The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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The Fibonacci Quarterly

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and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ON RECIPROCAL SUMS OF CHEBYSHEV RELATED SEQUENCES

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University of Technology, Sydney 2007, Australia
(*Submitted July 1993*)

1. INTRODUCTION

Define the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ for any real number p by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \; U_1 = 1, \; n \geq 2, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \; V_1 = p, \; n \geq 2. \end{cases} \tag{1.1}$$

They can be extended to negative subscripts by the use of the recurrence relation. The Binet forms are

$$U_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \tag{1.2}$$

$$V_n = \gamma^n + \delta^n, \tag{1.3}$$

where

$$\gamma = \frac{p + \sqrt{p^2 + 4}}{2}, \quad \delta = \frac{p - \sqrt{p^2 + 4}}{2}. \tag{1.4}$$

These sequences are generalizations of the Fibonacci and Lucas sequences and as such have appeared frequently in the literature. See, for example, [5], [9], [10], [19], and [24]. Lucas [23] studied the following generalizations of the sequences (1.1):

$$U_n = pU_{n-1} - qU_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \quad n \geq 2,$$
$$V_n = pV_{n-1} - qV_{n-2}, \quad V_0 = 2, \quad V_1 = p, \quad n \geq 2.$$

Let $\{T_n(x)\}_{n=0}^{\infty}$ and $\{S_n(x)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$S_n(x) = \sin n\theta / \sin\theta, \; x = \cos\theta, \; n \geq 0, \tag{1.5}$$

$$T_n(x) = \cos n\theta, \; x = \cos\theta, \; n \geq 0. \tag{1.6}$$

These polynomial sequences have been studied extensively. See, for example, [1], [11], and [22]. Lucas states (see [23], p. 189) that Jean Bernoulli, in 1701, expressed $\sin n\theta / \sin\theta$ and $\cos n\theta$ in powers of $\sin\theta$ and $\cos\theta$. Lucas also states (p. 208) that Viète (sometimes spelled Vièta), in 1646, was the first to express $\sin n\theta / \sin\theta$ and $\cos n\theta$ as sums of powers of $\cos\theta$. Chebyshev's contributions came much later since he lived during the years 1821-1894. Lucas also quotes (p. 195) a continued fraction expansion for $\sin(n+1)\theta / \sin n\theta$ in which $2\cos\theta$ is repeated n times.

Writing $U_n = S_n(x)$, $V_n = 2T_n(x)$, $p = 2x$, we have the familiar recurrences (see [20], [22]),

$$\begin{cases} U_n = pU_{n-1} - U_{n-2}, & U_0 = 0, \; U_1 = 1, \; n \geq 2, \\ V_n = pV_{n-1} - V_{n-2}, & V_0 = 2, \; V_1 = p, \; n \geq 2. \end{cases} \tag{1.7}$$

Because of their connection with the Chebyshev polynomials, the sequences (1.7) merit closer attention. We have studied them independently of the Chebyshev polynomials, taking p to be a real number. In particular, we have noticed that many properties of the sequences (1.1) have interesting analogs for the sequences (1.7).

In this paper we focus on results for the sequences (1.1) involving certain reciprocal sums and obtain analogs for the sequences (1.7).

Unless otherwise stated, we assume throughout that in (1.7) p is real and $|p|>2$. The Binet forms for the sequences (1.7) are then

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.8}$$

$$V_n = \alpha^n + \beta^n, \tag{1.9}$$

where

$$\alpha = \frac{p+\sqrt{p^2-4}}{2}, \quad \beta = \frac{p-\sqrt{p^2-4}}{2}. \tag{1.10}$$

We use the $U_n - V_n$ notation throughout to refer to the sequences (1.1) and to the sequences (1.7). There will be no ambiguity since we shall always indicate the set to which we are referring.

2. THE RESULTS OF GOOD AND GRIEG

Good [12] showed that

$$\sum_{i=0}^{n} \frac{1}{F_{2^i}} = 3 - \frac{F_{2^n-1}}{F_{2^n}}, \quad n \geq 1, \tag{2.1}$$

from which he obtained the corresponding infinite sum by noting that $\frac{F_{n-1}}{F_n} \to \frac{\sqrt{5}-1}{2}$.

In a comprehensive paper, Hoggatt and Bicknell [16] indicate eleven methods of obtaining the corresponding infinite sum, highlighting the contributions of various authors. Bruckman and Good [8] give several interesting sums involving reciprocals. They state that Hoggatt, in an unpublished note (December 1974) obtained the sum

$$\sum_{i=1}^{\infty} \frac{1}{F_{k2^i}}.$$

Later Hoggatt and Bicknell [17] generalized this by evaluating

$$\sum_{i=0}^{n} \frac{1}{F_{k2^i}}.$$

Greig [15] gave a different version of the last sum by showing that

$$\sum_{i=0}^{n} \frac{1}{F_{k2^i}} = C_k - \frac{F_{k2^n-1}}{F_{k2^n}}, \quad n, k \geq 1, \tag{2.2}$$

where C_k is independent of n and is given by

$$C_k = \begin{cases} \dfrac{1+F_{k-1}}{F_k}, & k \text{ even}, \\ \dfrac{1+F_{k-1}}{F_k} + \dfrac{2}{F_{2k}}, & k \text{ odd}. \end{cases} \tag{2.3}$$

By using a certain partition of the natural numbers, Greig obtained

$$\sum_{i=1}^{\infty} \frac{1}{F_i} = \sum_{k=0}^{\infty} \left(C_{2k+1} - \frac{1}{\phi} \right), \quad \phi = \frac{1+\sqrt{5}}{2}, \tag{2.4}$$

and other variant forms. He did this by observing that, as k and m take on all integer values such that $k \geq 0$ and $m \geq 0$, $(2k+1)2^m$ generates each natural number once. Then he used the following rearrangement theorem in conjunction with (2.2) and (2.3) to obtain (2.4) and its variants.

Theorem 1:

$$\sum_{i=1}^{\infty} f(i) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((2k+1)2^n)$$

for an arbitrary function f provided only that the series on the left converges absolutely.

Gould [13], commenting on Greig's paper, remarked that Theorem 1 "seems to be common knowledge in the mathematical community, but its use in forming interesting series rearrangements does not seem to be widely known or appreciated." He then used Theorem 1 to prove that, for $|x| < 1$,

$$\sum_{i=0}^{\infty} \frac{x^{2^i}}{1 - x^{2^{i+1}}} = \frac{x}{1-x}.$$

Indeed, Bromwich (see [7], p. 24) attributed this formula to De Morgan. The paper by Bruckman and Good [8] was based on generalizations of this formula. Gould then stated a generalized version of Theorem 1 and gave interesting applications to summations involving the Riemann zeta function, Euler's ϕ-function, and the Trigonometric and Hyperbolic functions. We will use Theorem 1 subsequently.

In a further paper, Greig [14] obtained formulas analogous to (2.1)-(2.4) for the sequence $\{U_n\}$ given in (1.1). Gould [13] commented that similar results for L_n seem to be unattainable using the method of Greig. Horadam [18] made the same comment in relation to the Pell-Lucas numbers (i.e., the sequence $\{V_n\}$ in (1.1) with $p = 2$). Badea [4] proved that

$$\sum_{i=0}^{\infty} \frac{1}{L_{2^i}}$$

is irrational, and recently André-Jeannin [2] proved that

$$\sum_{i=1}^{\infty} \frac{\varepsilon^i}{L_{2^i}},$$

where $\varepsilon = \pm 1$, does not belong to $Q(\sqrt{5})$. These results suggest that a sum corresponding to (2.1) for L_n does not exist.

We now obtain results analogous to (2.1)-(2.4) for the sequence $\{U_n\}$ given in (1.7), where, as stated earlier, we assume that $|p|>2$.

