), \tag{3.5}$$

for some integer $d(a, b, c)$. In view of (3.4)

$$d(a, b, c) = f(a, b, c) - f(c, b, a).$$

By (3.4) this gives

$$d(a, b, c) = (c-a)(a+b+c+1). \tag{3.6}$$

Thus (3.5) becomes

$$q^{a(N+1)}F(a, b, c) = q^{c(N+1)}F(c, b, a), \; N = a + b + c. \tag{3.7}$$

We shall show below (§5) by a combinatorial argument that (3.7) is indeed correct.

SECTION 4

Turning now to the general situation (1.6), we define the operators E_1^{-1}, E_2^{-1}, ..., E_n^{-1} by means of

$$E_j^{-1}\phi(\boldsymbol{e}) = \phi(\boldsymbol{e} - \delta_j), \tag{4.1}$$

where the notation is that in (1.5). We also put

$$\Omega = \sum_{j=1}^{n} q^{jN}E_j^{-1}\phi(\boldsymbol{e}) \qquad (N = e_1 + \cdots + e_n), \tag{4.2}$$

so that (1.6) becomes

$$F(\boldsymbol{e}) = \Omega F(\boldsymbol{e}) \qquad (N > 0). \tag{4.3}$$

Iteration of (4.3) gives

$$F(\boldsymbol{e}) = \Omega^m F(\boldsymbol{e}) \qquad (0 \le m \le N). \tag{4.4}$$

Generalizing (2.8), we write

$$\Omega^m = \sum_{\Sigma r_j = m} q^{\omega(\boldsymbol{r})N} C(\boldsymbol{r}) E_1^{-r_1} \ \dots \ E^{-r_n}, \tag{4.5}$$

where

$$= (r_1, r_2, \dots, r_n), \ \omega(r) = r_1 + 2r_2 + \cdots + nr_n \tag{4.6}$$

and $C(\boldsymbol{r})$ is independent of N. Then, in the first place, for $m = N$, (4.5) yields

$$F(\boldsymbol{e}) = q^{\omega(\boldsymbol{e})N} C(\boldsymbol{e}), \tag{4.7}$$

so that (3.5) becomes

$$\Omega^m = \sum_{\Sigma r_j = m} q^{\omega(\boldsymbol{r})(N-m)} C(\boldsymbol{r}) E_1^{-r_1} \ \dots \ E^{-r_n}. \tag{4.8}$$

It then follows from (4.5) and (4.8) that

$$F(\boldsymbol{e}) = \sum_{\Sigma r_j = m} q^{\omega(\boldsymbol{r})(N-m)} F(\boldsymbol{r}) F(\boldsymbol{e} - \boldsymbol{r}). \tag{4.9}$$

This result can be written in the more symmetrical form

$$F(\boldsymbol{e}) = \sum_{\substack{\Sigma r_j = m \\ r_j + r_j' = e_j}} q^{\omega(\boldsymbol{r})(N-m)} F(\boldsymbol{r}) F(\boldsymbol{r}'). \tag{4.10}$$

The remark about the equivalence of (2.13) and (2.14) is easily extended to (4.10).

As a simple application of (4.10) we take

$$F(a\delta_j) = q^{j(a-1)} F(\delta_j) F\big((a-1)\delta_j\big).$$

Then, since $F(\delta_j) = q^j$, we get

$$F(a\delta_j) = q^{\frac{1}{2}ja(a+1)} \qquad (1 \le j \le n). \tag{4.11}$$

This is evidently in agreement with (2.15).

Next

$$F(a, 1, 0, \dots, 0) = q^a F(1, 0, 0, \dots, 0) F(a-1, 1, 0, \dots, 0)$$
$$+ q^{2a} F(0, 1, 0, \dots, 0) F(a, 0, 0, \dots),$$

which reduces to

$$F(a, 1, 0, \ldots, 0) = q^{a+1}F(a-1, 1, 0, \ldots) + q^{\frac{1}{2}a(a+1)+2a+2}.$$

This gives

$$F(a, 1, 0, \ldots, 0) = q^{\frac{1}{2}(a^2+3a+4)}(1 + q + \cdots + q^a). \tag{4.12}$$

For example

$$F(1, 1, 0, \ldots, 0) = q^4(1+q),\ F(2, 1, 0, \ldots, 0) = q^7(1+q+q^2),$$

in agreement with earlier results.

Clearly $F(\boldsymbol{e})$ is a polynomial in q with non-negative integral coefficients. Put

$$d(\boldsymbol{e}) = \deg F(\boldsymbol{e}), \tag{4.13}$$

the degree of $F(\boldsymbol{e})$. Then by (1.6)

$$d(\boldsymbol{e}) = \max_{1 \le j \le n} \left\{ jN + d(\boldsymbol{e} - \delta_i) \right\}. \tag{4.14}$$

Thus, by (4.11)

$$\begin{aligned} d(\boldsymbol{e}) \ge\ & n\big(N + (N-1) + \cdots + (N-e_n+1)\big) \\ & + (n-1)\ \big((N-e_n) + (N-e_n-1) + \cdots + (N-e_n-e_{n-1}+1)\big) \\ & + \cdots + 2\big((N-e_n-\cdots-e_3) + \cdots + (N-e_n-\cdots-e_2+1)\big) \\ & + \frac{1}{2}e_1(e_1+1). \end{aligned}$$

After some manipulation this becomes

$$d(\boldsymbol{e}) \ge \frac{1}{2}N_1 + \frac{1}{2}N_2\ , \tag{4.15}$$

where

$$N_1 = \sum_{j=1}^{n} je\ ,\ N_2 = \sum_{i,j=1}^{n} \max\ (i, j)e_i e_j\,. \tag{4.16}$$

We shall show that indeed

$$d(\boldsymbol{e}) = \frac{1}{2}N_2 + \frac{1}{2}N_1\ . \tag{4.17}$$

To prove (4.17) it suffices to show that

$$N + d(\boldsymbol{e} - \delta_k) - d(\boldsymbol{e} - \delta_{k-1}) = e_k + e_{k+1} + \cdots + e_n \qquad (k = 2, 3, \ldots, n)$$

under the assumption that (4.17) holds up to and including $N-1$. Making use of (4.17) we find that

$$d(\boldsymbol{e} - \delta_k) - d(\boldsymbol{e} - \delta_{k-1}) = \sum_{j=1}^{k-1} e_j \qquad (k = 2, 3, \ldots, n)$$

and (4.18) follows.

Corresponding to

$$\boldsymbol{e} = (e_1, e_2, \ldots, e_n)$$

we define

$$\boldsymbol{e}' = (e_n, e_{n-1}, \ldots, e_1).$$

Clearly

$$N = \sum_{j=1}^{n} e_j = \sum_{j=1}^{n} e_{n-j+1}.$$

However

$$N_1' = \sum_{j=1}^{n} j e_{n-j+1} = \sum_{j=1}^{n} (n-j+1) e_j = (n+1)N - N_1.$$

Thus

$$d(\boldsymbol{e}') = \frac{1}{2}(N+1)\big((n+1)N - N_1\big), \tag{4.19}$$

so that

$$d(\boldsymbol{e}) - d(\boldsymbol{e}') = \frac{1}{2}(N+1)\big(2N_1 - (n+1)N\big). \tag{4.20}$$

In particulr, for $n = 3$, (4.20) reduces to

$$d(\boldsymbol{e}) - d(\boldsymbol{e}') = (N+1)(c-a)$$

in agreement with (3.6).

We shall show by a combinatorial argument in §5 that

$$F(\boldsymbol{e}) = q^{\frac{1}{2}(N+1)(2N_1-(n+1)N)} F(\boldsymbol{e}'). \tag{4.21}$$

SECTION 5

The combinatorial problem alluded to at the beginning of the paper is the following. Put

$$\boldsymbol{e} = (e_1, e_2, \ldots, e_n), \quad N = e_1 + e_2 + \cdots + e_n, \tag{5.1}$$

where the e_j are non-negative integers. Consider sequences of length N:

$$\sigma = (a_1, a_2, \ldots, a_N),$$

where the a_j are in $Z_n = \{1, 2, \ldots, n\}$ and each element i occurs exactly e_i times; $\boldsymbol{e}$ is called the *signature* of σ. We define the weight $\omega(\sigma)$ of σ by means of

$$\omega(\sigma) = \sum_{j=1}^{n} j\alpha_j. \tag{5.3}$$

We seek $f(\boldsymbol{e}, k)$, the number of sequences σ from Z_n of signature $\boldsymbol{\sigma}$ and weight k. It is convenient to define a refinement of $f(\boldsymbol{e}, k)$. For $1 \le j \le n$, we let $f_j(\boldsymbol{e}, k)$ denote the number of sequences σ from Z_n of signature $\boldsymbol{\sigma}$, weight k, and with last element $\alpha_N = j$. It follows immediately from the definition that

$$f(\boldsymbol{e}, k) = \sum_{j=1}^{n} f_j(\ , k). \tag{5.4}$$

Moreover

$$f_j(\boldsymbol{e}, k) = \sum_{i=1}^{n} f_i(\boldsymbol{e} - \delta_j, k - jN), \tag{5.5}$$

where δ_j has the same meaning as above.

Put

$$\begin{cases} F(\boldsymbol{e}) = F(\boldsymbol{e}, q) = \sum_k f(\boldsymbol{e}, k) q^k \\ F_j(\boldsymbol{e}) = F_j(\boldsymbol{e}, q) = \sum_k f_j(\boldsymbol{e}, k) q^k, \end{cases} \tag{5.6}$$

so that

$$F(\boldsymbol{e}) = \sum_{j=1}^{k} F_j(\boldsymbol{e}). \tag{5.7}$$

Multiplying both sides of (5.5) by q^k and summing over k, we get

$$\begin{aligned} F_j(\boldsymbol{e}) &= \sum_k \sum_{i=1}^{n} f_i(\boldsymbol{e} - \delta_j, k - jN) q^k \\ &= q^{jN} \sum_k f(\boldsymbol{e} - \delta_j, k) q^k \\ &= q^{jN} F(\boldsymbol{e} - \delta_j). \end{aligned}$$

Hence, summing over j, it is clear that

$$F(\boldsymbol{e}) = \sum_{j=1}^{n} q^{jN} F(\boldsymbol{e} - \delta_j). \tag{5.8}$$

This is identical with the recurrence (1.6); also $F(\boldsymbol{e})$ satisfies the same initial conditions as in §1.

The polynomial $F(\boldsymbol{e})$ also satisfies a second recurrence. To find this recurrence we let $\bar{f}_j(\boldsymbol{e}, k)$ denote the number of sequences σ from Z_n with signature $\boldsymbol{e}$, weight k, and *first* element $e_1 = j$. Then of course

$$f(\boldsymbol{e}, k) = \sum_{j=1}^{n} \bar{f}_j(\boldsymbol{e}, k). \tag{5.9}$$

We have also

$$\bar{f}_j(\boldsymbol{e}, k) = \sum_{i=1}^{n} \bar{f}_i(\boldsymbol{e} - \delta_j,\ k - N_1 + j) = f(\boldsymbol{e} - \delta_j,\ k - N_1 + j), \tag{5.10}$$

where

$$N_1 = e_1 + 2e_2 + \cdots + ne_n. \tag{5.11}$$

Hence, by (5.9)

$$F(\boldsymbol{e}) = \sum_{j=1}^{n} F(\boldsymbol{e} - \delta_j) q^{N_1 - j}.$$

Now put

$$\boldsymbol{e}' = (e_n,\ e_{n-1},\ \ldots,\ e_1) \tag{5.12}$$

and

$$\sigma' = (a'_N,\ a'_{N-1},\ \ldots,\ a'_1), \tag{5.13}$$

where

$$a'_j = n - a_j + 1 \qquad (j = 1,\ 2,\ \ldots,\ N).$$

Corresponding to (5.11), we put

$$N'_1 = e_n + 2e_{n-1} + \cdots + ne_1. \tag{5.14}$$

Thus

$$N_1 + N'_1 = (n+1)N. \tag{5.15}$$

The weight of σ' is evidently

$$\begin{aligned}\omega(\sigma') &= \sum_{j=1}^{N} j a'_{N-j+1} = \sum_{j=1}^{N} (N-j+1) a'_j = \sum_{j=1}^{N} (N-j+1)(n-a_j+1) \\ &= (n+1)N(N+1) - \tfrac{1}{2}(n+1)N(N+1) - (N+1)\sum_{j=1}^{N} a_j + \sum_{j=1}^{N} j a_j.\end{aligned}$$

This gives

$$\omega(\sigma') = \tfrac{1}{2}(n+1)N(N+1) - (N+1)N_1 + \omega(\sigma). \tag{5.16}$$

Thus there is a 1-1 correspondence between sequences σ of signature $\boldsymbol{e}$ and weight k, and sequences σ' of signature $\boldsymbol{e}'$ and weight

$$\tfrac{1}{2}(N+1)\big((n+1)N - 2N_1\big) + k,$$

so that

$$f(\boldsymbol{e}, k) = f\left(\boldsymbol{e}, \tfrac{1}{2}(N+1)\left((n+1)N - 2N_1\right) + k\right).$$

This yields

$$F(\boldsymbol{e}) = q^{\frac{1}{2}(N+1)(2N_1-(n+1)N)}F(\boldsymbol{e}'), \tag{5.17}$$

so that we have proved (4.21).