Theorem 2:

$$\sum_{i=0}^{n} \frac{1}{U_{2^i}} = 1 + \frac{U_{2^n-1}}{U_{2^n}}.$$

Now, since U_{n-1}/U_n approaches $\frac{1}{\alpha}(p>2)$ or $\frac{1}{\beta}(p<-2)$, we have as a corollary of Theorem 2

Theorem 3:

$$\sum_{i=0}^{\infty} \frac{1}{U_{2^i}} = \begin{cases} 1+\dfrac{1}{\alpha}, & p>2, \\ 1+\dfrac{1}{\beta}, & p<-2. \end{cases}$$

We now give a generalization of Theorem 2. The proof relies on the following result which can be established using Binet forms:

$$U_{2j-1}U_j - U_{2j}U_{j-1} = U_j. \tag{2.5}$$

We mention in passing that the corresponding identity for the Fibonacci numbers is

$$F_{2j-1}F_j - F_{2j}F_{j-1} = (-1)^{j-1}F_j.$$

Theorem 4: For $k \geq 1$ an integer,

$$\sum_{i=0}^{n} \frac{1}{U_{k2^i}} = \frac{1-U_{k-1}}{U_k} + \frac{U_{k2^n-1}}{U_{k2^n}}.$$

Proof: We proceed by induction. When $n=0$, both sides reduce to $1/U_k$. The inductive step requires us to prove that

$$U_{k2^{n+1}-1}U_{k2^n} - U_{k2^{n+1}}U_{k2^n-1} = U_{k2^n}$$

This is achieved by putting $j = k2^n$ in (2.5) and the proof of Theorem 4 is complete. □

As a corollary, we have

Theorem 5: For $k \geq 1$ an integer,

$$\sum_{i=0}^{\infty} \frac{1}{U_{k2^i}} = \begin{cases} \dfrac{1-U_{k-1}}{U_k}+\dfrac{1}{\alpha}, & p>2, \\ \dfrac{1-U_{k-1}}{U_k}+\dfrac{1}{\beta}, & p<-2. \end{cases}$$

We now use Theorem 1 to obtain an interesting bisection result for the sum of the reciprocals of $\{U_n\}_{n=1}^{\infty}$. We shall need the following result which is easily established using Binet forms:

$$\frac{1}{\alpha} - \frac{U_{k-1}}{U_k} = \frac{1}{\alpha^k U_k}, \quad k \geq 1. \tag{2.6}$$

Using Theorem 1 we have, for $p > 2$,

$$\sum_{i=1}^{\infty}\frac{1}{U_i}=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{U_{(2k+1)2^n}}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{U_{2k+1}}+\frac{1}{\alpha}-\frac{U_{2k}}{U_{2k+1}}\right)\quad\text{(by Theorem 5)}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{U_{2k+1}}+\frac{1}{\alpha^{2k+1}U_{2k+1}}\right)\quad\text{[by (2.6)]}$$

and so

$$\sum_{i=1}^{\infty}\frac{1}{U_i}=\sum_{i=0}^{\infty}\frac{1}{U_{2i+1}}+\sum_{i=0}^{\infty}\frac{1}{\alpha^{2i+1}U_{2i+1}}. \tag{2.7}$$

Comparing both sides of (2.7) leads to

$$\sum_{i=1}^{\infty}\frac{1}{U_{2i}}=\sum_{i=0}^{\infty}\frac{1}{\alpha^{2i+1}U_{2i+1}}. \tag{2.8}$$

For $p < -2$, simply replace α by β.

Numerical examples suggest that the right side of (2.8) converges much faster than the left side. For example, taking $p = 3$ gives $U_n = F_{2n}$, and (2.8) becomes

$$\sum_{i=1}^{\infty}\frac{1}{F_{4i}}=\sum_{i=0}^{\infty}\frac{1}{\alpha^{2i+1}F_{4i+2}}, \tag{2.9}$$

where $\alpha = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$.

Taking twelve terms on the left side gives the value 0.389083066... but only six terms on the right are needed to achieve this value.

Note: Since $|p| > 2$, use of the ratio test shows that the left side of (2.7) is absolutely convergent, so that the use of Theorem 1 is valid.

3. THE RESULTS OF ANDRÉ-JEANNIN

We now examine reciprocal sums of a different nature. For the sequences (1.1), André-Jeannin [3] proved the following.

Theorem 6: If k is an odd integer and $p > 0$, then

$$\sum_{i=1}^{\infty}\frac{1}{U_{ki}U_{k(i+1)}}=\frac{2(\gamma-\delta)}{U_k}[L(\delta^{2k})-2L(\delta^{4k})+2L(\delta^{8k})]+\frac{\delta^k}{U_k^2},$$

$$\sum_{i=1}^{\infty}\frac{1}{V_{ki}V_{k(i+1)}}=\frac{2}{(\gamma-\delta)U_k}[L(\delta^{2k})-2L(\delta^{8k})]+\frac{\delta^k}{(\gamma-\delta)U_kV_k}.$$

Here, $L(x)$ is the Lambert series defined by

$$L(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n},\quad |x|<1.$$

Information about the Lambert series can be found in Knopp [21] and in Borwein and Borwein [6]. In the latter reference there are numerous reciprocal sums for the Fibonacci and Lucas numbers involving the Lambert series and the theta functions of Jacobi. On pages 94-95 in [6], the sums $\sum_{n=0}^{\infty}F_{2n+1}^{-1}$ and $\sum_{n=1}^{\infty}F_{2n}^{-1}$, due to Landau, are stated.

We now obtain sums analogous to those in Theorem 6 for the sequences (1.7). Interestingly, these sums do not involve $L(x)$, and the requirement that k be odd is not needed. For the remainder of this section, U_n and V_n are as in (1.7). We shall need the following lemma.

Lemma 1: For integers k and n,

$$\alpha^k U_{k(n+1)}-U_{kn}=\alpha^{k(n+1)}U_k, \tag{3.1}$$

$$\alpha^k V_{k(n+1)}-V_{kn}=(\alpha-\beta)\alpha^{k(n+1)}U_k. \tag{3.2}$$

Proof: We prove only (3.2) since the proof of (3.1) is similar. Recalling that $\alpha\beta=1$, we have

$$\begin{aligned}\alpha^k V_{k(n+1)}-V_{kn}&=\alpha^k(\alpha^{k(n+1)}+\beta^{k(n+1)})-(\alpha^{kn}+\beta^{kn})\\&=\alpha^{kn+2k}-\alpha^{kn}\\&=\alpha^{kn}(\alpha^{2k}-\alpha^k\beta^k)\\&=\alpha^{k(n+1)}(\alpha^k-\beta^k)\\&=(\alpha-\beta)\alpha^{k(n+1)}U_k.\end{aligned}$$

Before proceeding we note that, for $|p|\ge 2$, $\{|U_n|\}_{n=0}^{\infty}$ is an increasing sequence and, for $|p|>2$, $\{|V_n|\}_{n=0}^{\infty}$ is an increasing sequence.

Theorem 7: For $|p|\ge 2$, $k\ne 0$ an integer,

$$\sum_{i=1}^{n}\frac{1}{U_{ki}U_{k(i+1)}}=\frac{1}{\alpha^k U_k}\left[\frac{1}{U_k}-\frac{1}{\alpha^{kn}U_{k(n+1)}}\right], \tag{3.3}$$

$$\sum_{n=1}^{\infty}\frac{1}{U_{ki}U_{k(i+1)}}=\frac{1}{\alpha^k U_k^2}. \tag{3.4}$$

Proof:

$$\begin{aligned}\frac{1}{\alpha^{ki}U_{ki}}-\frac{1}{\alpha^{k(i+1)}U_{k(i+1)}}&=\frac{\alpha^k U_{k(i+1)}-U_{ki}}{\alpha^{k(i+1)}U_{ki}U_{k(i+1)}}\\&=\frac{U_k}{u_{ki}U_{k(i+1)}}\quad\text{[by (3.1)]}.\end{aligned}$$

Letting $i = 1, 2, \ldots, n$ and summing both sides proves (3.3). Letting $n \to \infty$ in (3.3) establishes (3.4). □

Making use of (3.2) and proceeding in the same manner, we obtain

Theorem 8: $|p| > 2$, $k \neq 0$ an integer,

$$\sum_{i=0}^{n} \frac{1}{V_{ki}V_{k(i+1)}} = \frac{1}{(\alpha-\beta)U_k}\left[\frac{1}{2} - \frac{1}{\alpha^{k(n+1)}V_{k(n+1)}}\right], \tag{3.5}$$

$$\sum_{i=0}^{\infty} \frac{1}{V_{ki}V_{k(i+1)}} = \frac{1}{2(\alpha-\beta)U_k}. \tag{3.6}$$