It is proved in (4.17) that

$$\deg F(\boldsymbol{e}) = \frac{1}{2}(N_1 + N_2), \tag{5.18}$$

which implies

$$f(\boldsymbol{e}, k) = 0 \qquad \left(k > \frac{1}{2}(N_1 + N_2)\right). \tag{5.19}$$

Also the proof of (4.17) gives

$$f\left(\boldsymbol{e}, \frac{1}{2}(N_1 + N_2)\right) = 1. \tag{5.20}$$

In the next place, define

$$\overline{\sigma} = (a_N, a_{N-1}, \ldots, a_1),$$

so that

$$\omega(\overline{\sigma}) = \sum_{j=1}^{N} j a_{N-j+1} = \sum_{j=1}^{N} (N-j+1)a_j = (N+1)N_1 - \omega(\sigma). \tag{5.21}$$

It therefore follows from (5.19) and (5.20) that

$$f(\boldsymbol{e}, k) = \begin{cases} 1 & \left(k = NN_1 + \frac{1}{2}(N_1 - N_2)\right) \\ 0 & \left(k < NN_1 + \frac{1}{2}(N_1 - N_2)\right). \end{cases} \tag{5.22}$$

Thus

$$\begin{cases} \omega_{\max}(\sigma) = \frac{1}{2}(N_1 + N_2) \\ \omega_{\min}(\sigma) = NN_1 + \frac{1}{2}(N_1 - N_2). \end{cases} \tag{5.23}$$

Finally it is evident from (5.21) that

$$F(\boldsymbol{e}, q) = q^{(N+1)N_1} F(\boldsymbol{e}, q^{-1}), \tag{5.24}$$

where we are using the fuller notation, $F(\boldsymbol{e}, q) = F(\boldsymbol{e})$.

Put

$$f_n(N, k) = \sum_{e_1 + \cdots + e_n = N} f(\boldsymbol{e}, k),$$

so that $f_n(N, k)$ is the number of sequences from Z_n of length N and weight k. Also put

$$F_n(N, q) = \sum_k f_n(N, k)q^k.$$

Then it follows almost immediately from the definition of $f_n(N, k)$ that

$$F_n(N, q) = q^{\frac{1}{2}N(N+1)} \prod_{j=1}^{N} \frac{1 - q^{nj}}{1 - q^j}. \tag{5.25}$$

Indeed it suffices to observe that the right-hand side of (5.25) is equal to

$$\prod_{j=1}^{N} (q^j + q^{2j} + \cdots + q^{nj}).$$

A curious partition identity is implied by (5.25). Put

$$\prod_{j=1}^{N} (1 - q\ )^{-1} = \sum_{m=0}^{\infty} p(m, N)q^m,$$

so that $p(m, N)$ is the number of partitions of m into parts $\leq N$. Now rewrite (5.25) in the form

$$q^{\frac{1}{2}N(N+1)} \sum_{k=0}^{\infty} p(k, N)q^k = \sum_{m=0}^{\infty} p(m, N)\, q^{mn} \sum_k f_n(N, k)q^k.$$

Then, equating coefficients of q^k, we get

$$p\left(k - \frac{1}{2}N(N+1)\right) = \sum_{mn \leq k} p(m, N) f_n(N, k - mn). \tag{5.26}$$

Another identity is obtained by replacing n by $2n$ in (5.25):

$$F_{2n}(N, q) = q^{\frac{1}{2}N(N+1)} \prod_{j=1}^{N} \frac{1 - q^{2nj}}{1 - q^j}.$$

Then by division

$$F_{2n}(N, q) = F_n(N, q) \prod_{j=1}^{N} (1 + q^{nj}).$$

Hence, if we put

$$\prod_{j=1}^{N} (1 + q^j) = \sum_{m=0}^{\frac{1}{2}N(N+1)} \overline{p}(m, N)q^m,$$

so that $\overline{p}(m, N)$ is the number of partitions of m into distinct parts $\leq N$, we get

$$\sum_k f_{2n}(N, k)q^k = \sum_k f_n(N, k)q^k \sum_{m=0}^{\frac{1}{2}N(N+1)} \overline{p}(m, N)q^{mn}.$$

Therefore

$$f_{2n}(N, k) = \sum_{mn \leq k} \overline{p}(m, N) f_n(N, k - mn). \tag{5.27}$$

For references to other enumerative problems involving sequences see [1].

REFERENCE

[1] L. Carlitz, "Permutations, Sequences and Special Functions," *SIAM Review* Vol. 17 (1975), pp. 298-322.

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SOME REMARKS ON A COMBINATORIAL IDENTITY

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SECTION 1

Let k, p, q, v be positive integers, $q < p < k$, n a non-negative integer and $\{\lambda_0 = 1, \lambda_1, \lambda_2, \ldots\}$ a sequence of indeterminates. Let $s(k, j)$ be the (signed) Stirling number of the first kind defined by

$$\sum_{j=0}^{k} s(k, j)x^j = x(x - 1) \ldots (x - k + 1).$$

Put

$$L(v, p, q) = \sum r_1 r_2 \ldots r_v \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v}, \tag{1.1}$$

where the summation is over all sets of integers $r_1, r_2, \ldots, r_v$ such that

$$p = r_0 \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q, \tag{1.2}$$

and

$$d_j = r_{j-1} - r_j \qquad (j = 1, 2, \ldots, v). \tag{1.3}$$

A. Ran [2] proved that

$$\sum_{j=0}^{k} s(k, j)L(j + n, p, q) \equiv 0 \tag{1.4}$$

identically, that is, for arbitrary $\lambda_1, \lambda_2, \lambda_3, \ldots$.

Hanani [1] has recently given another proof of (1.4). Hanani's proof is elementary but makes use of a rather difficult lemma.

The purpose of the present note is first to give another proof of (1.4) that makes use of the familiar recurrence

$$s(k + 1, j) = s(k, j - 1) - k \cdot s(k, j) \tag{1.5}$$

and the recurrence (2.2) below satisfied by $L(v, p, q)$. We show also that a result like (1.4) can be obtained for the more general sum

$$L_t(v, p, q) = \sum (r_1 r_2 \ldots r_v)^t \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} \tag{1.6}$$

where again the summation is over all $r_1, r_2, \ldots, r_v$ that satisfy (1.2) and (1.3).

We have been unable to find a simple generating function for $L(v, p, q)$. However, we do give an operational formula for the sum

$$F_v(y, z) = \sum_{p=0}^{\infty} \sum_{q=0}^{p} L(v, p, q) y^p z^q. \tag{1.7}$$

See (4.5) below.

SECTION 2

In view of (1.2) and (1.3) we can rewrite (1.1) in the following form:

$$L(v, p, q) = \sum_{d_1 + \cdots + d_v = q} (p - d_1)(p - d_1 - d_2) \ldots (p - d_1 - \ldots - d_v)\lambda_{d_1}\lambda_{d_2} \ldots \lambda_{d_v} \tag{2.1}$$

where the summation is over all nonnegative integers $d_1, d_2, \ldots d_v$ satisfying $d_1 + \cdots + d_v = q$. Thus

$$\begin{aligned} L(v+1, p, q) &= \sum_{d + d_1 + \cdots + d_v = q} (p - d)(p - d - d_1) \ldots (p - d - d_1 - \ldots - d_v)\lambda_d \lambda_{d_1} \ldots \lambda_{d_v} \\ &= \sum_{d=0}^{q} (p - d)\lambda \sum_{d_1 + \cdots + d_v = q - d} (p - d - d_1) \ldots (p - d - d_1 - \ldots - d_v)\lambda_{d_1} \ldots \lambda_{d_v} \\ &= \sum_{d=0}^{q} (p - d)\lambda_d L(v, p - d, q - d), \end{aligned}$$

so that

$$L(v+1, p, q) = \sum_{d=0}^{q} (p - d)\lambda_d L(v, p - d, q - d). \tag{2.2}$$

In the next place, by (1.5) and (2.2),

$$\begin{aligned} \sum_{j=0}^{k+1} s(k+1, j) L(j+n, p, q) &= \sum_j \left\{ s(k, j-1) - k \cdot s(k, j) \right\} L(j+n, p, q) \\ &= -k \sum_{s=0}^{j} s(k, j) L(j+n, p, q) \\ &\quad + \sum_{j=0}^{k} s(k, j) L(j+n+1, p, q) \\ &= -k \sum_{s=0}^{j} s(k, j) L(j+n, p, q) \\ &\quad + \sum_{j=0}^{k} s(k, j) \sum_{d=0}^{q} (p - d)\lambda_d L(j+n, p - d, q - d) \end{aligned}$$

(continued)

$$= -k\sum_{s=0}^{j} s(k, j)L(j+n, p, q)$$
$$+ \sum_{d=0}^{q}(p-d)\lambda \sum_{j=0}^{k} s(k, j)L(j+n, p-d, q-d).$$

Hence, if we put

$$R(k, n, p, q) = \sum_{j=0}^{k} s(k, j)L(j+n, p, q), \tag{2.3}$$

it is clear that we have proved that

$$R(k+1, n, p, q) = -kR(k, n, p, q) + \sum_{d=0}^{q}(p-d)\lambda_d R(k, n, p-d, q-d). \tag{2.4}$$

In particular, for $k = p$, (2.4) reduces to

$$R(p+1, n, p, q) = \sum_{d=1}^{q}(p-d)\lambda_d R(p, n, p-d, q-d). \tag{2.5}$$

Taking $q = 0$ in (2.1) we get

$$L(v, p, 0) = \sum_{d_1+\cdots+d_v=0}(p-d_1)\ \ldots\ (p-d_1-\ldots-d_v)\lambda_{d_1}\lambda_{d_2}\ \ldots\ \lambda_{d_v} = p^v$$

as is also clear from (1.1). Thus substitution in (2.3) gives

$$R(k, n, p, 0) = \sum_{j=0}^{k} s(k, j)p^{j+n} = p^n \cdot p(p-1)\ \ldots\ (p-k+1),$$

so that

$$R(k, n, p, 0) = 0 \qquad (k > p), \tag{2.6}$$

while

$$R(k, n, p, 0) = \frac{p^n \cdot p!}{(p-k)!} \qquad (k \le p). \tag{2.7}$$

Finally, by (2.6) and repeated application of (2.4) and (2.5), we have

$$R(k, n, p, q) = 0 \qquad (k > p \ge q \ge 0). \tag{2.8}$$

SECTION 3

The above proof of (1.4) suggests the following generalization. Let $t \ge 1$ and define generalized Stirling numbers of the first kind by means of

$$\sum_{s=0}^{k} s_t(k, j)x^j = x(x-1^t)(x-2^t)\ \ldots\ \left(x-(k-1)^t\right). \tag{3.1}$$

Put

$$L_t(v, p, q) = \sum (r_1 r_2 \ldots r_v)^t \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v}, \tag{3.2}$$

where the summation is over all $r_1, r_2, \ldots, r_v$ such that

$$p = r_0 \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q,$$

and

$$d_j = r_{j-1} - r_j \qquad (j = 1, 2, \ldots, v).$$

Then

$$\sum_{j=0}^{k} s_t(k, j) L_t(j+n, p, q) = 0, \tag{3.3}$$

where

$$n \geq 0, \; k > p \geq q > 0. \tag{3.4}$$

The proof is exactly like the proof of (2.8) and will be omitted.

SECTION 4

Put

$$F_v(y, z) = \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} L(v, p, q) y^p z^q, \; F_0(y, z) = \frac{1}{1-y} \tag{4.1}$$

and

$$\Lambda(z) = \sum_{d=0}^{\infty} \lambda_d z^d. \tag{4.2}$$

By (2.1),

$$L(v, p, q) = \sum_{d_1 + \cdots + d_v = q} (p - d_1)(p - d_1 - d_2) \ldots (p - d_1 - \ldots - d_v) \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v},$$

so that

$$F_v(y, z) = \sum_{d_1, \cdots, d_v = 0}^{\infty} \sum_{p \geq d_1 + \cdots + d_v} (p - d_1)(p - d_1 - d_2) \ldots (p - d_1 - \ldots - d_v) \cdot \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} y^p z^{d_1 + \cdots + d_v}$$

$$= \sum_{d_1, \ldots, d_v = 0}^{\infty} \sum_{p \geq d_2 + \cdots + d_v} p(p - d_2) \ldots (p - d_2 - \ldots - d_v) \cdot \lambda_{d_1} \lambda_{d_2} \ldots \lambda_{d_v} y^p (yz)^{d_1} z^{d_2 + \cdots + d_v}$$

$$= \Lambda(yz) \sum_{d_2,\ldots,d_v=0}^{\infty} \sum_{p \geq d_2+\cdots+d_v} p(p-d_2)\ldots(p-d_2-\ldots-d_v) \cdot \lambda_{d_2}\ldots\lambda_{d_v} y^p z^{d_2+\cdots+d_v}.$$

Since

$$\sum_{d_2,\ldots,d_v=0}^{\infty} \sum_{p \geq d_2+\cdots+d_v} p(p-d_2)\ldots(p-d_2-\ldots-d_v)\lambda_{d_2}\ldots\lambda_{d_v} y^p z^{d_2+\cdots+d_v}$$

$$= (yD_y) \sum_{d_2,\ldots,d_v=0}^{\infty} \sum_{p \geq d_2+\cdots+d_v} (p-d_2)\ldots(p-d_2-\ldots-d_v)\lambda_{d_2}\ldots\lambda_{d_v} y^p z^{d_2+\cdots+d_v},$$

where $D_y = \partial/\partial y$, it follows that

$$F_v(y, z) = \left(y\Lambda(yz)D_y\right)F_{v-1}(y, z). \tag{4.3}$$

Iteration of (4.3) gives

$$F_v(y, z) = \left(y\Lambda(yz)D_y\right)^{v-1}F_1(y, z) \qquad (v \geq 1).$$

Moreover, by (2.1) and (4.1),

$$F_1(y, z) = \sum_{d=0}^{\infty}\sum_{p=d}^{\infty}(p-d)\lambda_d y^p z^d = \sum_{d=0}^{\infty}\sum_{p=0}^{\infty} p\lambda_d y^p (yz)^d$$
$$= \frac{y}{(1-y)^2}\Lambda(yz) = \left(y\Lambda(yz)D_y\right)F_0(y, z).$$

Hence we get

$$F_v(y, z) = \left(y\Lambda(yz)D_y\right)^v F_0(y, z) \qquad (v \geq 0) \tag{4.4}$$

and more generally

$$F_{v+n}(y, z) = \left(y\Lambda(yz)D_y\right)^v F_n(y, z) \qquad (v \geq 0,\ n \geq 0). \tag{4.5}$$

By (2.3)

$$R(k, n, p, q) = \sum_{j=0}^{k} s(k, j)L(j+n, p, q).$$

Thus

$$G_{k,n}(y, z) \equiv \sum_{q=0}^{\infty}\sum_{p=q}^{\infty} R(k, n, p, q)y^p z^q = \sum_{j=0}^{k} s(k, j)F_{j+n}(y, z)$$
$$= \sum_{j=0}^{k} s(k, j)\left(y\Lambda(yz)D_y\right)^j \cdot F_n(y, z).$$

Hence if we put

$$z^{(k)} = z(z-1)\ldots(z-k+1) = \sum_{j=0}^{k} s(k, j)z^j,$$

we have

$$G_{k,n}(y, z) = \left(y\Lambda(yz)D_y\right)^k \cdot F(y, z), \tag{4.6}$$

where by (2.8),

$$G_{k,n}(y, z) = \sum_{q=0}^{\infty}\sum_{\substack{p=q\\ p\geq k}}^{\infty} R(k, n, p, q)y^p z^q. \tag{4.7}$$

We remark that in the special case

$$\lambda_n = 1 \qquad (n = 0, 1, 2, \ldots), \tag{4.8}$$

(1.1) reduces to

$$L(v, p, q) = \sum r_1 r_2 \ldots r_n, \tag{4.9}$$

where the summation is over all $r_1, r_2, \ldots, r_n$ such that

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q.$$

It is proved in the following article, "Enumeration of Certain Weighted Sequences," that, when (4.8) holds, $L(v, p, q)$ satisfies

$$L(v, p, q) = \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1} \qquad (v \geq 1;\ p \geq q \geq 0). \tag{4.10}$$

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ENUMERATION OF CERTAIN WEIGHTED SEQUENCES

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SECTION 1

The following problem has occurred somewhat incidentally in the preceding paper [1]. A complicated solution is implicit in the results of the paper. In the present paper we give a simple direct solution.

Let $v \geq 1$ and $p \geq q \geq 0$. Let $L(v, p, q)$ denote the sum

$$\sum r_1 r_2 \ldots r_v, \tag{1.1}$$

where the summation is over all $r_1, r_2, \ldots, r_v$ satisfying

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_v = p - q. \tag{1.2}$$

To get a recurrence for $L(v, p, q)$, we observe that, for $v > 1$,

$$L(v, p, q) = (p - q)\sum r_1 r_2 \ldots r_{v-1},$$

where now

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_{v-1} \geq p - q.$$

Hence

$$L(v, p, q) = (p - q)\sum_{k=0}^{q}\sum r_1 r_2 \ldots r_{v-1},$$

where, in the inner sum

$$p \geq r_1 \geq r_2 \geq \ldots \geq r_{v-1} = p - k.$$

It follows that

$$L(v, p, q) = (p - q)\sum_{k=0}^{q} L(v - 1, p, k) \qquad (v > 1). \tag{1.3}$$

Replacing q by $q - 1$ in (1.3), we get

$$L(v, p, q-1) = (p-q+1)\sum_{k=0}^{q-1} L(v-1, p, k).$$

Combining this with (1.3) we get the recurrence

$$(p-q+1)L(v, p, q) - (p-q)L(v, p, q-1) = (p-q)(p-q+1)L(v-1, p, q). \tag{1.4}$$

We shall now think of p as an indeterminate and define

$$M(v, p, q) = \frac{L(v, p, q)}{p - q}. \tag{1.5}$$

Then (1.4) yields

$$M(v, p, q) = M(v, p, q-1) + (p-q)M(v-1, p, q) \qquad (v > 1), \tag{1.6}$$

together with the initial conditions

$$\begin{cases} M(1, p, q) = 1 & (q = 0, 1, 2, \ldots) \\ M(v, p, 0) = p^{v-1} & (v = 1, 2, 3, \ldots). \end{cases} \tag{1.7}$$

Clearly $M(v, p, q)$ is uniquely determined by (1.6) and (1.7). The first few values are easily computed

$v \backslash q$	0	1	2	3
1	1	1	1	1
2	p	$2p - 1$	$3p - 3$	$4p - 6$
3	p^2	$3p^2 - 3p + 1$	$6p^2 - 12p + 7$	$10p^2 - 30p + 25$
4	p^3	$4p^3 - 6p^2 + 4p - 1$	$10p^3 - 30p^2 + 35p - 15$	$20p^3 - 90p^2 + 150p - 90$

We shall show that generally

$$M(v, p, q) = \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1}. \tag{1.8}$$

For $v = 1$, (1.8) reduces to

$$\begin{aligned} M(1, p, q) &= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^q \\ &= \frac{1}{q!}\sum_{s=0}^{q}(-1)^{q-s}\binom{q}{s}s^q = 1 \qquad (q = 0, 1, 2, \ldots), \end{aligned}$$

by well-known results from finite differences. Also by (1.8),

$$M(v, p, 0) = p^{v-1} \qquad (v = 1, 2, 3, \ldots).$$

Thus (1.7) is verified.

Now assume that (1.8) holds for all v, q such that

$$v + q < m. \tag{1.9}$$

Then, for $v + q = m$, we have

$$M(v, p, q-1) + (p-q)M(v-1, p, q)$$

$$= \frac{1}{(q-1)!}\sum_{s=0}^{q-1}(-1)^s\binom{q-1}{s}(p-s)^{v+q-2} + \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-2}$$

$$= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(q-s)(p-s)^{v+q-2} + \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-2}$$

$$= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-2}\big((q-s)+(p-q)\big)$$

$$= \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1}$$

$$= M(v, p, q).$$

Hence (1.8) holds for $v + q = m$, thus completing the induction.

Finally, by (1.5) and (1.8), we have

$$L(v, p, q) = \frac{p-q}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(p-s)^{v+q-1} = \frac{p-q}{q!}\Delta_p^q(p-q)^{v+q-1} \tag{1.10}$$

where Δ_p^q denotes the finite difference operator defined by

$$\Delta_p f(p) = f(p+1) - f(p), \quad \Delta_p^q f(p) = \Delta_p \cdot \Delta_p^{q-1} f(p).$$

For $p \geq q \geq 0$, $v \geq 1$, (1.10) evaluates the sum (1.1).

SECTION 2

For $p = q$, (1.8) reduces to

$$M(v, q, q) = \frac{1}{q!}\sum_{s=0}^{q}(-1)^s\binom{q}{s}(q-s)^{v+q-1} = \frac{1}{q!}\sum_{s=0}^{q}(-1)^{q-s}\binom{q}{s}^{v+q-1},$$

so that

$$M(v, q, q) = S(v+q-1, q) \qquad (v \geq 1), \tag{2.1}$$

a Stirling number of the second kind. Generally, it follows from (1.8) that

$$M(v, p, q) = \frac{1}{q!}\sum_{s=0}(-1)^s\binom{q}{s}\sum_{t=0}^{v+q-1}\binom{v+q-1}{t}(p-q)^{v+q-t-1}(q-s)^t,$$

which gives

$$M(v, p, q) = \sum_{t=q}^{v+q-1} \binom{v+q-1}{t} (p-q)^{v+q-t-1} S(t, q). \tag{2.2}$$

It follows from (1.8) that

$$\sum_{n=0}^{\infty} M(n-q+1, p, q)\frac{z^n}{n!} = \frac{1}{q!}\sum_{s=0}^{\infty}(-1)^s\binom{q}{s}e^{(p-s)z} = \frac{1}{q!}e^{(p-q)z}(e^z-1)^q,$$

so that

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n} M(n-q+1, p+q, q)x^q\frac{z^n}{n!} = e^{pz} \exp\left\{x(e^z - 1)\right\}. \tag{2.3}$$

For additional properties of the sum

$$\sum_{k=0}^{q}(-1)^k\binom{q}{k}(p-k)^n,$$

see [2, Ch. 1].

SECTION 3

The results of §1 can be generalized in the following way. Let $t \geq 1$ and put

$$L\ (v, p, q) = \sum (r_1 r_2 \ \dots\ r_v)^t, \tag{3.1}$$

where the summation is over all r , r , ..., r_v satisfying

$$p \geq r_1 \geq r_2 \geq \dots \geq r_v = p - q.$$

Then, in the first place

$$L_t(v, p, q) = (p-q)^t \sum_{k=0}^{q} L_t(v-1, p, k) \qquad (v > 1). \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned}(p-q+1)^t L_t(v, p, q) &- (p-q)^t L_t(v, p, q-1)\\ &= (p-q)^t (p-q+1)^t L_t(v-1, p, q) \qquad (v > 1).\end{aligned} \tag{3.3}$$

Hence

$$M_t(v, p, q) = M_t(v, p, q-1) + (p-q)M_t(v-1, p, q) \qquad (v > 1), \tag{3.4}$$

where

$$M_t(v, p, q) = \frac{L_t(v, p, q)}{(p-q)^t}$$

and

$$\begin{aligned} M_t(1, p, q) &= 1 \qquad (q = 0, 1, 2, \ldots) \\ M_t(v, p, 0) &= p^{t(v-1)} \qquad (v = 1, 2, 3, \ldots). \end{aligned} \tag{3.5}$$

As in §1, we are again thinking of p as an indeterminate.

By means of (3.4) and (3.5) it is easy to show that

$$M_t(v+1, p, q) = \sum_{i_0+i_1+\cdots+i_q=v} p^{i_0 t}(p-1)^{i_1 t} \ldots (p-q)^{i_q t}. \tag{3.6}$$

It then follows that

$$\sum_{v=0}^{\infty} M_t(v+1, p, q) z^v = \sum_{j=0}^{q} \left(1 - (p-j)^t z\right)^{-1}. \tag{3.7}$$

Now put

$$\prod_{j=0}^{q} \left(1 - (p-j)^t z\right)^{-1} = \sum_{j=0}^{q} \frac{A_j^{(t)}}{1 - (p-j)^t z}$$

where the $A_j^{(t)}$ are independent of z. Then

$$A_j^{(t)} = \prod_{\substack{i=0 \\ i \neq j}}^{q} \left(1 - (p-i)^t (p-j)^{-t}\right)^{-1} = (p-j)^{qt} \prod_{\substack{i=0 \\ i \neq j}}^{q} \left((p-j)^t - (p-i)^t\right)^{-1}. \tag{3.8}$$

Finally, we have

$$M_t(v+1, p, q) = \sum_{j=0}^{q} A_j (p-j)^{tv},$$

with $A_j^{(t)}$ given by (3.8).

For $t = 1$, (3.8) reduces to

$$A_j^{(1)} = (p-j)^q \prod_{\substack{i=0 \\ i \neq j}}^{q} (i-j)^{-1} = \frac{(-1)^j (p-j)^q}{j!\,(q-j)!} = \frac{(-1)^j}{q!} \binom{q}{j} (p-j)^q.$$

Hence (3.9) becomes

$$M_1(v+1, p, q) = \frac{1}{q!} \sum_{j=0}^{q} (-1)^j \binom{q}{j} (p-j)^{q+v},$$

in agreement with (1.8).

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THE NUMBER OF DERANGEMENTS OF A SEQUENCE WITH GIVEN SPECIFICATION

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SECTION 1

Consider sequences

$$\sigma = (a_1, a_2, \ldots, a_N), \tag{1.1}$$

where $a_j \in Z_k = \{1, 2, \ldots, k\}$. The sequence is said to have *specification* $[n_1, n_2, \ldots, n_k]$, where the n_j are non-negative integers, $N = n_1 + n_2 + \cdots + n_k$, if each element j, $1 \le j \le k$, occurs in σ exactly n_j times. The sequence is called a *derangement* provided no element is in a position occupied by it in the sequence

$$(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, k, k, \ldots, k). \tag{1.2}$$

Let $P(n_1, n_2, \ldots, n_k)$ denote the number of possible derangements. Even and Gil is [1] (see also Jackson [2]) have proved the following result.

$$P(n_1, n_2, \ldots, n_k) = (-1)^{n_1+n_2+\cdots+n_k} \cdot \int_0^\infty e^{-x}\left\{\prod_{j=1}^{k} L_{n_j}(x)\right\} dx, \tag{1.3}$$

where $L_n(x)$ is the Laguerre polynomial defined by

$$L_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\frac{x^j}{j!}. \tag{1.4}$$

The object of the present note is to give a simple proof of (1.3) along the lines of the standard proof of the case $n_1 = n_2 = \cdots = n_k = 1$ [3, p. 59]. We also prove some related results.

SECTION 2

Let
$$P(\boldsymbol{n}, \boldsymbol{m}) = P(n_1, \ldots, n_k; m_1, \ldots, m_k), \tag{2.1}$$

where $0 \le m_j \le n_j$, denote the number of sequences (1.1) in which, for each j, exactly m_j of the values remain in their original position in (1.2). It follows at once from the definition that

$$P(\boldsymbol{n}, \boldsymbol{m}) = P(\boldsymbol{n} - \boldsymbol{m}, \boldsymbol{0})\prod_{j=1}^{k}\binom{n_j}{m_j} = P(\boldsymbol{n} - \boldsymbol{m})\prod_{j=1}^{k}\binom{n_j}{m_j}, \tag{2.2}$$

where $P(\boldsymbol{n}) = P(n_1, n_2, \ldots, n_k)$.

Clearly

$$\sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} P(\boldsymbol{n}, \boldsymbol{m}) = (n_1, n_2, \ldots, n_k) = \frac{N!}{n_1! n_2! \ldots n_k!},$$

where

$$\sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} \equiv \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \cdots \sum_{m_k=0}^{n_k}.$$

Thus, by (2.2),

$$\sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} P(m) = (n_1, n_2, \ldots, n_k).$$

This relation is equivalent to

$$\begin{aligned} P(\boldsymbol{n}) &= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^{N-M} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} (m_1, \ldots, m_k) \qquad (2.3) \\ &= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^{N-M} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{M!}{m_1! \ldots m_k!}, \end{aligned}$$

where $M = m_1 + \cdots + m_k$.

SECTION 3

To verify that (2.3) is in agreement with (1.3), we take

$$\begin{aligned} \int_0^\infty e^{-x} \left\{ \prod_{j=1}^{k} L_{n_j}(x) \right\} dx &= \int_0^\infty e^{-x} \left\{ \prod_{j=1}^{k} \sum_{m=0}^{n_j} (-1)^{m_j} \binom{n_j}{m_j} \frac{x^{m_j}}{m_j!} \right\} dx \\ &= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^M \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{1}{m_1! \ldots m_k!} \int_0^\infty e^{-x} x^M dx \\ &= \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^M \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{M!}{m_1! \ldots m_k!}. \end{aligned}$$

This evidently proves the equivalence of (1.3) and (2.3).