As an application of our results we note that, when $p = 2$,

$$\{U_n\}_{n=0}^{\infty} = \{0, 1, 2, 3, \ldots\} \quad \text{and} \quad \alpha = \beta = 1.$$

Although the Binet form for U_n degenerates, (3.1) remains valid, and Theorem 7 yields the familiar sums:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}, \tag{3.7}$$

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1. \tag{3.8}$$

As another application, taking $p = 3$ gives $U_n = F_{2n}$, $V_n = L_{2n}$, and (3.4) and (3.6) become, respectively,

$$\sum_{i=1}^{\infty} \frac{1}{F_{2ki}F_{2k(i+1)}} = \frac{2^k}{(3+\sqrt{5})^k F_{2k}^2}, \tag{3.9}$$

$$\sum_{i=0}^{\infty} \frac{1}{L_{2ki}L_{2k(i+1)}} = \frac{\sqrt{5}}{10F_{2k}}. \tag{3.10}$$

4. CONCLUDING COMMENTS

We feel that there is more scope for developing results for the sequences (1.7) which parallel existing results for the sequences (1.1). For example, Brugia, Di Porto, and Filipponi [9] investigated the sum

$$S = \sum_{i=0}^{\infty} \frac{U_i}{r^i}, \quad r \neq 0,$$

$$= \frac{r}{r^2 - pr - 1}, \quad |r| > \gamma,$$

where U_n is as in (1.1). They established the following theorem.

Theorem 9: If U_n is as in (1.1), where p is integral, the rational values of r for which S is integral are given by

$$r = \frac{U_{2n+1}}{U_{2n}} \quad (n = 1, 2, 3, \ldots),$$

and the corresponding value of S is given by

$$S = U_{2n}U_{2n+1} \quad (n = 1, 2, 3, \ldots).$$

We have investigated precisely the same sum for U_n as in (1.7), with $|r| > \alpha$, and have found the following parallel result.

Theorem 10: If U_n is as in (1.7), where $p > 2$ is integral, the rational values of r for which S is integral are given by

$$r = \frac{U_{n+1}}{U_n} \quad (n = 1, 2, 3, \ldots),$$

and the corresponding value of S is given by

$$S = U_nU_{n+1} \quad (n = 1, 2, 3, \ldots).$$

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SOME INVARIANT AND MINIMUM PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

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The Stirling numbers of the second kind $S(n, k)$ have been studied extensively. This note was motivated by the enumeration of pairwise disjoint finite sequences of random natural numbers. The two main results presented in this note demonstrate some invariant and minimum properties of the Stirling numbers of the second kind.

Combinatorial arguments are used to establish these results; hence, it would be helpful to recall that $S(n, k)$ counts the number of ways to partition a set of n elements into k nonempty subsets. The first main result is

Theorem 1: Let $\mathbf{r} = (r_1, \ldots, r_m)$ be an m-tuple of positive integers, and let N be a positive integer. Denote by $f(N; \mathbf{r})$ the number of m pairwise disjoint finite sequences of rolling an N-faced die, in which the i^{th} sequence consists of r_i trials. Then

$$f(N; \mathbf{r}) = \sum_{\substack{t_1, \ldots, t_m \geq 1 \\ t_1 + \cdots + t_m = N}} \frac{N!}{t_j!} t_j^{r_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} S(r_i, t_i)$$

for any j where $1 \leq j \leq m$.

Proof: Assume there are t_i distinct outcomes from the i^{th} sequence, where $i \neq j$, then the j^{th} sequence consists of at most $t_j = N - \sum_{i \neq j} t_i$ distinct outcomes. There are $\binom{N}{t_1, \ldots, t_m}$ ways to select the possible outcomes. For each $i \neq j$, and a fixed set of t_i outcomes, there are $t_i! S(r_i, t_i)$ ways to roll the die. The j^{th} sequence can be formed in $t_j^{r_j}$ ways. Thus, the total number of ways to roll the die in m disjoint sequences is

$$\sum_{\substack{t_1, \ldots, t_m \geq 1 \\ t_1 + \cdots + t_m = N}} \binom{N}{t_1, \ldots, t_m} t_j^{r_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} t_i! S(r_i, t_i) = f(N; \mathbf{r}).$$

This completes the proof of the claim and the theorem. □

The special case of $m = 2$ appeared in [1]. Its solution (see [2]) can be extended easily to provide another proof of Theorem 1 which is similar to the above proof in spirit. The following corollaries are immediate.

Corollary 2: For $m \geq N$, we have

$$f(N; \mathbf{r}) = \begin{cases} N! & \text{if } m = N, \\ 0 & \text{if } m > N. \end{cases}$$

Corollary 3: The probability that m finite sequences $S_1, S_2, \ldots, S_m$ of rolling a fair N-faced die are pairwise disjoint is $f(N; \mathbf{r}) / N^S$, where $\mathbf{r} = (|S_1|, \ldots, |S_m|)$ and $S = \sum_{k=1}^m |S_k|$.

Corollary 4: Given a permutation σ of $\{1, 2, \ldots, m\}$, define $\sigma(\mathbf{r})$ to be $(r_{\sigma(1)}, \ldots, r_{\sigma(m)})$. Then $f(N; \mathbf{r}) = f(N; \sigma(\mathbf{r}))$ for all σ.

In the proof of Theorem 1, if we assume there are t_i distinct outcomes from the i^{th}-sequence for each i, we have

Corollary 5: Let $\mathbf{r}$ and $f(N; \mathbf{r})$ be defined as in Theorem 1. Then

$$f(N; \mathbf{r}) = \sum_{\substack{t_1, \ldots, t_m \geq 1 \\ t_1 + \cdots + t_m \leq N}} \frac{N!}{\left(N - \sum_{i=1}^m t_i\right)!} \prod_{i=1}^m S(r_i, t_i).$$

We note that $f(N; \mathbf{r})$ can be expressed in an interesting form which possesses a nice commutative property. Let $(N)_i$ denote the falling factorial, $N(N-1) \cdots (N-i+1)$. Given a natural number p, define $N(p)$ as an operator with base N and index p that operates on a polynomial $f(N)$ by the rule

$$N(p) * f(N) = \sum_{j=1}^{\min(p, N)} S(p, j)(N)_j f(N-j).$$

Consider $m = 2$. Suppose the first trial consists of j distinct outcomes, then S_1 and S_2 can be formed in $S(r_1, j)(N)_{r_i}$ and $(N - r_1)^{r_2}$ ways, respectively. Thus,

$$f(N; (r_1, r_2)) = N(r_1) * N^{r_2}.$$

The general result follows by induction:

Theorem 6: The value of $f(N; \mathbf{r})$ has the commutative property

$$f(N; \mathbf{r}) = N(r_1) * \cdots * N(r_{m-1}) * N^{r_m} = N(r_{\sigma(1)}) * \cdots * N(r_{\sigma(m-1)}) * N^{r_{\sigma(m)}}$$

for any permutation σ of $\{1, 2, \ldots, m\}$.

Our second main result is

Theorem 7: Let p and q be positive integers such that $p + q = C$ for some constant C. Then, for a fixed positive integer N, the value of

$$f(N; p, q) = \sum_{i=1}^{\min(p, N)} S(p, i) N(N-1) \cdots (N-i+1)(N-i)^q$$

attains its minimum when $|p - q| \leq 1$.