SECTION 4

Put

$$P_k(N) = \sum_{n_1 + \cdots + n_k = N} P(\boldsymbol{n}). \qquad (4.1)$$

Thus $P(n)$ denotes the total number of derangements from Z_k of length N. Then by (2.3) we have

$$P_k(n) = \sum_{n_1 + \cdots + n_k = N} \sum_{\boldsymbol{m}=0}^{\boldsymbol{n}} (-1)^{N-M} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \frac{M!}{m_1! \ldots m_k!}$$

$$= \sum_{m_1+\cdots+m_k=N} (-1)^{N-M} \frac{M!}{m_1!\ \ldots\ m_k!} \sum_{n_1+\cdots+n_k=N} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k},$$

where as above $M = m_1 + \cdots + m_k$. Since the inner sum on the extreme right is equal to

$$\binom{N+k-1}{M+k-1},$$

we get

$$P_k(N) = \sum_{m_1+\cdots+m_k\le N} (-1)^{N-M} \frac{M!}{m_1!\ \ldots\ m_k!} \binom{N+k-1}{M+k-1}$$

$$= \sum_{M=0}^{N} (-1)^{N-M} \binom{N+k-1}{M+k-1} \sum_{m_1+\cdots+m_k=M} \frac{M!}{m_1!\ \ldots\ m_k!}.$$

By the multinomial theorem

$$\sum_{m_1+\cdots+m_k=M} \frac{M!}{m_1!\ \ldots\ m_k!} = k^M,$$

so that

$$P_k(N) = \sum_{M=0}^{N} (-1)^{N-M} \binom{N+k-1}{M+k-1} k^M \tag{4.2}$$

It follows from (4.2) that

$$k^{k-1}P_k(N) = \sum_{m=k-1}^{N+k-1} (-1)^{N+k-m-1} \binom{N+k-1}{m} k^m$$

$$= \sum_{m=0}^{N+k-1} (-1)^{N+k-m-1} \binom{N+k-1}{m} k^m - \sum_{j=0}^{k-2} (-1)^{N+k-j-1} \binom{N+k-1}{j} k^j$$

and therefore

$$P_k(N) = k^{1-k}\left\{(k-1)^{N+k-1} - \sum_{j=0}^{k-2} (-1)^{N+k-j-1} \binom{N+k-1}{j} k^j\right\} \qquad (k \ge 1). \tag{4.3}$$

It follows from (4.3) that, for fixed $k > 2$,

$$P_k(N) \sim k^{1-k}(k-1)^{N+k-1} \qquad (N \to \infty). \tag{4.4}$$

On the other hand, if N is fixed and $k \to \infty$, it is evident from (4.2) that

$$P_k(N) = \sum_{M=0}^{N} (-1)^M \binom{N+k-1}{M} k^{N-M} \sim \sum_{M=0}^{N} (-1)^M \frac{k^M}{M!} k^{N-M},$$

so that

$$P_k(N) \sim k^N \sum_{M=0}^{N} \frac{(-1)^M}{M!} \qquad (k \to \infty). \tag{4.5}$$

SECTION 5

Fairly simple generating functions are implied by (4.2). We have first

$$\begin{aligned}\sum_{N=0}^{\infty} x^N \sum_{M=0}^{N} (-1)^{N-M} \binom{N+k-1}{M+k-1} k^M &= \sum_{M=0}^{\infty} k^M x^M \sum_{N=0}^{\infty} (-1)^N \binom{N+M+k-1}{M+k-1} x^N \\ &= \sum_{M=0}^{\infty} k^M x^M (1+x)^{-M-k} \\ &= (1+x)^{-k} \left(1 - \frac{kx}{1+x}\right)^{-1}.\end{aligned}$$

Hence

$$\sum_{N=0}^{\infty} P_k(N) x^N = (1+x)^{-k+1} (1+x-kx)^{-1}. \tag{5.1}$$

In the next place

$$\begin{aligned}\sum_{N=0}^{\infty} P_k(N) \frac{x^N}{(N+k-1)!} &= \sum_{N=0}^{\infty} \frac{x^N}{(N+k-1)!} \sum_{M=0}^{N} (-1)^{N-M} \binom{N+k-1}{M+k-1} k^M \\ &= \sum_{M=0}^{\infty} \frac{k^M x^M}{(M+k-1)!} \sum_{N=0}^{\infty} (-1)^N \frac{x^N}{N!} \\ &= e^{-x} \sum_{M=0}^{\infty} \frac{k^M x^M}{(M+k-1)!}.\end{aligned}$$

Thus

$$\sum_{N=0}^{\infty} P_k(N) \frac{x^N}{(N+k-1)!} = (kx)^{-k+1} e^{-x} \left\{ e^{kx} - \sum_{m=0}^{k-2} \frac{k^m x^m}{m} \right\} \qquad (k \geq 1). \tag{5.2}$$

It is easily seen that (4.3) is implied by (5.2).

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ENUMERATION OF PERMUTATIONS BY SEQUENCES

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SECTION 1

André [2] discussed the enumeration of permutations by number of sequences; his results are reproduced in Netto [5, pp. 105-112]. Let $P(n, s)$ denote the number of permutations of $Z_n = \ldots 1, 2, \ldots, n \ldots$ with s ascending or descending sequences. For example, the permutation 24315 has the ascending sequences 24, 15 and the descending sequence 431; the permutation 613254 has the ascending sequences 13, 25 and the descending sequences 61, 32, 54. The total number of sequences is five. Generally, a permutation of Z_n has at most $n - 1$ sequences; such a permutation is called an *up-down* or *down-up* permutation according as it begins with an ascending or a descending sequence. Clearly, in this case all the sequences are of length two.

It is convenient to put

$$P(0, s) = \delta_{0,s}, \; P(1, s) = \delta_{0,s} . \tag{1.1}$$

André proved that $P(n, s)$ satisfies the recurrence

$$P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2) \qquad (n \geq 2). \tag{1.2}$$

With the convention $P(1, s) = \delta_{0,s}$, (1.2) holds for $n \geq 1$.

$P(n, s)$:

n \ s	0	1	2	3	4	5
1	1					
2		2				
3		2	4			
4		2	12	10		
5		2	28	58	32	
6		2	60	236	300	122

Let $A(n)$ denote the number of up-down and $B(n)$ the number of down up permutations of Z_n. Then

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$$A(n) = B(n) = \frac{1}{2}P(n, n-1) \qquad (n \geq 2). \tag{1.3}$$

Moreover, André [1] showed that

$$\sum_{n=0}^{\infty} A(n)\frac{z^n}{n!} = \sec z + \tan z, \tag{1.4}$$

with $A(0) = A(1) = 1$. Thus, a generating function for $P(n, n-1)$ is known; also, (1.4) yields an explicit formula for $A(n)$ and, therefore, also for $P(n, n-1)$.

A generating function for $P(n, s)$ has apparently not been found. We shall show that

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\,\frac{z^n}{n!}\sum_{s=0}^{n}P(n+1, s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2} + \sin z}{x - \cos z}\right)^2. \tag{1.5}$$

We have been unable to find an explicit formula for $P(n, s)$. However, it follows from (1.2) and (1.3) that

$$P(n, n-2) = 2A(n+1) - 4A(n) \qquad (n \geq 2),$$
$$P(n, n-3) = A(n+2) - 4A(n+1) - (n-5)A(n) \qquad (n \geq 3),$$

and so on. Generally, we have

$$P(n, n-s) = \sum_{j=1}^{s} f_{sj}(n)A(n+s-j) \qquad (n \geq s > 0),$$

where the $f_{sj}(n)$ are polynomials in n, $f_{s1}(n) = 1$. However, the $f_{sj}(n)$ are not evaluated.

If we let $P(n, r, s)$ denote the number of permutations of Z_n with r ascending and s descending sequences, it is easy to show that

$$\begin{cases} P(n, r, r) = P(n, 2r) \\ P(n, r, r-1) = P(n, r-1, r) = \frac{1}{2}P(n, 2r-1). \end{cases}$$

Moreover, $P(n, r, s) = 0$ unless $r = s$, $s+1$, or $s-1$. Also, permutations can be classified further according as they begin or end with either an ascending or descending sequence. This suggests the four enumerants

$$P_{++}(n, r, s), \quad P_{+-}(n, r, s), \quad P_{-+}(n, r, s), \quad P_{--}(n, r, s);$$

for precise definitions, see §5 below.

It is also of some interest to adapt another point of view. We define $P(n, r, s)$ as the number of permutations π of Z_n with r ascending and s descending sequences in which we count an additional ascending sequence if π begins with a descending sequence, also an additional descending sequence if π ends with an ascending sequence. For the relation of $P(n, r, s)$ to the other enumerants and a generating function, see §§5 and 6.

SECTION 2

Put

$$P_n(x) = \sum_{s=0}^{n-1} P(n, s)x^s \qquad (n \geq 1) \tag{2.1}$$

and

$$G(x, z) = \sum_{n=0}^{\infty} P_{n+1}(x)\frac{z^n}{n!}. \tag{2.2}$$

By (1.2) and (2.1),

$$\begin{aligned} P_{n+2}(x) &= \sum_{s=0}^{n+1} P(n+2, s)x^s \\ &= \sum_{s=0}^{n+1} \left\{sP(n+1, s) + 2P(n+1, s-1) + (n-s+2)P(n+1, s-2)\right\}x^s \\ &= xP'_{n+1}(x) + 2xP_{n+1}(x) + \sum_{s=0} (n-x)P(n+1, s)x \\ &= xP'_{n+1}(x) = 2xP_{n+1}(x) + nx^2P_{n+1}(x) - x^3P'_{n+1}(x). \end{aligned}$$

Hence

$$P_{n+2}(x) = (nx^2 + 2x)P_{n+1}(x) - (x^3 - x)P'_{n+1}(x) \qquad (n \geq 0). \tag{2.3}$$

It now follows from (2.2) that

$$\begin{aligned} \frac{\partial G(x, z)}{\partial z} &= \sum_{n=0}^{\infty} P_{n+2}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} \left\{(nx^2 + 2x)P_{n+1}(x) - (x^3 - x)P'_{n+1}(x)\right\}\frac{z^n}{n!} \\ &= 2xG(x, z) + x^2z\frac{\partial G(x, z)}{\partial z} - (x^3 - x)\frac{\partial G(x, z)}{\partial x}. \end{aligned}$$

Thus

$$(x^3 - x)\frac{\partial G(x, z)}{\partial x} - (x^2z - 1)\frac{\partial G(x, z)}{\partial z} = 2xG. \tag{2.4}$$

The system

$$\frac{dx}{x^3 - x} = \frac{dz}{-x^2z + 1} = \frac{dG}{2xG} \tag{2.5}$$

has the integrals

$$z\sqrt{x^2 - 1} + \arcsin \frac{1}{x}, \ \frac{x+1}{x-1}G. \tag{2.6}$$

It follows that

$$\frac{x+1}{x-1}G(x,\ z) = \phi\left(z\sqrt{x^2-1} + \arcsin\frac{1}{x}\right), \tag{2.7}$$

for some $\phi(u)$.

It is convenient to replace x by x^{-1} and z by xz, so that (2.7) becomes

$$\frac{1+x}{1-x}G(x^{-1},\ xz) = \phi\left(z\sqrt{1-x^2} + \arcsin x\right). \tag{2.8}$$

For $z = 0$, (2.8) reduces to

$$\frac{1+x}{1-x}G(x^{-1},\ 0) = \phi(\arcsin x).$$

Since $G(x^{-1},\ 0) = 1$, it follows at once that

$$\phi(u) = \frac{1+\sin u}{1-\sin u}. \tag{2.9}$$

Hence (2.8) becomes, on replacing z by $z/\sqrt{1-x^2}$,

$$\frac{1+x}{1-x}G\left(x^{-1},\ \frac{xz}{\sqrt{1-x^2}}\right) = \frac{1+\sin(z+\arcsin x)}{1-\sin(z+\arcsin x)}.$$

It can be verified that the right member is equal to

$$\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2.$$

Therefore, we have

$$H(x,\ z) = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2 \tag{2.10}$$

where

$$H(x,\ z) = G\left(x^{-1},\ \frac{xz}{\sqrt{1-x^2}}\right) = \sum_{n=0}^{\infty}(1-x^2)^{-n/2}\frac{z^n}{n!}\sum_{s=0}^{n}P(n+1,\ s)x^{n-s}. \tag{2.11}$$

SECTION 3

For $x = 0$, (2.10) reduces to

$$\sum_{n=0}^{\infty}P(n+1,\ n)\frac{z^n}{n!} = \frac{(1+\sin z)^2}{\cos^2 z} = 2\sec^2 z + 2\sec z\tan z - 1. \tag{3.1}$$

By (1.4),

$$\sum_{n=0}^{\infty}A(n+1)\frac{z^n}{n!} = \sec z\tan z + \sec^2 z \tag{3.2}$$

while, by (1.3),

$$\sum_{n=0}^{\infty}P(n+1,\ n)\frac{z^n}{n!} = 1+2\sum_{n=1}^{\infty}A(n+1)\frac{z^n}{n!} = -1+2\sum_{n=0}^{\infty}A(n+1)\frac{z^n}{n!}.$$

Hence (3.1) and (3.2) are in agreement.

We may rewrite (2.10) in the form

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{s=0}^{n}P(n+1,\ s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin(z\sqrt{1-x^2})}{x-\cos(z\sqrt{1-x^2})}\right)^2 \tag{3.3}$$

n n n n n

It is clear from the definition that

$$\sum_{s=0}^{n}P(n+1,\ s) = (n+1)! \tag{3.4}$$

Hence, for $x = 1$, the left-hand side of (3.3) should reduce to

$$\sum_{n=0}^{\infty}(n+1)z^n = (1-z)^{-2}.$$

As for the right-hand side of (3.3), we have

$$\frac{1-x}{1+x}\left\{\frac{(1-x^2)^{\frac{1}{2}}+z(1-x^2)^{\frac{1}{2}}-\frac{1}{3!}z\ (1-x^2)^{\frac{3}{2}}+\cdots}{x-1+\frac{1}{2!}z^2(1-x^2)-\frac{1}{4!}z^4(1-x^2)^2+\cdots}\right\}^2$$

$$= \left\{\frac{1+z-\frac{1}{3!}z^3(1-x^2)+\cdots}{1-\frac{1}{2!}z^2(1+x)+\cdots}\right\}^2,$$

which reduces to

$$\left(\frac{1+z}{1-z^2}\right)^2 = (1-z)^{-2}. \tag{3.5}$$

Note also that for $x = -1$, we get $(1+z)^2$. It therefore follows from (3.3) that

$$\sum_{s=0}^{n}(-1)^{n-s}P(n+1,\ s) = 0 \qquad (n > 2). \tag{3.6}$$

This is a known result [2], [5].

Combining (3.6) with (3.4) gives

$$\sum_{2s\le n}P(n+1,\ 2s) = \sum_{2s\le n}P(n+1,\ 2s+1) = \frac{1}{2}(n+1)! \tag{3.7}$$

If we take $s = n$ in (1.2) we get $P(n+1, n) = 2P(n, n-1) + P(n, n-2)$. Thus it follows from (1.3) that

$$P(n,\ n-2) = 2A(n+1) - 4A(n) \qquad (n \ge 2). \tag{3.8}$$

Taking $s = n - 1$, we get

$$P(n+1,\ n-1) = (n-1)P(n,\ n-1) + 2P(n,\ n-2) + 2P(n,\ n-3),$$

which gives

$$P(n,\ n-3) = A(n+2) - 4A(n+1) - (n-5)A(n) \qquad (n \geq 3). \tag{3.9}$$

Next, taking $s = n - 2$, we get

$$P(n,\ n-4) = A(n+3) - 6A(n+2) - (3n-16)A(n+1) + (6n-18)A(n) \tag{3.10}$$
$$(n \geq 4).$$

Thus it appears that

$$P(n,\ n-s) = \sum_{j=1}^{s} f_{sj}(n)A(n+s-j) \qquad (n \geq s > 0), \tag{3.11}$$

where the $f_{sj}(n)$ are polynomials in n, $f_{s1}(n) = 1$. Indeed, using (1.2), we find that

$$sf_{s+1,j}(n) = f_{s,j}(n+1) - (n-s+1)f_{s-1,j-2}(n) - 2f_{s,j-1}(n). \tag{3.12}$$

However, it is not evident how to evaluate the $f_{s,j}(n)$ from this recurrence.

Returning to (2.10), if we replace x by $\cos x$, we get

$$\sum_{n=0}^{\infty} \frac{(z/\sin x)^n}{n!} \sum_{s=0}^{n} P(n+1,\ s)\cos^{n-s} x = \frac{1-\cos x}{1+\cos x}\left(\frac{\sin x + \sin z}{\cos x - \cos z}\right)^2.$$

Hence

$$\cot \frac{1}{2}x \sum_{n=0}^{\infty} \frac{(z/\sin x)^n}{n!} \sum_{s=0}^{n} P(n+1,\ s)\cos^{n-s} x = \cot^2 \frac{1}{2}(x-z). \tag{3.13}$$

Since the right-hand side of (3.13) is symmetric in x, z, it follows that

$$\frac{1}{2}x \sum_{n=0}^{\infty} \frac{(z/\sin x)^n}{n!} \sum_{s=0}^{n} P(n+1,\ \ )\cos^{n-s} x \tag{3.14}$$
$$= \cot \frac{1}{2}z \sum_{n=0}^{\infty} \frac{(x/\sin z)^n}{n!} \sum_{s=0}^{n} P(n+1,\ x)\cos^{n-s} z.$$

It would be interesting to know whether there is some combinatorial result equivalent to (3.14).

SECTION 5

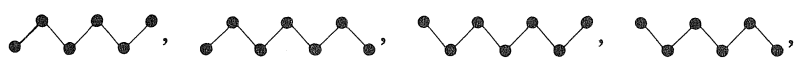
As a refinement of $P(n,\ s)$ we define $P(n,\ r,\ s)$ as the number of permutations of Z_n with r ascending and s descending sequences. It is evident that $P(n,\ r,\ s) = 0$ unless $r = s$, $s+1$, or $s-1$. Moreover, since a permutation can be read from left to right or right to left, we have

$$P(n,\ r,\ r-1) = P(n,\ r-1,\ r).$$

It accordingly follows that

$$\begin{cases} P(n,\ r,\ r) = P(n,\ 2r) \\ P(n,\ r,\ r-1) = P(n,\ r-1,\ r) = \frac{1}{2}P(n,\ 2r) \end{cases} \tag{5.1}$$

Now divide the permutations of Z into four nonoverlapping classes according as they begin or end with ascending or descending sequence. We denote the classes by C_{++}, C_{+-}, C_{-+}, C_{--}. The permutations in these classes have the appearance



(5.2)

respectively. Denote the corresponding enumerants by

$$P_{++}(n,\ r,\ s),\quad P_{+-}(n,\ r,\ s),\quad P_{-+}(n,\ r,\ s),\quad P_{--}(n,\ r,\ s).$$

Then we have the following equalities:

$$P_{++}(n,\ r,\ s) = P_{--}(n,\ s,\ r) \tag{5.3}$$

and

$$P_{+-}(n,\ r,\ s) = P_{-+}(n,\ s,\ r).$$

These relations follow on applying the transformation

$$b_i = n - a_i + 1 \qquad (i = 1,\ 2,\ \ldots,\ n)$$

to any permutation $(a_1,\ a_2,\ \ldots,\ a_n)$ of Z_n. Alternatively (5.3) follows on first reading a permutation of C_{++} from left to right and then from right to left.

In the next place, it is evident from (5.2) that $r = s + 1$ in C_{++}, $r = s$ in C_{++} or C_{--}, $r = s - 1$ in C_{--}. Thus

$$P_{+-}(n,\ r,\ s) = P_{-+}(n,\ r,\ s) = 0 \qquad (r \neq s), \tag{5.5}$$

$$P_{++}(n,\ r,\ s) = 0 \qquad (r \neq s + 1), \tag{5.6}$$

$$P_{--}(n,\ r,\ s) = 0 \qquad (r \neq s - 1). \tag{5.7}$$

Hence

$$\begin{cases} P_{+-}(n,\ r,\ r) = P_{-+}(n,\ r,\ r) = \frac{1}{2}P(n,\ 2r) \\ P_{++}(n,\ r,\ r-1) = P_{--}(n,\ r-1,\ r) = \frac{1}{2}P(n,\ 2r-1). \end{cases} \tag{5.8}$$

In view of (5.8), generating functions for the four enumerants are implied by (2.10).

Another point of view is of some interest. Given a permutation $(a_1, a_2, \ldots, a_n)$ of Z_n, we adjoin virtual elements $0, 0'$: $(0, a_1, a_2, \ldots, a_n, 0')$. If $a_1 > a_2$, then $0a_1$ is counted as an additional ascending sequence; if however $a_1 < a_2$, the number of ascending sequences is unchanged. Similarly, if $a_{n-1} < a_n$, then $a_n 0'$ is counted as an additional descending sequence; if $a_{n-1} > a_n$, the number of descending sequences is unchanged. Also, let $P(n, r, s)$ denote the number of permutations of Z_n with r ascending and s descending sequences using these conventions. It follows at once that

$$\overline{P}(n, r, s) = 0 \qquad (r \neq s). \tag{5.9}$$

Moreover we have, by (5.8)

$$\begin{aligned}\overline{P}(n, r, r) = P_{+-}(n, r, r) &+ P_{-+}(n, r-1, r-1) \\ &+ P_{++}(n, r, r-1) + P_{--}(n, r-1, r).\end{aligned} \tag{5.10}$$

To illustrate (5.10), take $n = 4$, $r = 2$. The permutations are:

$$C_{++}\begin{cases}1 & 3 & 2 & 4\\ 1 & 4 & 2 & 3\\ 2 & 3 & 1 & 4\\ 2 & 4 & 1 & 3\\ 3 & 4 & 1 & 2\end{cases} \qquad C_{--}\begin{cases}2 & 1 & 4 & 3\\ 3 & 1 & 4 & 2\\ 3 & 2 & 4 & 1\\ 4 & 1 & 3 & 2\\ 4 & 2 & 3 & 1\end{cases}$$

For $n = 3$, $r = 2$, the permutations are:

$$C_{-+}\begin{cases}2 & 1 & 3\\ 3 & 1 & 2\end{cases}$$

For $n = 3$, $r = 1$:

$$C_{+-}\begin{cases}1 & 3 & 2\\ 2 & 3 & 1\end{cases}.$$

It follows form (5.8) and (5.10) that

$$\overline{P}(n, 2r) = P_{+-}(n, r, r) + P_{-+}(n, r-1, r-1) + P(n, 2r-1). \tag{5.11}$$

We have also

$$\overline{P}_n(x) = P_n^{+-}(x) + x^{-2}P_n^{-+}(x) + x^{-1}P_n^{++}(x) + x^{-1}P_n^{--}(x) \tag{5.12}$$

and

$$P_n(x) = P_n^{+-}(x) + P_n^{-+}(x) + P_n^{++}(x) + P_n^{--}(x), \tag{5.13}$$

where

$$P_n(x) = \sum_r P(n, r)x^{n-k}, \quad \overline{P}_n(x) = \sum_r \overline{P}(n, r, r)x^{n-2r},$$

$$P_n^{+-}(x) = \sum_r P_{+-}(n, r, r)x^{n-2r},$$

$$P_n^{++}(x) = \sum_r P_{++}(n, r, r-1)x^{n-2r-1}, \text{ etc.}$$

Note that $P_n(x)$ is not the same as the $P_n(x)$ of (2.1).

Comparison of (5.13) with (5.12) gives

$$\overline{P}_n(x) - x^{-1}P_n(x) = (1 - x^{-1})^2 P_n^{+-}(x). \tag{5.14}$$

SECTION 6

A generating function for $P(n, r, r)$ can be obtained rapidly by using a known result on the enumeration of permutations by maxima. Given the permutation $(a_1, a_2, \ldots, a_n)$ of Z_n, then a_k, $1 < k < n$, is a maximum if $a_{k-1} < a_k$, $a_k > a_{k-1}$. In addition, a_1 is a maximum if $a_1 > a_2$; a_n is a maximum if $a_{n-1} < a_n$. Let $M(n, m)$ denote the number of permutations of Z with m maxima.

Clearly if a permutation has m maxima in accordance with this definition, then it has exactly m ascending and m descending sequences and conversely. Thus

$$\overline{P}(n, r, r) = M(n, r). \tag{6.1}$$

A generating function for $M(n, k)$ is furnished by [3], [4]:

$$\sum_{n,k=0}^{\infty} M(n + 2k + 1, k + 1)\frac{u^n v^{2k}}{(n + 2k)!} = \left\{\cosh\sqrt{u^2 - v^2} - \frac{u}{\sqrt{u^2 - v^2}}\sinh\sqrt{u^2 - v^2}\right\}^{-2}. \tag{6.2}$$

Making some changes in notation, this becomes

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{2j \le n} M(n+1, j+1)x = \frac{1 - x^2}{(\sqrt{1 - x^2}\cos z - x\sin z)^2}. \tag{6.3}$$

Finally, in view of (6.1), we have

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{2j \le n} P(n+1, j+1, j+1)x = \frac{1 - x^2}{(\sqrt{1 - x^2}\cos z - x\sin z)^2}. \tag{6.4}$$

If we put

$$H(x, z) = \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}P_{n+1}(x), \quad H(x, z) = \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\overline{P}_{n+1}(x),$$

$$H^{+-}(x, z) = \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}P_{n+1}^{+-}(x),$$

it follows from (5.14) that

$$x\overline{H}(x, z) - x^{-1}H(x, z) = (1 - x^{-1})^2 H^{+-}(x, z). \tag{6.5}$$

Therefore, by (2.10) and (5.14), we get

$$x^{-1}(1-x^2)H^{+-}(x, z) = \frac{x^2(1+x)^2}{\left(\sqrt{1-x^2}\cos z - x\sin z\right)^2} - \left(\frac{\sqrt{1-x^2} + \sin z}{x - \cos z}\right)^2. \tag{6.6}$$

Values of $P(n, r, s)$ for $n = 2, 3, 4$ follow.

$n = 2$:

$r \backslash s$	0	1
0	•	1
1	1	•

$n = 3$:

$r \backslash s$	0	1
0	•	1
1	1	4

$n = 4$:

$r \backslash s$	0	1	2
0	•	1	•
1	1	12	5
2	•	5	•

$n = 5$:

$r \backslash s$	0	1	2
0	•	1	•
1	1	28	29
2	•	29	32

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GENERALIZED TRIBONACCI NUMBERS AND THEIR CONVERGENT SEQUENCES

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1. INTRODUCTION

In this note we consider sequences $\{a_n\}$ generated by the third-order recurrence relation

$$a_{n+3} = \alpha a_{n+2} + \beta a_{n+1} + \gamma a_n, \quad n = 1, 2, 3, \ldots, \tag{1}$$

with real parameters α, β, γ and arbitrary real numbers a_1, a_2, a_3. Sequences like these have been considered by [3], [4], [5] with the TRIBONACCI sequence for $\alpha = \beta = \gamma = 1$ and $a_1 = a_2 = 1$, $a_3 = 2$ as a special case [1].

In this paper we give in the second section a general real representation for a_n using the roots of the auxiliary equation

$$P_3(x) := x^3 - \alpha x^2 - \beta x - \gamma \tag{2}$$

in all possible cases. In the third section we characterize convergent sequences, give their limits and, finally, in the fourth section we consider various series with a_n as terms and give their limits by the use of a generating function.