Proof: From the proof of Theorem 1, we know that $f(N; p, q)$ counts the number of ways to choose from $\{1, 2, \ldots, N\}$ two disjoint sequences S_1 and S_2 of length p and q, respectively. Let $v = 1 - 1/N$ represent the probability that some specific number does not turn up in a single toss of a fair N-faced die. Because of Corollary 3, it suffices to study

$$\begin{aligned}\Pr((r \in S_1 \cup S_2) \wedge (r \notin S_1 \cap S_2)) &= \Pr((r \in S_1) \wedge (r \notin S_2)) + \Pr((r \in S_2) \wedge (r \notin S_1)) \\ &= (1-v^p)v^q + (1-v^q)v^p \\ &= v^p + v^q - 2v^C,\end{aligned}$$

which is minimum if $v^p + v^q$ is. Since this is the sum of two positive numbers whose product v^C is a constant, it follows that it is minimum when $|p-q| \le 1$. □

Not surprisingly, Theorem 7 can be generalized. Let $S = \sum_{k=1}^m r_k$. Then

$$\begin{aligned}\Pr\left(\left(r \in \textstyle\bigcup_{k=1}^m S_k\right) \wedge (r \notin S_i \cap S_j,\ 1 \le i < j \le m)\right) &= \sum_{k=1}^m (1-v^{r_k})v^{S-r_k} \\ &= v^S \sum_{k=1}^m 1/v^{r_k} - mv^S.\end{aligned}$$

For fixed S, N, m, this probability will be minimum if $L = \sum_{k=1}^m 1/v^{r_k}$ is minimum. But L is the sum of positive numbers whose product $1/v^S$ is a constant; therefore, minimum probability is obtained when the r_i are as nearly equal as possible. In other words, we have $|r_i - r_j| \le 1$ for any distinct pair of integers i and j. This is equivalent to saying that R of the values r_i equal $Q+1$ and the remaining $m-R$ values equal Q, where $S = mQ+R$, $0 \le R < m$. This is the same condition under which $f(N; \mathbf{r})$ is minimum.

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The authors would like to dedicate this article to Dr. Zia Ulddin Ahman, Ex-Vice Chancellor, Aligarh Muslim University, Aligarh.

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A NOTE ON SIERPINSKI NUMBERS

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In [5], W. Sierpinski proved that there are infinitely many odd integers k (Sierpinski numbers) such that $k \cdot 2^n + 1$ is composite for all $n \geq 0$. In his proof, Sierpinski used as a covering set the set of primes $\{3, 5, 17, 257, 641, 65537, 6700417\}$. A "covering set" for any k means here a finite set of primes such that every integer $k \cdot 2^n + 1$, $n \geq 0$, is divisible by at least one of them. There are other covering sets (see [4], [6]). In 1962, J. L. Selfridge (unpublished manuscript) discovered that $\{3, 5, 7, 13, 19, 37, 73\}$ is a covering set for 78557.

In this note, we prove that there are infinitely many Sierpinski numbers of the new kind. We find those k such that $k \cdot 2^n + 1$, for n of the form $n = 4m + 2$, has an easy algebraic decomposition while, for other n, we have the covering set $\{3, 17, 257, 641, 65537, 6700417\}$ from Sierpinski's set.

Theorem 1: Let the positive integer t be any solution of the system of congruences

$$\begin{cases} t \equiv 1 \pmod{2}, \\ t \equiv 1 \text{ or } 2 \pmod{3}, \\ t \equiv 0 \pmod{5}, \\ t \equiv 1, 4, 13, \text{ or } 16 \pmod{17}, \\ t \equiv 1, 16, 241, \text{ or } 256 \pmod{257}, \\ t \equiv 1, 256, 65281, \text{ or } 65536 \pmod{65537}, \\ t \equiv 1, 65536, 6634881, \text{ or } 6700416 \pmod{6700417}, \\ t \equiv 256, 318, 323, \text{ or } 385 \pmod{641}. \end{cases} \tag{1}$$

Then $k = t^4$ is a Sierpinski number.

Proof: By (1),

$$\begin{cases} k \equiv 1 \pmod{2}, \\ k \equiv 1 \pmod{3}, \\ k \equiv 0 \pmod{5}, \\ k \equiv 1 \pmod{17}, \\ k \equiv 1 \pmod{257}, \\ k \equiv 1 \pmod{65537}, \\ k \equiv 1 \pmod{6700417}, \\ k \equiv -1 \pmod{641}. \end{cases} \tag{2}$$

So we have

$$k \cdot 2^{2m+1} + 1 \equiv 0 \pmod{3},$$
$$k \cdot 2^{8m+4} + 1 \equiv 0 \pmod{17},$$
$$k \cdot 2^{16m+8} + 1 \equiv 0 \pmod{257},$$

$$k \cdot 2^{32m+16} + 1 \equiv 0 \pmod{65537},$$
$$k \cdot 2^{64m+32} + 1 \equiv 0 \pmod{6700417},$$
$$k \cdot 2^{64m} + 1 \equiv 0 \pmod{641},$$

for $m \geq 0$. For $n = 4m+2$, we have

$$\begin{aligned} k \cdot 2^n + 1 &= t^4 \cdot 2^{4m+2} + 1 \\ &= 4(t \cdot 2^m)^4 + 1 \\ &= (t^2 \cdot 2^{2m+1} + t \cdot 2^{m+1} + 1)(t^2 \cdot 2^{2m+1} - t \cdot 2^{m+1} + 1). \end{aligned}$$

Since $t > 1$, $t^2 \cdot 2^{2m+1} - t \cdot 2^{m+1} + 1 > 1$ for $m \geq 0$. Therefore, $k \cdot 2^{4m+2} + 1$ is composite for all $m \geq 0$. Note that $k \cdot 2^{4m+2} + 1 \equiv 1 \pmod 5$, so $\{3, 5, 17, 257, 641, 65537, 6700417\}$ is not a covering set for k.

Are there other Sierpinski numbers analogous to Theorem 1?

The problem of determining the least value k_0 of k such that $k \cdot 2^n + 1$ is always composite was posed by Sierpinski [5], again by Guy [2], and was considered in [1] and [3]. The least known k is Selfridge's $k = 78557$ with covering set $\{3, 5, 7, 13, 19, 37, 73\}$. Perhaps k_0 has no covering set.

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CONJECTURES CONCERNING IRRATIONAL NUMBERS AND INTEGERS

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Let r be an irrational number between one and two. Every positive integer n can be represented in terms of r in a very simple way (Theorem 1) that perhaps deserves to be better known than it is. To get started, recall the customary notation [7] associated with the continued fraction for r:

$$r = [a_0, a_1, a_2, \ldots], \tag{1}$$

$$p_{-2} = 0,\ p_{-1} = 1,\ p_i = a_i p_{i-1} + p_{i-2}$$

and

$$q_{-2} = 1,\ q_{-1} = 0,\ q_i = a_i q_{i-1} + q_{i-2},$$

for $i = 0, 1, 2, \ldots$. The rational numbers p_i / q_i are in reduced form, and their limit is r. Moreover,

$$1 = q_0 \le q_1 < q_2 < \cdots < q_i < \cdots . \tag{2}$$

Theorem 1: Every positive integer n has a representation

$$n = \sum_{i=0}^{u} c_i q_i, \tag{3}$$

where the c_i are integers satisfying

$$0 \le c_i \le a_{i+1} \text{ for } 0 \le i \le u, \text{ and } c_u \ge 1. \tag{4}$$

Proof: For given n, let u be the index for which $q_u \le n < q_{u+1}$. By the division algorithm, there exist integers c_u and n_{u-1} such that $n = c_u q_u + n_{u-1}$, where $0 \le n_{u-1} < q_u$. Now

$$(a_{u+1} + 1) q_u \ge a_{u+1} q_u + q_{u-1} = q_{u+1} > n,$$

so that $c_u \le a_{u+1}$. If $n_{u-1} > 0$ then, similarly, $n_{u-1} = c_{u-1} q_{u-1} + n_{u-2}$, where $0 \le n_{u-2} < q_{u-1}$ and $c_{u-1} \le a_u$, so that $n = c_u q_u + c_{u-1} q_{u-1} + n_{u-2}$. If $n_{u-2} > 0$, we continue to strip away terms of the form $c_i q_i$ until reaching the representation (3). □

The proof of Theorem 1 occurs within a proof of a deeper theorem [3, p. 125] which is not primarily concerned with representing integers. (Theorem 1 may be viewed as a corollary to a more general representation theorem; see [1], [8, Ch. 8], and [4].) We abbreviate the representation (3) as $CF(r, n)$ and the set of all such representations for given r as $CF(r, \cdot)$. By construction, $CF(r, \cdot)$ is a unique representation in the sense that the coefficients c_i are the only positive integers satisfying

$$0 \le n - \sum_{i=s}^{u} c_i q_i < q_s \tag{4}$$

for $s = 0, 1, \ldots, u$.