2. REAL REPRESENTATION FOR $\{a_n\}$

According to the general theory of recurrence relations $\{a_n\}$ can be represented by

$$a_n = Aq_1^{n-1} + Bq_2^{n-1} + Cq_3^{n-1}, \quad n = 1, 2, 3, \ldots, \tag{3}$$

where q_1, q_2, q_3 are the roots of the auxiliary equation $P_3(x) = 0$. The constants A, B, C are given by the linear equations system from (3) for $n = 1, 2, 3$ with prescribed "start" numbers a_1, a_2, a_3. The determinant of this system is the VANDERMONDE determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{vmatrix} = \prod_{\substack{i,k=1 \\ i>k}} (q_i - q_k) = (q_2 - q_1)(q_3 - q_1)(q_3 - q_2)$$

which does not vanish for distinct q_1, q_2, q_3. In this case we get

$$A = \frac{q_2q_3a_1 - (q_2 + q_3)a_2 + a_3}{(q_3 - q_1)(q_2 - q_1)} \qquad B = \frac{-q_1q_3a_1 + (q_1 + q_3)a_2 - a_3}{(q_3 - q_2)(q_2 - q_1)}$$

$$C = \frac{q_1q_2a_1 - (q_1 + q_2)a_2 + a_3}{(q_3 - q_2)(q_3 - q_1)}$$

So we have by (3) a real representation for a_n, if the roots of $P_3(x)$ are distinct and real. If two roots are equal, e.g., $q_2 = q_3$, we get from (3) the limit as q_3 approaches q_2

$$a_n = Dq_1^{n-1} + E_nq_2^{n-1},\ n = 1, 2, 3, \ldots, \tag{4}$$

with

$$D = \frac{q_2^2a_1 - 2q_2a_2 + a_3}{(q_2 - q_1)^2},$$

$$E_n = \frac{1}{(q_2 - q_1)^2}\left\{[(n - 3)q_2 - (n - 2)q_1]a_1q_1 - \left[(n - 3)q_2 - (n - 1)\frac{q_1^2}{q_2}\right]a_2 + \left[(n - 2) - (n - 1)\frac{q_1}{q_2}\right]a_3\right\}.$$

If all roots are equal, we get from (4) the limit as q_2 approaches q_1

$$a_n = F_nq_1^{n-1},\ u = 1, 2, 3, \ldots, \tag{5}$$

with

$$F_n = \frac{(n - 2)(n - 3)}{2}a_1 - \frac{(n - 1)(n - 3)}{q_1}a_2 + \frac{(n - 1)(n - 2)}{2q_1^2}a_3.$$

In the last of the possible cases for the roots of $P_3(x) = 0$ we have one real root q_1 and two conjugate complex ones q_2, q_3. Writing $q_2 = re^{i\varphi}$, $q_3 = \overline{q}_2 = re^{-i\varphi}$ we get

$$a_n = Gq_1^{n-1} + H_nr^{n-1},\ n = 1, 2, 3, \ldots, \tag{6}$$

with

$$G = \frac{r^2a_1 - 2r\cos\varphi a_2 + a_3}{r^2 - 2rq_1\cos\varphi + q_1^2},$$

$$H_n = \frac{(a_1q_1 - a_2)r\sin(n-3)\varphi + (a_3 - a_1q_1^2)\sin(n-2)\varphi + (a_2q_1 - a_3)\frac{q_1}{r}\sin(n-1)\varphi}{\sin\varphi(r^2 - 2rq_1\cos\varphi + q_1^2)},$$

$$r = \sqrt{q_1^2 - \alpha q_1 - \beta}, \quad \varphi = 2 \arctan \sqrt{\frac{2r + q_1 - \alpha}{2r - q_1 + \alpha}},$$

where q_1 can be computed by the formula of CARDANO for the reduced form of $P_3(x)$ (without the quadratic term).

3. CONVERGENT SEQUENCES $\{a_n\}$

In the two-dimensional case, that means $\gamma = 0$ in (1), we were able to characterize convergent sequences immediately from the real representation for a_n [2]. Some similar considerations yield in the three-dimensional case:

Theorem 1: The sequences $\{a_n\}$ defined by (1) are convergent if and only if the parameters α, β, γ are points of the three-dimensional region

(7) $\vartheta := \left\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \,\middle|\, \alpha + \beta + \gamma \leq 1,\ -\alpha + \beta - \gamma < 1,\ \gamma^2 - \alpha\gamma - \beta < 1\right\}$ (Fig. 1)

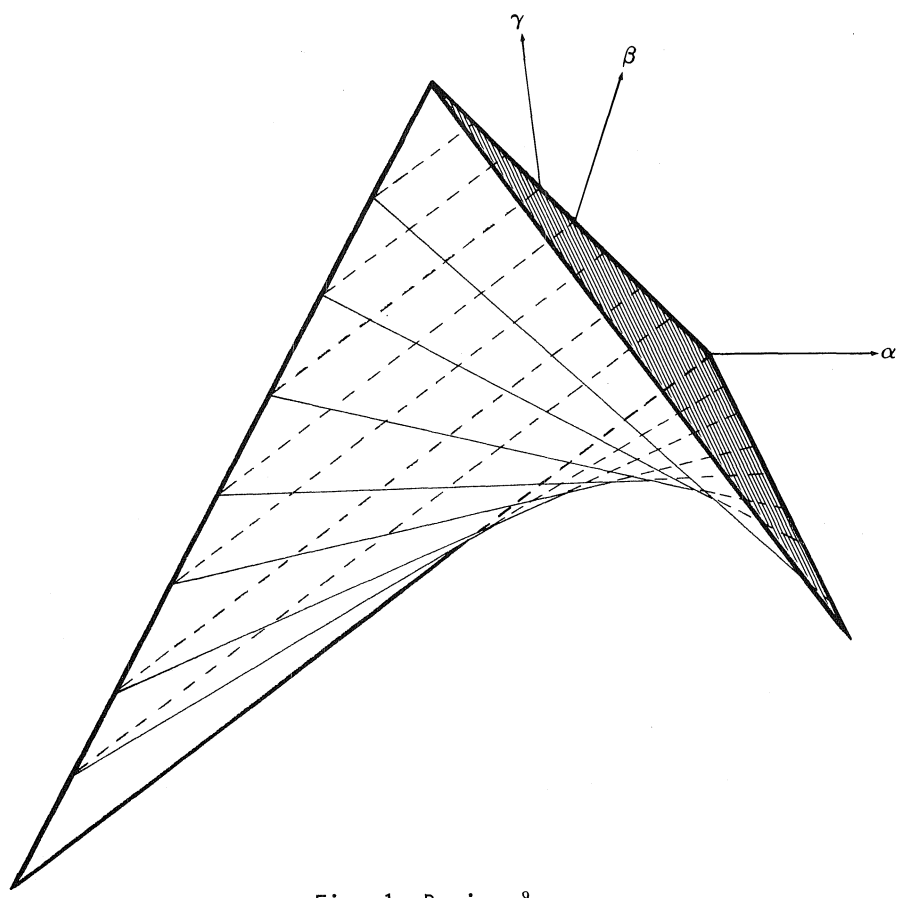

Fig. 1 Region ϑ

for all real numbers a_1, a_2, a_3. In the interior $\underline{\vartheta}$ of the region ϑ the sequences $\{a_n\}$ converge to zero. On the boundary $\alpha + \beta + \gamma = 1$ of ϑ the limit of a_n is given by

$$a := \lim_{n\to\infty} a = \frac{\gamma a_1 + (1-\alpha)a_2 + a_3}{2+\gamma-\alpha}, \quad 2+\gamma-\alpha \neq 0. \tag{8}$$

Proof: From the real representations for a_n we obtain the following necessary and sufficient conditions for convergence.

1. All roots of $P_3(x) = 0$ are distinct and real:

$$-1 < q_1,\ q_2,\ q_3 \leq 1 \tag{9a}$$

2. Two distinct real roots:

$$-1 < q_1 \leq 1,\ -1 < q_2 = q_3 < 1 \tag{9b}$$

3. All roots are equal:

$$-1 < q_1 = q_2 = q_3 < 1 \tag{9c}$$

4. One real root and two conjugate complex ones:

$$-1 < q_1 \leq 1,\ 0 < q_2 q_3 = r^2 < 1. \tag{9d}$$

This means, for the polynomial $P_3(x)$, that

$$\begin{aligned} P_3(-1) &= -1 - \alpha + \beta - \gamma < 0, \\ P_3(1) &= 1 - \alpha - \beta - \gamma \geq 0. \end{aligned} \tag{10}$$

We have the following relations between the coefficients and the roots of $P_3(x)$ (VIETA):

$$\begin{aligned} q_1 + q_2 + q_3 &= \alpha, \\ q_1q_2 + q_2q_3 + q_1q_3 &= -\beta, \\ q_1q_2q_3 &= \gamma. \end{aligned} \tag{11}$$

We start with the case $\gamma > 0$. Then q_1 may be the smallest of the positive roots, the only positive of the real roots, or the only real root of $P_3(x) = 0$. It follows from the last equation (11) with $0 < q_2q_3 < 1$ from (9a)-(9d):

$$0 < \gamma < q_1.$$

We can conclude that, in the interval $[0, \gamma]$, there is no further root of $P_3(x)$; which, using the continuity of $P_3(x)$, means that $P_3(0)$ and $P_3(\gamma)$ have the same signs. So with $P_3(0) = -\gamma < 0$, $P_3(\gamma) = \gamma(\gamma^2 - \alpha\gamma - \beta - 1) < 0$, or with $\gamma > 0$,

$$\gamma^2 - \alpha\gamma - \beta < 1. \tag{12}$$

The case $\gamma = 0$ leads to the known two-dimensional case [2] and corresponds to the fact that one or more roots are zero. There we have convergence for points $(\alpha, \beta) \in R^2$ which satisfy the inequalities

$$\alpha + \beta \leq 1, \ -\alpha + \beta < 1, \ \beta > -1. \tag{13}$$

If $\gamma < 0$, then q_1 may be the greatest negative of the negative roots, the only negative of the real roots or the only real root of $P_3(x) = 0$. It follows from the last equation (11) with $0 < q_2 q_3 < 1$ that

$$q_1 < \gamma < 0.$$

We conclude, as in the first case, that $P_3(0)$ and $P_3(\gamma)$ have the same signs. We have with $P_3(0) = -\gamma > 0$, $P_3(\gamma) = \gamma(\gamma^2 - \alpha\gamma - \beta - 1) > 0$ or because of $\gamma < 0$

$$\gamma^2 - \alpha\gamma - \beta < 1. \tag{14}$$

So we have convergence in all cases if and only if $(\alpha, \beta, \gamma) \in R^3$ satisfy the inequalities (10), (12), (13), and (14), which define the required region ϑ (Fig. 1).

If (α, β, γ) are points of $\underline{\vartheta}$, the interior of ϑ, we have $|q_\nu| < 1$, $\nu = 1, 2, 3$, and it follows with the limits

$$\lim_{n\to\infty} n^\mu q_\nu^n = 0, \ \mu = 0, 1, 2; \ \nu = 1, 2, 3,$$

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} (q_2 q_3)^{\frac{n}{2}} = 0$$

from the real representation (3)-(6) for a_n, that a_n converges to zero. If $P_3(1) = 1 - \alpha - \beta - \gamma = 0$, we are on the boundary of ϑ (shaded area in Fig. 1). This means that 1 is a root of $P_3(x)$. We set $q_1 = 1$ and get, from (3), (4), or (6),

$$a := \lim_{n\to\infty} a_n = A = \frac{\gamma a_1 - (\alpha - 1)a_2 + a_3}{2 + \gamma - \alpha} = \frac{(1 - \alpha - \beta)a_1 - (\alpha - 1)a_2 + a_3}{3 - 2\alpha - \beta} = G.$$

Also, if $q_2 = 1$, we have, from (11), $q_3 = \alpha - 2$, $2q_3 + 1 = -\beta$, and $q_3 = \gamma$, so that $\gamma^2 - \alpha\gamma - \beta = q_3 - (q_3 + 2)q_3 + 2q_3 + 1 = 1$, which contradicts the inequalities (12), (14); thus, $q_1 = 1$ must be a single root. Dividing $P_3(x)$ by the linear term $(x - 1)$, we get $P_2(x) := x^2 + (1 - \alpha)x + 1 - \alpha - \beta$. Since $q_1 = 1$ is a single root, we obtain $P_2(1) \neq 0$, so that $2 + \gamma - \alpha = 3 - 2\alpha - \beta \neq 0$, as stated in (8).

4. CONVERGENT SERIES

By the use of the generating function

$$\frac{a_1 + (a_2 - \alpha a_1)x + (a_3 - \alpha a_2 - \beta a_1)x^2}{1 - \alpha x - \beta x^2 - \gamma x^3} = \sum_{\nu=0}^{\infty} a_{\nu+1} x^{\nu} \tag{15}$$

we will give some limits of infinite series with a_ν, $\nu = 1, 2, \ldots$, as terms. First, we determine the radius of convergence ρ of the power series in (15). It is given by the smallest absolute value of the roots of

$$Q_3(x) := 1 - \alpha x - \beta x^2 - \gamma x^3 = 0. \tag{16}$$

Substituting in $Q_3(x)y = \frac{1}{x}$, $x \neq 0$, we get

$$Q_3\left(\frac{1}{y}\right) = \frac{1}{y^3}(y^3 - \alpha y^2 - \beta y - \gamma) = \frac{1}{y^3}P_3(y). \tag{17}$$

Using the notation of §3, with q_ν, $\nu = 1, 2, 3$, as the roots of $P_3(x)$ for the radius of convergence, we get

$$\rho = \min\left\{\frac{1}{|q_1|}, \frac{1}{|q_2|}, \frac{1}{|q_3|}\right\},$$

or as a further result,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{\rho} = \max\left\{|q_1|, |q_2|, |q_3|\right\}. \tag{18}$$

If $(\alpha, \beta, \gamma) \in \underline{\vartheta}$ we have $|q_\nu| < 1$, $\nu = 1, 2, 3$, so that

$$\rho > 1, \ (\alpha, \beta, \gamma) \in \underline{\vartheta}.$$

Especially, we have convergence in (15) for $x = 1$. So we get for $x = 1$:

$$\sum_{\nu=1}^{\infty} a_\nu = \frac{(1 - \alpha - \beta)a_1 + (1 - \alpha)a_2 + a_3}{1 - \alpha - \beta - \gamma}, \ (\alpha, \beta, \gamma) \in \underline{\vartheta}; \tag{19}$$

for $x = -1$:

$$\sum_{\nu=1}^{\infty} (-1)^{\nu-1} a_\nu = \frac{(1 + \alpha - \beta)a_1 - (1 + \alpha)a_2 + a_3}{1 + \alpha - \beta + \gamma}, \ (\alpha, \beta, \gamma) \in \underline{\vartheta}. \tag{20}$$

Addition or subtraction of (19) and (20) and division by 2 yields

$$(21)\quad \sum_{\nu=1}^{\infty} a_{2\nu-1} = \frac{[(1-\beta)^2 - \alpha^2 - 2\alpha\gamma]a_1 + (\gamma + \alpha\beta)a_2 + (1-\beta)a_3}{(1-\beta)^2 - (\alpha+\gamma)^2}, \quad (\alpha, \beta, \gamma) \in \underline{\vartheta}$$

or

$$\sum_{\nu=1}^{\infty} a_{2\nu} = \frac{\gamma(1-\beta)a_1 - [1 - \beta - \alpha(\alpha+\gamma)]a_2 + (\alpha+\gamma)a_3}{(1-\beta)^2 - (\alpha+\gamma)^2}, \quad (\alpha, \beta, \gamma) \in \underline{\vartheta}.$$

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KNIGHT'S TOUR REVISITED

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ABSTRACT

This paper shows that on any big enough rectangular chessboard there is a knight's path. If the number of squares is even there is a circuit.