Note that in (2) the base numbers are distinct except perhaps for $q_1 = q_0$. We shall show that when this happens either $c_0 = 0$ or else $c_1 = 0$; that is, the base number 1 occurs at most once in each evaluation of (3). For a proof, suppose that the proposition is false for some r, and let n be the least positive integer having $CF(r,n)$ of the form

$$n = c_0 \cdot 1 + c_1 \cdot 1 + c_2 \cdot q_2 + \cdots + c_u \cdot q_u$$

with c_0 and c_1 both nonzero. Let $n' = n - c_2 q_2 - \cdots - c_u q_u$. If $c_1 \le a_2 - 1$, then $1 \cdot 1 + c_1 \cdot 1$ and $0 \cdot 1 + (c_1 + 1) \cdot 1$ are distinct representations of n', contrary to the uniqueness of $CF(r, n')$. On the other hand, if $c_1 = a_2$, then $c_0 = 1$ since $c_0 \le a_1 = 1$, so that $c_0 + c_1 = a_2 + 1$. However, $a_2 + 1 = q_2$, so that $1 \cdot 1 + a_2 \cdot q_1 = 0 \cdot q_0 + 0 \cdot q_1 + 1 \cdot q_2$, contrary to the uniqueness of $CF(r, q_2)$.

Let s_j $[= s_j(r)]$ be the j^{th} positive integer n for which $c_1 \ne 0$ in the representation $CF(r,n)$. That is, s_j is the j^{th} positive integer n for which the smallest base number appearing in (3) is 1. Our first conjecture is that the sequence $\{s_j\}$ is "almost" an arithmetic sequence.

Conjecture 1: There exists a number $f = f(r)$ such that $|s_j - jf| \le 2$ for all $j \ge 1$.

In order to state a second conjecture about the sequence $\{s_j\}$, we recall a definition introduced by I. Niven [6]. Suppose $\Lambda = \{\lambda_j\}$ is a sequence of integers. For any integers k and $m \ge 2$, let $\Lambda(J, k, m)$ be the number of indices j that satisfy $1 \le j < J$ and $\lambda_j \equiv k \pmod{m}$. If the limit

$$\lim_{J\to\infty} \frac{1}{J} \Lambda(J, k, m)$$

exists and equals $1/m$ for all k satisfying $1 \le k \le m$, then Λ is *uniformly distributed* (*mod m*). If Λ is uniformly distributed (mod m) for every integer $m \ge 2$, then Λ is *uniformly distributed*.

Conjecture 2: $\{s_j\}$ is uniformly distributed.

Conjectures 1 and 2 extend to other sequences. Let $s(i, j)$ be the j^{th} positive integer n for which the least base number appearing in (3) is q_j.

Conjecture 3: There exist numbers $f_i = f_i(r)$ and $B_i = B_i(r)$ such that $|s(i,j) - jf_i| \le B_i$ for all $j \ge 1$.

Conjecture 4: For each $i \ge 1$, the sequence $\{s(i,j)\}_{j=1}^{\infty}$ is uniformly distributed.

The simplest representations $CF(r,\cdot)$ are for $r = (1+\sqrt{5})/2$, for in this case $a_i = 1$ for all $i \ge 0$, so that (3) gives the well-studied Zeckendorf representation of n. Moreover, the array $\{s(i,j)\}$ is the Zeckendorf array, which is proved identical in [2] to the Wythoff array introduced in [5]. For general r, we suggest that $CF(r,\cdot)$ be called the *r-Zeckendorf representation of n* and that the array $\{s(i,j)\}$ be called the *r-Zeckendorf array*.

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ON CERTAIN ARITHMETIC PROPERTIES OF FIBONACCI AND LUCAS NUMBERS

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0. INTRODUCTION

In [1] the authors discussed, inter alia, certain striking resemblances between the arithmetic behavior of Fibonacci and Lucas numbers on the one hand, and certain numbers arising from (generalized) paper-folding on the other. To describe the latter, let t, u be mutually prime positive integers (we may assume $t > u$) and set

$$M_n = \frac{t^n - u^n}{t-u}, \quad P_n = t^n + u^n, \quad n \geq 1. \tag{0.1}$$

Then the sequence $\{M_n\}$ shares arithmetical properties with the Fibonacci sequence $\{F_n\}$, while the sequence $\{P_n\}$ is similarly related to the Lucas sequence $\{L_n\}$. In particular, we have

$$\gcd(M_a, M_b) = M_d, \text{ where } d = \gcd(a, b), \tag{0.2}$$

mirroring a well-known feature of Fibonacci numbers (see Theorem 2.5).

It was pointed out in [1] that (0.2) could itself be used to *disprove* the corresponding assertion for the lcm; precisely, if $\text{lcm}(a, b) = \ell$, then $\text{lcm}(M_a, M_b) = M_\ell$ only in the trivial cases $a|b$ or $b|a$. The argument rested on a uniqueness theorem for the expression of rational numbers as a ratio of two members of the $\{M_n\}$ sequence. However, the authors did not establish the corresponding negative results for $\text{lcm}(F_a, F_b)$, $\text{lcm}(L_a, L_b)$.

In this paper the gap is filled, precisely by establishing the relevant uniqueness statements for ratios of Fibonacci numbers and Lucas numbers. It turns out that much of the work can be done for arbitrary sequences $\{u_n\}$ of positive integers satisfying the recurrence relation $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$.

Such sequences are, in a sense, *classified* by their initial values u_1, u_2. However, to discuss the classification, it is better to take the sequences backward with respect to n, that is, to allow n to take *any* integer value, although the principal results are all to be concerned with positive values of n. Then the Fibonacci sequence $\{F_n\}$ belongs to the special class given by $u_0 = 0$. Another interesting class, from our point of view, is given by $0 < u_0 \leq u_1$. The Lucas sequence $\{L_n\}$ seems, to us, to belong to a singleton class.

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It is interesting to note that the Fibonacci sequence plays a special role in the study of the whole class of sequences $\{u_n\}$ (see Theorem 1.1).

This note closes with a related result concerning the lcm in the mixed case, i.e., $\text{lcm}(F_a, L_b)$. The corresponding result for $\gcd(F_a, L_b)$ (see Theorem 2.9) is to be found in [1] and [2]. It would be interesting to seek the analogous result concerning $\gcd(M_a, P_b)$.

We use (r, s) for the open interval $r < x < s$ and $(r, s]$ for the half-open interval $r < x \le s$.

We do not claim originality for the statements of Corollary 2.3 and Theorem 2.4—though we have not ourselves succeeded in finding them in the literature. However, we do believe that our results on the lcm are entirely new.

1. GENERAL PROPERTIES

We assume here that we have a sequence $\{u_n, n \ge 0\}$ of integers such that $u_0 \ge 0$, $u_1 > 0$, and

$$u_{n+2} = u_{n+1} + u_n,\ n \ge 0. \tag{1.1}$$

Of course, both the Fibonacci numbers $\{F_n\}$ and the Lucas numbers $\{L_n\}$ meet these conditions. We prove the following:

Theorem 1.1: Let $\ell \ge 1$. Then

$$\frac{u_{k+\ell}}{u_k} \in (F_{\ell+1}, F_{\ell+2}),\ k \ge 3;\ \frac{u_{2+\ell}}{u_2} \in (F_{\ell+1}, F_{\ell+2}].$$

Proof: We argue by induction on ℓ. If $\ell = 1$, then

$$\frac{u_{k+1}}{u_k} > 1 \text{ if } k \ge 2;\ \frac{u_2}{u_1} \ge 1.$$

On the other hand, $u_{k+1}/u_k = 1 + u_{k-1}/u_k < 2$ if $k \ge 3; u_3/u_2 = 1 + u_1/u_2 \le 2$. Hence

$$\frac{u_{k+1}}{u_k} \in (1, 2) = (F_2, F_3),\ k \ge 3;\ \frac{u_3}{u_2} \in (1, 2] = (F_2, F_3].$$

Now let $\ell = 2$. Then

$$\frac{u_{k+2}}{u_k} = \frac{u_{k+1}}{u_k} + 1 \in (2, 3) = (F_3, F_4),\ k \ge 3;\ \text{and}\ \frac{u_4}{u_2} = \frac{u_3}{u_2} + 1 \in (2, 3] = (F_3, F_4].$$

We now carry out the inductive step. We assume the theorem is true for $\ell - 1, \ell - 2, \ell \ge 3$. Then

$$\begin{aligned}\frac{u_{k+\ell}}{u_k} &= \frac{u_{k+\ell-1}}{u_k} + \frac{u_{k+\ell-2}}{u_k} \in (F_\ell, F_{\ell+1}) + (F_{\ell-1}, F_\ell) \\ &= (F_\ell + F_{\ell-1}, F_{\ell+1} + F_\ell) = (F_{\ell+1}, F_{\ell+2}),\ \text{if}\ k \ge 3;\end{aligned}$$

and

$$\begin{aligned}\frac{u_{2+\ell}}{u_2} &= \frac{u_{1+\ell}}{u_2} + \frac{u_\ell}{u_2} \in (F_\ell, F_{\ell+1}] + (F_{\ell-1}, F_\ell] \\ &= (F_\ell + F_{\ell-1}, F_{\ell+1} + F_\ell] = (F_{\ell+1}, F_{\ell+2}],\end{aligned}$$

proving the theorem.

Remarks:

(a) Notice that, for the Fibonacci numbers, $F_3 / F_2 = 2 = F_3$. Indeed $u_3 / u_2 = F_3$ precisely when $u_0 = 0$.

(b) There is no need in this theorem for u_n to be an integer.

Theorem 1.2: $u_{k+\ell} = F_\ell u_{k+1} + F_{\ell-1} u_k$, for all k, ℓ.

Proof: We hold k fixed and prove this for two successive values of ℓ; plainly, this suffices. Now $F_{-1} = 1, F_0 = 0, F_1 = 1$, so it is plain that the formula holds* for $\ell = 0, 1$.

Theorem 1.3: Let $\frac{u_{k+1}}{u_k} = \frac{u_{m+1}}{u_m}$ for some positive k, m. Then $k = m$.

Proof: Plainly,

$$\frac{u_{k+1}}{u_k} = \frac{u_{m+1}}{u_m} \Leftrightarrow \frac{u_{k-1}}{u_k} = \frac{u_{m-1}}{u_m} \Leftrightarrow \frac{u_k}{u_{k-1}} = \frac{u_m}{u_{m-1}}.$$

Thus, if $k \neq m$, we have $h \geq 3$ such that

$$\frac{u_h}{u_{h-1}} = \frac{u_2}{u_1}. \tag{1.2}$$

We now proceed backwards, that is, we allow negative values of n in u_n. We look at the sequence $\{u_n, 2 \geq n > -\infty\}$. There are three possibilities:

(i) As n decreases, this sequence remains positive. This, however, is impossible, since, if all terms are positive, it follows from (1.1) that the sequence is decreasing; but there is no strictly decreasing infinite sequence of positive integers.

(ii) As n decreases, this sequence remains positive until it takes the value 0.

(iii) As n decreases, this sequence remains positive until it takes a negative value.

Thus there must be a first value n for which u_n is nonpositive (as n *decreases*). It follows from (1.2) that

$$\frac{u_{r+1}}{u_r} = \frac{u_{n+1}}{u_n} \tag{1.3}$$

with $r > n$. Thus $u_{r+1} u_n = u_r u_{n+1}$, with u_{r+1}, u_r, u_{n+1} positive and u_n nonpositive. This contradiction implies $k = m$.

From Theorems 1.2 and 1.3, we infer

Theorem 1.4: Let $\frac{u_{k+\ell}}{u_k} = \frac{u_{m+\ell}}{u_m}$ for some positive k, m, and $\ell \geq 1$. Then $k = m$.

Proof: By Theorem 1.2, we infer that

$$\frac{F_\ell u_{k+1}}{u_k} + F_{\ell-1} = \frac{F_\ell u_{m+1}}{u_m} + F_{\ell-1}.$$

* We may, of course, continue the sequence $\{u_n\}$ backwards, using (1.1). In particular, we may define F_n, L_n for n negative.

Since $\ell \geq 1, F_\ell \neq 0$, so $u_{k+1}/u_k = u_{m+1}/u_m$. We apply Theorem 1.3.

We now put together Theorems 1.4 and 1.1 to infer

Theorem 1.5: If $\dfrac{u_{k+\ell}}{u_k} = \dfrac{u_{m+n}}{u_m}$ with k, $m \geq 2$, and $\ell, n \geq 1$, then $k = m$, $\ell = n$.

Proof: By Theorem 1.1,

$$\frac{u_{k+\ell}}{u_k} \in (F_{\ell+1}, F_{\ell+2}], \quad \frac{u_{m+n}}{u_m} \in (F_{n+1}, F_{n+2}].$$

Now if $\ell, n \geq 1$, then $(F_{\ell+1}, F_{\ell+2}]$ and $(F_{n+1}, F_{n+2}]$ are disjoint unless $\ell = n$. Thus, $\ell = n$ and

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{m+\ell}}{u_m}, \ \ell \geq 1, \text{ so that, by Theorem 1.4, } k = m.$$

Remark: In fact, it is only in the proof of Theorem 1.3 that we use the fact that $\{u_n\}$ is a sequence of integers. However, this, of course, implies that Theorems 1.4 and 1.5 are only proved under this assumption. Note that the conclusion of Theorem 1.3 is false if $u_n = \phi^n$, where ϕ is the golden section (so that $\phi^2 = \phi + 1$).

2. SPECIAL PROPERTIES OF FIBONACCI AND LUCAS NUMBERS

It is noteworthy that, in Theorem 1.5, we must exclude the possibility that $k = 1$ or $m = 1$. Of course, if we require $\ell, n \geq 1$, then the conclusion of Theorem 1.5 trivially follows if $k = m = 1$, for then $u_{\ell+1} = u_{n+1}$ with $\ell+1, n+1 \geq 2$, and the sequence $\{u_n, n \geq 2\}$ is strictly increasing. However, we also have to consider the possibilities $k = 1, m \geq 2$ and $m = 1, k \geq 2$.

In considering these possibilities, we are content largely to confine our attention to the sequences $\{F_n\}$ and $\{L_n\}$ of Fibonacci and Lucas numbers, since it is to arithmetic properties of these sequences that we will be applying our enhanced form of Theorem 1.5. However, notice that we get the enhanced form for certain sequences $\{u_n\}$ immediately, by the following observation.

Suppose that, in fact, $0 < u_0 \leq u_1$. Set $v_n = u_{n-1}, n \geq 0$, noticing that $u_{-1} = u_1 - u_0 \geq 0$. Thus we may apply Theorem 1.5 to the sequence $\{v_n\}$, obtaining

$$\frac{v_{k+\ell}}{v_k} = \frac{v_{m+n}}{v_m}, \text{ with } k, m \geq 2, \ \ell, n \geq 1 \Rightarrow k = m, \ \ell = n. \tag{2.1}$$

Rewrite (2.1) in terms of the original sequence $\{u_n\}$, writing $k+1$ for k, $m+1$ for m; we obtain

Theorem 2.1: If, in addition, $0 < u_0 \leq u_1$, then

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{m+n}}{u_m}, \text{ with } k, m \geq 1, \ \ell, n \geq 1 \Rightarrow k = m, \ \ell = n. \tag{2.2}$$

We must proceed differently in seeking the enhanced form of Theorem 1.5 in the case of the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$. For, of course, neither sequence satisfies

$0 < u_0 \le u_1$. Indeed, $F_0 = 0$ and $L_0 > L_1$. Thus we first attach the supplementary condition $u_0 = 0$; this means that $u_n = u_1 F_n$. We prove

Theorem 2.2: If, in addition, $u_0 = 0$, then

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{m+n}}{u_m}, \text{ with } k, m \ge 1,\ \ell, n \ge 1 \Rightarrow k = m,\ \ell = n;$$
$$\text{or} \quad k = 1,\ m = 2,\ \ell = n+1;$$
$$\text{or} \quad k = 2,\ m = 1,\ n = \ell + 1.$$

Proof: We may assume $k = 1, m \ge 2$ in light of Theorem 1.5 and the opening remark of this section. Then

$$\frac{u_{\ell+1}}{u_1} = \frac{u_{m+n}}{u_m} \text{ or } F_{\ell+1} = \frac{F_{m+n}}{F_m}.$$

But $m \ge 2$ so that, by Theorem 1.1,

$$F_{\ell+1} = \frac{F_{m+n}}{F_m} \in (F_{n+1}, F_{n+2}],\ \ell, n \ge 1.$$

This forces $\ell = n+1$, so that

$$\frac{F_{m+n}}{F_m} \notin (F_{n+1}, F_{n+2}),$$

forcing $m = 2$. Of course, the case $m = 1, k \ge 2$ is treated similarly.