A board is big enough if its smaller dimension is at least 5.

KNIGHT'S TOUR REVISITED

Almost two hundred years ago Euler investigated the problem of whether it was possible to move a knight through every square of a chessboard once and return to the starting square. Euler demonstrated that this knight's tour was possible by displaying a chessboard with the required sequence of moves. He also generalized the problem by showing that there were other size boards on which the knight's tour was possible.

We must recall at this point that the standard chessboard has 64 squares arranged in 8 equal rows and columns. The knight is the chess piece that often looks like a horse. If the knight is on square (i, j), it is allowed to move to one of the eight possible squares $(i \pm 1, j \pm 2)$ and $(i \pm 2, j \pm 1)$, if these squares are on the board.

We first became interested in knight's tour when we wanted some examples to test the behavior of a heuristic algorithm on graphs that had Hamiltonian paths. A graph is a pair (V, E), where V is a finite set of objects called vertices or nodes, and E, the set of edges, is a subset of $V \times V$ such that if $(i, j) \in E$, then $(j, i) \in E$. A Hamiltonian path (named after the famous Irish mathematician William Rowan Hamilton), is a sequence of vertices $v_1, v_2, \ldots, v_N$ that includes each vertex once, and such that $(v_i, v_{i+1}) \in E$. The path is a Hamiltonian circuit if (v_N, v_1) is also in E. Hamilton demonstrated that the dodecahedron has a Hamiltonian circuit. It is suspected [2] and [4] that determining whether or not a given graph has a Hamiltonian circuit or path is difficult in the sense that there might be no easier way than looking at all the $N!$ permutations of the vertex set and testing each for the circuit or path property.

Thus it is of some interest to have a large class of graphs that have the Hamiltonian path or circuit property. The knight's problem is: For what (n, m) does the graph derived from the $n \times m$ chessboard by the allowed knight's moves have a Hamiltonian path or circuit? Restated, the problem is: Does the $n \times m$ chessboard have a knight's path or a knight's circuit?

It is easy to show that some chessboards do not have a knight's circuit. Let us recall that the chessboard has squares of two colors, usually red and black, such that two squares that have a side in common are of different color. This implies that the knight must move in one step from one color to the other color. Thus if the board has an odd number of squares, a knight's circuit is impossible, since there are more squares of one color than the other color. But this does not rule out the possibility of a knight's path.

When we needed some examples of graphs with Hamiltonian paths, we used $n \times n$ chessboards with $n \geq 5$. We assumed that it was well known that all such boards have the required paths. But when we were asked to produce a reference, we had none. A search of the literature was called for. The standard books on recreational mathematics were little help. Kraitchik [6] had a diagram which proved that if $n \equiv 1 \pmod 4$, then the $n \times n$ chessboard has a knight's path. Ball [1] had a technique that he claimed would show that if $n \equiv 0 \pmod 4$, then the $n \times n$ chessboard has a knight's circuit. But we must confess that we were unable to fill in the details and we have doubts that the technique works. Dudeney [3] boldly states that if $n \geq 5$, then the $n \times n$ board has a knight's path, and if n is even there is a knight's circuit. Unfortunately, he neither gives a proof nor gives a reference to a proof. We were delighted to find that Kraitchik [5] had written a monograph on the knight's problem. But when we obtained a copy we were disappointed to find no proof of the general statement. Instead, there is a large collection of paths and circuits with various degrees of symmetry, the diagram for the case $n \equiv 1 \pmod 4$, and a detailed discussion of $4 \times n$ boards.

Unable to find a proof in the literature, we were forced to construct our own. The proof that follows may be of some interest to others.

In what follows, it is often necessary to refer to a particular square on a board. We may do so in either of two ways. We can refer to a square by a pair of integers (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq m$ for an $n \times m$ board. The square (1,1) will be in the lower left-hand corner of the board. If we have displayed a knight's path on a board, we may instead refer to a square by a single integer i, such that square i is the ith square visited on a particular path. This second method is used throughout the figures.

Lemma 1: The $5 \times m$ board with $m \geq 5$ has a knight's path that starts in the lower left and exits at either the lower right or at the upper right.

Proof: The lemma is true for $5 \times r$ boards with $r \in \{5, 6, 7, 8, 9\}$, as shown in Figure 1. For any $m \geq 10$, partition the board into a sequence of 5×5 boards ending with a $5 \times r$ board. Clearly, we can take the knight's path for the first 5×5 board, starting in the lower left and exiting into the lower left of the next 5×5 board, and continue doing so until we land in the lower left of the $5 \times r$ board. But then we can take the knight's path for the $5 \times r$ board that ends in either the upper right or the lower right, and the lemma is proved.

Fig. 1

3	22	13	16	5
12	17	4	21	14
23	2	15	6	9
18	11	8	25	20
1	24	19	10	7

5 x 5

7	12	15	20	5
16	21	6	25	14
11	8	13	4	19
22	17	2	9	24
1	10	23	18	3

5 x 5

7	14	21	28	5	12
22	27	6	13	20	29
15	8	17	24	11	4
26	23	2	9	30	19
1	16	25	18	3	10

5 x 6

3	10	29	20	5	12
28	19	4	11	30	21
9	2	17	24	13	6
18	27	8	15	22	25
1	16	23	26	7	14

5 x 6

3	12	21	30	5	14	23
20	29	4	13	22	31	6
11	2	19	32	7	24	15
28	33	10	17	26	35	8
1	18	27	34	9	16	25

5 x 7

3	32	11	34	5	26	13
10	19	4	25	12	35	6
31	2	33	20	23	14	27
18	9	24	29	16	7	22
1	30	17	8	21	28	15

5 x 7

3	12	37	26	5	14	17	28
34	23	4	13	36	27	6	15
11	2	35	38	25	16	29	18
22	33	24	9	20	31	40	7
1	10	21	32	39	8	19	30

5 x 8

33	8	17	38	35	6	15	24
18	37	34	7	16	25	40	5
9	32	29	36	39	14	23	26
30	19	2	11	28	21	4	13
1	10	31	20	3	12	27	22

5 x 8

9	4	11	16	23	42	33	36	25
12	17	8	3	32	37	24	41	34
5	10	15	20	43	22	35	26	29
18	13	2	7	38	31	28	45	40
1	6	19	14	21	44	39	30	27

5 x 9

9	4	11	16	27	32	35	40	25
12	17	8	3	36	41	26	45	34
5	10	15	20	31	28	33	24	39
18	13	2	7	42	37	22	29	44
1	6	19	14	21	30	43	38	23

5 x 9

Lemma 2: Every $6 \times m$ board with $m \geq 5$ has a knight's circuit and a knight's path starting in the lower left and exiting at the upper left.

Proof: Looking at the 6 x 5 board in Figure 2, we can see that we can connect two 6 x 5 boards together so that we start in square 1 of the first, follow the indicated numbers until square 27, then jump to square 1 of the next board, and follow its knight's path ending in square 30, from which we can jump back to square 28 of the first board and complete its knight's path. This construction does not depend on the boards being 6 x 5; in fact, it will work on $6 \times k$ boards as long as the starting square is in the lower left (1, 1), the end square is diagonally below the upper left (5, 2), and the squares $(2, k-1)$ and $(4, k)$ are adjacent. From Figure 2, this is true for $6 \times r$ boards with $r \in \{5, 6, 7, 8, 9\}$. Thus every $6 \times m$ board has a knight's path beginning in the lower left and ending in the upper left, since we can partition the $6 \times m$ board into a series of 6 x 5 boards and one $6 \times r$ board, and connect them following the construction.

For the knight's circuit, we note that for $6 \times r$ with $r \in \{5, 6, 7, 8, 9\}$ the circuits are given in Figure 3. If $m \geq 10$, we can partition the board into a 6 x 5 board and a $6 \times (m - 5)$ board. Looking at the circuit for the 6 x 5 board in Figure 3, we can start in square 1, follow the circuit until square 28, then jump to the $6 \times (m - 5)$ board. By the half of the lemma that has already been proved, this board has a knight's path that starts at (1, 1) and ends at (5, 2), but from 28 on the 6 x 5 board we can jump to (5, 2), then take the path backwards until (1, 1) is reached. From (1, 1) we can jump to 29 on the 6 x 5 board and complete the circuit.

Fig. 2

10	19	4	29	12
3	30	11	20	5
18	9	24	13	28
25	2	17	6	21
16	23	8	27	14
1	26	15	22	7

6 x 5

14	23	6	28	12	21
7	36	13	22	5	27
24	15	29	35	20	11
30	8	17	26	34	4
16	25	2	32	10	19
1	31	9	18	3	33

6 x 6

18	23	8	39	16	25	6
9	42	17	24	7	40	15
22	19	32	41	38	5	26
33	10	21	28	31	14	37
20	29	2	35	12	27	4
1	34	11	30	3	36	13

6 x 7

18	31	8	35	16	33	6	45
9	48	17	32	7	46	15	26
30	19	36	47	34	27	44	5
37	10	21	28	43	40	25	14
20	29	2	39	12	23	4	41
1	38	11	22	3	42	13	24

6 x 8

22	45	10	53	20	47	8	35	18
11	54	21	46	9	36	19	48	7
44	23	42	37	52	49	32	17	34
41	12	25	50	27	38	29	6	31
24	43	2	39	14	51	4	33	16
1	40	13	26	3	28	15	30	5

6 x 9

Fig. 3

16	9	6	27	18
7	26	17	14	5
10	15	8	19	28
25	30	23	4	13
22	11	2	29	20
1	24	21	12	3

6 x 5

4	25	34	15	18	7
35	14	5	8	33	16
24	3	26	17	6	19
13	36	23	30	9	32
22	27	2	11	20	29
1	12	21	28	31	10

6 x 6

26	37	8	17	28	31	6
9	18	27	36	7	16	29
38	25	10	19	30	5	32
11	42	23	40	35	20	15
24	39	2	13	22	33	4
1	12	41	34	3	14	21

6 x 7

30	35	8	15	28	39	6	13
9	16	29	36	7	14	27	38
34	31	10	23	40	37	12	5
17	48	33	46	11	22	41	26
32	45	2	19	24	43	4	21
1	18	47	44	3	20	25	42

6 x 8

14	49	4	51	24	39	6	29	22
3	52	13	40	5	32	23	42	7
48	15	50	25	38	41	28	21	30
53	2	37	12	33	26	31	8	43
16	47	54	35	18	45	10	27	20
1	36	17	46	11	34	19	44	9

6 x 9

Lemma 3: Every $8 \times m$ board with $m \geq 5$ has a knight's path starting in the lower left and ending in the upper left, and a knight's circuit.

Proof: From Figure 4, there are $8 \times r$ circuits for $r \in \{5, 6, 7, 8, 9\}$. Since in each of these circuits the two squares $(2, r-1)$ and $(4, r)$ are adjacent, we can join two boards together to form a larger circuit. Start in square 1 and follow the circuit until one of the squares $(2, r-1)$ or $(4, r)$ is reached. Then jump to the next board at either $(1, 1)$ or at $(3, 2)$. Follow the circuit on this board and when it is finished jump back to the first board at the square $(2, r-1)$ or $(4, r)$, whichever has not been visited, and complete the circuit. Since this construction can be carried out for any number of boards, there is always a circuit of the $8 \times m$ board if $m \geq 5$.

The required paths for $8 \times r$, $r \in \{5, 6, 7, 8, 9\}$ are displayed in Figure 5. If $m \geq 10$, partition the board into an 8×5 board and an $8 \times (m - 5)$ board. Start in the lower left of the 8×5 board, follow the path to 34, then jump to $(3, 2)$ of the $8 \times (m - 5)$ board. Take the circuit of the $8 \times (m - 5)$ board that ends in $(1, 1)$ and then jump back to 35 on the 8×5 board and complete the path.