Applying Theorem 2.2 to the sequence $\{F_n\}$, we have

Corollary 2.3: If

$$\frac{F_{k+\ell}}{F_k} = \frac{F_{m+n}}{F_m}, \text{ with } k, m \ge 1,\ \ell, n \ge 1 \text{ then } k = m,\ \ell = n;$$
$$\text{or} \quad k = 1,\ m = 2,\ \ell = n+1;$$
$$\text{or} \quad k = 2,\ m = 1,\ n = \ell + 1.$$

We now turn to the Lucas sequence $\{L_n\}$ and prove

Theorem 2.4: If $\dfrac{L_{k+\ell}}{L_k} = \dfrac{L_{m+n}}{L_m}$, with $k, m \ge 1,\ \ell, n \ge 1$, then $k = m,\ \ell = n$

Proof: As in the proof of Theorem 2.2, we observe that, effectively, we have only to show that the assumption $k = 1,\ m \ge 2$ leads to a contradiction. Thus we are given that

$$L_{\ell+1} = \frac{L_{m+n}}{L_m},\ m \ge 2.$$

By Theorem 1.1,

$$\frac{L_{m+n}}{L_m} \in (F_{n+1}, F_{n+2}].$$

But

$$L_{\ell+1} = F_\ell + F_{\ell+2} \in (F_{\ell+2}, F_{\ell+3}].$$

Thus we must have $n=\ell+1$, so that $L_mL_n=L_{m+n}$. Now it is easy to see that, for all m, n, $L_mL_n=L_{m+n}+(-1)^nL_{m-n}$. Since no Lucas number is zero, we have arrived at the hoped for contradiction.

We use Corollary 2.3 and Theorem 2.4 to obtain interesting results on lcms of Fibonacci and Lucas numbers. We have the well-known classical result

Theorem 2.5: Let $\gcd(a,b)=d$. Then $\gcd(F_a,F_b)=F_d$.

We prove

Theorem 2.6: Let $\operatorname{lcm}(a,b)=\ell$. Then $\operatorname{lcm}(F_a,F_b)=F_\ell \Leftrightarrow a|b$ or $b|a$.

Proof: Certainly, if $a|b$, then $F_a|F_b$, so $\operatorname{lcm}(F_a,F_b)=F_b$ and $b=\ell$. A similar argument applies if $b|a$. Now suppose that $\operatorname{lcm}(F_a,F_b)=F_\ell$. It then follows from Theorem 2.5 that

$$F_aF_b=F_\ell F_d \quad \text{or} \quad \frac{F_a}{F_d}=\frac{F_\ell}{F_b}.$$

Of course, $a\ge d$, $\ell\ge b$. Moreover, $a=d\Leftrightarrow \ell=b$, and this is the case $a|b$. If we are not in this case, then we may apply Corollary 2.3 to infer that

$$\begin{aligned} &a=\ell,\ d=b\\ \text{or}\ \ &d=1,\ b=2,\ a=\ell\\ \text{or}\ \ &d=2,\ b=1, a=\ell. \end{aligned}$$

However, the conjunction $d=1$, $b=2$, $a=\ell$ is absurd and the conjunction $d=2$, $b=1$, $a=\ell$ is even more absurd. Hence, we conclude that $a=\ell$, $d=b$, or $b|a$.

Now there is a result for Lucas numbers corresponding to Theorem 2.5 (see [1] and [2]); thus,

Theorem 2.7: Let $\gcd(a,b)=d$. Then

$$\gcd(L_a,L_b)=\begin{cases} L_d & \text{if } \ |a|_2=|b|_2,\\ 2 & \text{if } \ |a|_2\ne|b|_2 \text{ and } 3|d,\\ 1 & \text{if } \ |a|_2\ne|b|_2 \text{ and } 3\nmid d.\end{cases}$$

Here $|n|_2$ is the 2-valuation of n, i.e., the largest k such that $2^k|n$. Let us say ***a* divides *b* oddly** if $a|b$ and $|a|_2=|b|_2$, that is, if $a|b$ with odd quotient. Then we prove

Theorem 2.8: Let $\operatorname{lcm}(a,b)=\ell$. Then

$$\operatorname{lcm}(L_a,L_b)=L_\ell \Leftrightarrow a|b \text{ oddly or } b|a \text{ oddly or } a=1 \text{ or } b=1.$$

Proof: If $a|b$ oddly, then $L_a|L_b$, so $\operatorname{lcm}(L_a,L_b)=L_b=L_\ell$. A similar argument applies if $b|a$ oddly. Clearly if $a=1$ or $b=1$, then $\operatorname{lcm}(L_a,L_b)=L_\ell$. Now suppose conversely that $\operatorname{lcm}(L_a,L_b)=L_\ell$. Obviously we can have $a=1$ or $b=1$, so suppose $a,b\ge 2$. Then $L_a|L_\ell$, $L_b|L_\ell$, so, as an easy consequence of Theorem 2.7,

$$|a|_2=|\ell|_2=|b|_2.$$

Thus, by Theorem 2.7, $\gcd(L_a, L_b) = L_d$, whence $L_aL_b = L_dL_\ell$ or $L_a / L_d = L_\ell / L_b$. As in the proof of Theorem 2.6, we have $a \geq d$, $\ell \geq b$; and $a = d \Leftrightarrow \ell = b$, this being the case $a|b$ oddly. If we are not in this case, then $a > d$, $\ell > b$ so that, by Theorem 2.4, $a = \ell$, $d = b$, yielding $b|a$ oddly.

Remark: It is intriguing to see that, above, it is the fact that the gcd (of Fibonacci or Lucas numbers) *has* a certain desirable property which ensures that the lcm *cannot have* the corresponding property.

We close by proving a result similar to Theorems 2.6 and 2.8 in the *mixed* case. Here our argument is somewhat *ad hoc* (and could have been used in suitably adapted form to provide an alternative proof of Theorems 2.6 and 2.8). We first quote from [1] and [2].

Theorem 2.9: Let $\gcd(a, b) = d$. Then

$$\gcd(F_a, L_b) = \begin{cases} L_d & \text{if } |a|_2 > |b|_2, \\ 2 & \text{if } |a|_2 \leq |b|_2 \text{ and } 3|d, \\ 1 & \text{if } |a|_2 \leq |b|_2 \text{ and } 3\nmid d. \end{cases}$$

Now if $|a|_2 > |b|_2$ and $\ell = \operatorname{lcm}(a, b)$, then $F_a|F_\ell$, $L_b|F_\ell$ (since $2b|\ell$). Thus we may raise the question as to whether $\operatorname{lcm}(F_a, L_b) = F_\ell$. We prove

Theorem 2.10: Let $|a|_2 > |b|_2$. Then $\operatorname{lcm}(F_a, L_b) = F_\ell \Leftrightarrow b|a$.

Proof: If $b|a$, then $2b|a$, so $L_b|F_{2b}|F_a$ and $\operatorname{lcm}(F_a, L_b) = F_a = F_\ell$. Suppose conversely that $\operatorname{lcm}(F_a, L_b) = F_\ell$ and $b\nmid a$, that is, $b \neq d$, $a \neq \ell$. Then, by Theorem 2.9, $F_aL_b = F_\ell L_d$, so

$$L_b / L_d = F_\ell / F_a.$$

Suppose $d \geq 2$. Then, by Theorem 1.1,

$$\frac{L_b}{L_d} \in (F_{b-d+1}, F_{b-d+2}] \quad \text{and} \quad \frac{F_\ell}{F_a} \in (F_{\ell-a+1}, F_{\ell-a+2}].$$

Thus we conclude that $b - d = \ell - a$. This, however, is impossible, since it implies $a = d$ or $b = d$; and $a = d$ is excluded because $a\nmid b$.