Fig. 4

26	7	28	15	24
31	16	25	6	29
8	27	30	23	14
17	32	39	34	5
38	9	18	13	22
19	40	33	4	35
10	37	2	21	12
1	20	11	36	3

8 x 5

42	21	26	5	38	13
25	4	41	12	27	6
20	43	22	37	14	39
3	24	11	40	7	28
44	19	46	23	36	15
47	2	33	10	29	8
18	45	48	31	16	35
1	32	17	34	9	30

8 x 6

22	27	40	5	38	29	14
41	4	23	28	13	6	37
26	21	12	39	50	15	30
3	42	51	24	31	36	7
20	25	32	11	52	49	16
55	2	43	46	33	8	35
44	19	56	53	10	17	48
1	54	45	18	47	34	9

8 x 7

48	13	30	9	56	45	28	7
31	10	47	50	29	8	57	44
14	49	12	55	46	59	6	27
11	32	37	60	51	54	43	58
36	15	52	63	38	61	26	5
33	64	35	18	53	40	23	42
16	19	2	39	62	21	4	25
1	34	17	20	3	24	41	22

8 x 8

42	19	38	5	36	21	34	7	60
39	4	41	20	63	6	59	22	33
18	43	70	37	58	35	68	61	8
3	40	49	64	69	62	57	32	23
50	17	44	71	48	67	54	9	56
45	2	65	14	27	12	29	24	31
16	51	72	47	66	53	26	55	10
1	46	15	52	13	28	11	30	25

8 x 9

Fig. 5

28	7	22	39	26
23	40	27	6	21
8	29	38	25	14
37	24	15	20	5
16	9	30	13	34
31	36	33	4	19
10	17	2	35	12
1	32	11	18	3

8 x 5

42	11	26	9	34	13
25	48	43	12	27	8
44	41	10	33	14	35
47	24	45	20	7	28
40	19	32	3	36	15
23	46	21	6	29	4
18	39	2	31	16	37
1	22	17	38	5	30

8 x 6

38	19	6	55	46	21	8
5	56	39	20	7	54	45
18	37	4	47	34	9	22
3	48	35	40	53	44	33
36	17	52	49	32	23	10
51	2	29	14	41	26	43
16	13	50	31	28	11	24
1	30	15	12	25	42	27

8 x 7

Fig. 5—continued

24	11	37	9	26	21	39	7
36	64	25	22	38	8	27	20
12	23	10	53	58	49	6	28
63	35	61	50	55	52	19	40
46	13	54	57	48	59	29	5
34	62	47	60	51	56	41	18
14	45	2	32	16	43	4	30
1	33	15	44	3	31	17	42

8 x 8

32	47	6	71	30	45	8	43	26
5	72	31	46	7	70	27	22	9
48	33	4	29	64	23	44	25	42
3	60	35	62	69	28	41	10	21
34	49	68	65	36	63	24	55	40
59	2	61	16	67	56	37	20	11
50	15	66	57	52	13	18	39	54
1	58	51	14	17	38	53	12	19

8 x 9

Lemma 4: Every $n \times m$ board with n odd, $\min(n, m) \geq 5$, has a knight's path that starts in the upper left and exits at the upper right.

Proof: If $m \geq 10$, partition the board into an $n \times 5$ board and an $n \times (m - 5)$ board. If the lemma holds for each of these subboards, then it holds for the whole board. Thus the result holds by induction if we can show that it holds for all $n \times r$ boards with $r \in \{5, 6, 7, 8, 9\}$ and n odd. The cases $r = 6$ and $r = 8$ have been proved in the previous two lemmas.

For the $n \times 5$ case, we have as the base of an induction the boards that appear in Figure 6. If $n \geq 10$, we partition the board into a 5×5 board and an $(n - 5) \times 5$ board. Notice that in Figure 6 the squares 16 and 17 of the 5×5 board would command the squares (3, 2) and (1, 1), respectively, of the $(n - 5) \times 5$ board. If that board has a circuit, we could go from 1 to 16 on the 5×5 board, jump to (3, 2) on the $(n - 5) \times 5$ board, take the circuit ending in (1, 1), jump back to 17 in the 5×5 board and complete the path. If $n - 5 = 6$ or $n - 5 = 8$, we have shown that the required circuit exists in the previous two lemmas.

To show that there is a circuit if $n - 5 \geq 10$, we can again partition the $(n - 5) \times 5$ board into a 5×5 board and an $(n - 10) \times 5$ board. We know from Lemma 1 that there is a knight's path on the 5×5 board that starts at the lower left and exits at the lower right. To complete the circuit we need the $(n - 10) \times 5$ board to have a knight's path that starts at the upper right and exits at the upper left, i.e., into the starting square of the path on the 5×5 board. But interchanging left and right, this is what we are trying to prove. Thus we conclude by induction that the $5 \times n$ board has the required path.

The argument for the $7 \times n$ and $9 \times n$ boards is similar. We need as our induction base the 7×5 and 9×5 boards of Figure 1 and the 7×7, 7×9, 9×7, and 9×9 boards of Figure 7.

Fig. 6

7	20	9	14	5
10	25	6	21	16
19	8	15	4	13
24	11	2	17	22
1	18	23	12	3

5 × 5

17	14	25	6	19	8	29
26	35	18	15	28	5	20
13	16	27	24	7	30	9
34	23	2	11	32	21	4
1	12	33	22	3	10	31

5 × 7

7	12	37	42	5	18	23	32	27
38	45	6	11	36	31	26	19	24
13	8	43	4	41	22	17	28	33
44	39	2	15	10	35	30	25	20
1	14	9	40	3	16	21	34	29

5 × 9

Fig. 7

9	30	19	42	7	32	17
20	49	8	31	18	43	6
29	10	41	36	39	16	33
48	21	38	27	34	5	44
11	28	35	40	37	26	15
22	47	2	13	24	45	4
1	12	23	46	3	14	25

7 × 7

13	26	39	52	11	24	37	50	9
40	81	12	25	38	51	10	23	36
27	14	53	58	63	68	73	8	49
80	41	64	67	72	57	62	35	22
15	28	59	54	65	74	69	48	7
42	79	66	71	76	61	56	21	34
29	16	77	60	55	70	75	6	47
78	43	2	31	18	45	4	33	20
1	30	17	44	3	32	19	46	5

9 × 9

Fig. 7—continued

5	20	53	48	7	22	31
52	63	6	21	32	55	8
19	4	49	54	47	30	23
62	51	46	33	56	9	58
3	18	61	50	59	24	29
14	43	34	45	28	57	10
17	2	15	60	35	38	25
42	13	44	27	40	11	36
1	16	41	12	37	26	39

9 x 7

59	4	17	50	37	6	19	30	39
16	63	58	5	18	51	38	7	20
3	60	49	36	57	42	29	40	31
48	15	62	43	52	35	56	21	8
61	2	13	26	45	28	41	32	55
14	47	44	11	24	53	34	9	22
1	12	25	46	27	10	23	54	33

7 x 9

Theorem 1: An $n \times m$ board with nm even, $\min(n, m) \geq 5$, has a knight's circuit.

Proof: If n even, $n \geq 10$, partition the board into a $5 \times m$ board and an $(n - 5) \times m$ board. Choose as the starting square the upper left-hand corner of the $(n - 5) \times m$ board. Since $n - 5$ is odd, we know from Lemma 4 that there is a knight's path on this board that will end in a square accessible to the lower right-hand corner of the $5 \times m$ board. From Lemma 1 we know that there is a knight's path of the $5 \times m$ board that starts in the indicated corner and exits at the lower left. But this was the starting square for the $(n - 5) \times m$ board, so we have constructed a knight's circuit. Of course, the same construction works if m is even and $m \geq 10$, by switching rows and columns. The only other cases are $n = 6$ or $n = 8$, and we have demonstrated in Lemma 2 and Lemma 3 how to build circuits in these cases.

Theorem 2: Every $n \times m$ board with $\min(n, m) \geq 5$ has a knight's path.

Proof: By Lemma 4 this follows if the board has n or m odd. If n or m is even the previous theorem assures a circuit and thus a path.

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A FAMILY OF TRIDIAGONAL MATRICES

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Consider the sequence of tridiagonal determinants $\left\{P_n^{(k)}(a, b, c)\right\}_{n=1}^{\infty}$ defined by $P_n^{(k)}(a, b, c) = P_n^{(k)} = \left|(a_{ij})\right|$ where

$$a_{ij} = \begin{cases} a, & i = j \\ b, & i = j - k \\ c, & i = j + k \\ 0, & \text{otherwise} \end{cases}.$$

We shall assume $P_n^{(1)} \neq 0$. The determinant $P_n^{(k)}$ has a's down the main diagonal, b's down the diagonal k positions to the right of the main diagonal and c's down the diagonal k positions below the main diagonal.

In [1], the authors discuss $\left\{P_n^{(2)}\right\}_{n=1}^{\infty}$ and find its generating function. This note deals with a relationship that exists between

$$\left\{P_n^{(k)}\right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{P_n^{(1)}\right\}_{n=1}^{\infty} \quad \text{for } k \geq 2.$$

The first few terms of $\left\{P_n^{(1)}\right\}_{n=}^{\infty}$ with $P_0^{(1)}$ defined as one are:

$$\begin{aligned}
P_0^{(1)} &= 1 \\
P_1^{(1)} &= a \\
P_2^{(1)} &= a^2 - bc \\
P_3^{(1)} &= a^3 - 2abc \\
P_4^{(1)} &= a^4 - 3a^2bc + b^2c^2 \\
P_5^{(1)} &= a^5 - 4a^3bc + 3ab^2c^2 \\
P_6^{(1)} &= a^6 - 5a^4bc + 6a^2b^2c^2 - b^3c^3 \\
P_7^{(1)} &= a^7 - 6a^5bc + 10a^3b^2c^2 - 4ab^3c^3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

By induction on n, it can be shown that

$$\text{(A)} \qquad P_{n+2}^{(1)} = aP_{n+1}^{(1)} - bcP_n^{(1)}, \quad n \geq 1.$$

When $a = 1$ and $bc = -1$, we obtain the Fibonacci sequence. This result can also be found in [3] and [4].

The first few terms of $\left\{P_n^{(2)}\right\}_{n=1}^{\infty}$ can be found in [1] and are:

$$\begin{aligned}
P_1^{(2)} &= a = P_0^{(1)}P_1^{(1)} \\
P_2^{(2)} &= a^2 = \left[P_1^{(1)}\right]^2 \\
P_3^{(2)} &= a^3 - abc = P_1^{(1)}P_2^{(1)} \\
P_4^{(2)} &= (a^2 - bc)^2 = \left[P_2^{(1)}\right]^2 \\
P_5^{(2)} &= a^5 - 3a^2bc + 2ab^2c^2 = P_2^{(1)}P_3^{(1)} \\
P_6^{(2)} &= (a^3 - 2abc)^2 = \left[P_3^{(1)}\right]^2 \\
P_7^{(2)} &= a^7 - 5a^5bc + 7a^3b^2c^2 - 2ab^3c^3 = P_3^{(1)}P_4^{(1)} \\
P_8^{(2)} &= (a^4 - 3a^2bc + b^2c^2)^2 = \left[P_4^{(1)}\right]^2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

As with $\left\{P_n^{(1)}\right\}_{n=1}^{\infty}$, it can be shown by induction that

$$\text{(B)} \qquad P_n^{(2)} = aP_{n-1}^{(2)} - abcP_{n-3}^{(2)} + b^2c^2P_{n-4}^{(2)}, \quad n \geq 5.$$

Not until our investigation of $P_n^{(3)}$ did we become suspicious of the fact that

$$\text{(C)} \qquad P_n^{(2)} = \begin{cases} P_q^{(1)}P_{q-1}^{(1)}, & n = 2q - 1 \\ \left[P_q^{(1)}\right]^2, & n = 2q \end{cases}.$$

The proof of the result (C) is as follows. Multiply the first and second rows of $P_n^{(2)}$ by $-c/a$ and add the results respectively to the third and fourth rows. Evaluate the new determinant using the first two columns to obtain

$$P_n^{(2)} = \begin{vmatrix} P_2^{(1)} & 0 & bP_1^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & P_2^{(1)} & 0 & bP_1^{(1)} & 0 & 0 & \cdots \\ c & 0 & a & 0 & b & 0 & \cdots \\ 0 & c & 0 & a & 0 & b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Multiply the first and second rows of the new determinant by $-c/P_2^{(1)}$ and add the results respectively to the third and fourth rows. Evaluate the new determinant using the first two columns to obtain

$$P_n^{(2)} = \begin{vmatrix} P_3^{(1)} & 0 & bP_2^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & P_3^{(1)} & 0 & bP_2^{(1)} & 0 & 0 & \cdots \\ c & 0 & a & 0 & b & 0 & \cdots \\ 0 & c & 0 & a & 0 & b & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Repeating the process, using $-c/P_3^{(1)}$, we see that

$$P_n^{(2)} = \begin{vmatrix} P_4^{(1)} & 0 & bP_3^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & P_4^{(1)} & 0 & bP_3^{(1)} & 0 & 0 & \cdots \\ c & 0 & a & 0 & b & 0 & \cdots \\ 0 & c & 0 & a & 0 & b & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Let $n = 2q - 1$ and continue the technique above, evaluating by two columns at a time for $q - 1$ times until you obtain

$$P_n^{(2)} = \begin{vmatrix} P_{q-1}^{(1)} & 0 & bP_{q-2}^{(1)} \\ 0 & P_{q-1}^{(1)} & 0 \\ c & 0 & a \end{vmatrix} = P_q^{(1)} P_{q-1}^{(1)}.$$

If $n = 2q$ and we evaluate by two columns at a time for q times using the same technique as above we obtain

$$P_n^{(2)} = \begin{vmatrix} P_q^{(1)} & 0 \\ 0 & P_q^{(1)} \end{vmatrix} = \left[P_q^{(1)}\right]^2.$$

This procedure applied to $P_n^{(3)}$, where you evaluate by using three columns at a time instead of two, yields

$$P_n^{(3)} = \begin{cases} \left[P_{q-1}^{(1)}\right]^2 P_q^{(1)}, & n = 3q - 2 \\ P_{q-1}^{(1)}\left[P_q^{(1)}\right]^2, & n = 3q - 1 \\ \left[P_{q-1}^{(1)}\right]^3, & n = 3q \end{cases}.$$

In fact, it is easy to show if $n = kq - r$ that

(D) $$P_n^{(k)} = \left[P_{q-1}^{(1)}\right]^r \left[P_q^{(1)}\right]^{k-r} \text{ for } 0 \le r < k.$$

The authors found an alternate way of proving (C), but the technique did not apply if $k \ge 3$. This procedure is as follows. First show by induction, using (B) and (A), that

(E) $$P_{n+2}^{(2)} - bcP_n^{(2)} = P_{n+2}^{(1)}, \; n \ge 1.$$

Next apply the results of Horadam [2], where

$$P_n^{(1)} = aP_{n-1}^{(1)} - bcP_{n-2}^{(1)}$$

is $W_n(a, b: p, q)$ with $a = 1$, $b = a$, $p = a$, and $q = bc$ to obtain

$$\text{(F)} \qquad aP_{2n-1}^{(1)} + b^2c^2\left[P_{n-2}^{(1)}\right]^2 = \left[P_n^{(1)}\right]^2.$$

Finally, using (A), (B), (E), (F), and induction, you can show (C).

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