Suppose, finally, that $d = 1$. Then

$$L_b = \frac{F_\ell}{F_a} \in (F_{\ell-a+1}, F_{\ell-a+2}].$$

But $L_b \in (F_{b+1}, F_{b+2}]$, which implies $\ell - a = b$. This, in turn, implies $ab = a + b$ with a, b mutually prime, which is plainly absurd. This final contradiction establishes the theorem.

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SUMS OF ARITHMETIC PROGRESSIONS

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Readers of *Mathematical Spectrum* have recently indicated an interest in the problem of expressing a natural number n as a sum of (at least two) consecutive natural numbers (see [1]-[5]). Bob Bertuello showed that this is not possible when n is a power of 2. We prove here a general result which determines those natural numbers that can be expressed as a sum of natural numbers in arithmetic progression with common difference d.

A consequence of the case $d = 1$ is that a natural number is a sum of consecutive natural numbers if and only if it is not a power of 2. (We believe this result may already be known, but have not been able to trace it in the literature.) When $d = 2$, our theorem shows that a natural number n is a sum of natural numbers in arithmetic progression with common difference 2 if and only if n is not a prime. We shall also illustrate the result for the case $d = 3$.

Theorem: Let the natural number d be given. Then the natural number $n = 2^h m$. where m is odd and $n > 1$, is a sum of natural numbers which form an arithmetic progression with common difference d if and only if

(1) for d odd, n is not a power of 2 and either $m > d(2^{h+1} - 1)$ or $n > \frac{1}{2}dp(p-1)$, where p is the smallest odd prime factor of n,

(2) for d even, either n is even and $n > d$ or n is odd and $n > \frac{1}{2}dp(p-1)$, where again p is the smallest odd prime factor of n.

Proof: We first prove that the conditions given are necessary. Suppose that n is a sum of natural numbers which form an arithmetic progression with common difference d, say,

$$n = r + (r+d) + (r+2d) + \cdots + (r+sd)$$

for some natural numbers r and s. (It is always understood that there is more than one term in the sum.) Then

$$n = (s+1)\left(r + \frac{sd}{2}\right).$$

We consider four cases.

Case 1. d odd, s odd. Then

$$n = \frac{s+1}{2}(2r + sd)$$

and $2r + sd$ is an odd divisor of n. Hence, n is not a power of 2 and $\frac{s+1}{2} \geq 2^h$, i.e., $s \geq 2^{h+1} - 1$. Thus,

$$2^h m = n > \frac{s+1}{2} sd \geq 2^h d(2^{h+1} - 1),$$

whence $m > d(2^{h+1} - 1)$.

Case 2. d odd, s even. Then

$$n=(s+1)\left(r+\frac{s}{2}d\right)$$

and n has the odd divisor $s+1>1$. Hence, n is not a power of 2 and $s+1\geq p$, where p is the smallest odd prime factor of n. Thus,

$$n>(s+1)\frac{s}{2}d\geq\frac{1}{2}dp(p-1).$$

Case 3. d even, n even. Clearly, $n>d$.

Case 4. d even, n odd. Then

$$n=(s+1)\left(r+s\frac{d}{2}\right)$$

and n has the (odd) divisor $s+1>1$. The argument of Case 2 gives $n>\frac{1}{2}dp(p-1)$.

We now prove that the conditions are sufficient. Again, there are four cases.

Case 1. d odd, $m>d(2^{h+1}-1)$ (so that n is not a power of 2). Put $s=2^{h+1}-1$ and $r=\frac{1}{2}[m-d(2^{h+1}-1)]$. Then r and s are natural numbers and

$$\begin{aligned}r+(r+d)+(r+2d)+\cdots+(r+sd)&=(s+1)\left(r+\frac{1}{2}sd\right)\\&=2^{h+1}\left\{\frac{1}{2}[m-d(2^{h+1}-1)]+\frac{1}{2}d(2^{h+1}-1)\right\}\\&=2^hm=n.\end{aligned}$$

It is worth noting that, in this case, the arithmetic progression contains $s+1=2^{h+1}$ terms.

Case 2. d odd, n not a power of 2, and $n>\frac{1}{2}dp(p-1)$. Choose $s=p-1$ and $r=\frac{n}{p}-\frac{1}{2}d(p-1)$. Then r and s are natural numbers and

$$\begin{aligned}r+(r+d)+(r+2d)+\cdots+(r+sd)&=(s+1)\left(r+\frac{1}{2}sd\right)\\&=p\left\{\frac{n}{p}-\frac{1}{2}d(p-1)+\frac{1}{2}d(p-1)\right\}=n.\end{aligned}$$

In this case, the arithmetic progression contains p terms, where p is the smallest odd prime factor of n.

Case 3. d even, n even, and $n>d$. Choose $s=1$ and $r=\frac{1}{2}(n-d)$. Then r and s are natural numbers and $r+(r+d)=n$. In this case, there are just two terms in the arithmetic progression.

Case 4. d even, n odd, and $n>\frac{1}{2}dp(p-1)$. The argument is the same as for Case 2, and n is the sum of p terms in arithmetic progression.

This completes the proof of the general result. Finally, we consider what it looks like in the cases $d=1$, 2, and 3.

Corollary 1: A natural number > 1 is a sum of (at least two) consecutive natural numbers if and only if it is not a power of 2.

Proof: From the Theorem (with $d = 1$), if n is a sum of consecutive natural numbers, then it cannot be a power of 2.

Conversely, suppose n is not a power of 2. We can write $n = 2^h p^a q$, where $q \geq 1$ and the prime factors of q (if any) are all greater than p. If $q > 1$, then $q > p$ and $n > p^2 > \frac{1}{2}p(p-1)$ and, by the Theorem, n is expressible in the manner required. If $q = 1$ and $a > 1$, then $n \geq p^2 > \frac{1}{2}p(p-1)$ and, again, n is so expressible. If $q = 1$ and $a = 1$, then $n = 2^h p$. We need either $p > 2^{h+1} - 1$ or $2^h p > \frac{1}{2}p(p-1)$, i.e., either $p > 2^{h+1} - 1$ or $p < 2^{h+1} + 1$. One of these must hold, so that n is expressible as a sum of consecutive natural numbers.

Corollary 2: A natural number > 1 is a sum of natural numbers which form an arithmetic progression with common difference 2 if and only if it is not prime.

Proof: It is easy to see that, if n is prime, then it does not satisfy the conditions (2) in the Theorem with $d = 2$, so that n is not expressible in the way required. Suppose n is not prime. If n is even, it is greater than 2. If n is odd, say $n = pq$, for some odd integer $q \geq p$, then $n \geq p^2 > p(p-1)$. Hence, n is expressible in the manner required.

Corollary 3: A natural number > 1 is a sum of natural numbers which form an arithmetic progression with common difference 3 if and only if it is not one of the following:

(a) a power of 2;

(b) $2^h p$, where p is an odd prime such that $\frac{1}{3}(2^{h+1}+1) < p \leq 3(2^{h+1}-1)$.

Proof: If n is a power of 2, then, from the Theorem (with $d = 3$), it is not expressible in the way required. If $n = 2^h p$, where p is an odd prime, then n is expressible in the way required if and only if either $p > 3(2^{h+1}-1)$ or $2^h p > \frac{3}{2}p(p-1)$. The latter is equivalent to $p \leq \frac{1}{3}(2^{h+1}+1)$. Hence, n is not expressible in this way if and only if p lies between these two values, viz:

$$\frac{1}{3}(2^{h+1}+1) < p \leq 3(2^{h+1}-1).$$

If $n = 2^h pq$, where p is an odd prime and q is an odd number such that $q \geq p$, then, if $h = 0$ and $m = pq > 3(2^1 - 1)$, and if $h > 0$, then $n = 2^h pq \geq 2p^2 > \frac{3}{2}p(p-1)$, so that n is expressible in the way required.

Thus, examples of natural numbers *not* expressible as a sum of natural numbers in arithmetic progression with common difference 3 are:

$$\begin{aligned} h = 0&: \quad 2^0 \times 3, \\ h = 1&: \quad 2 \times 3, 2 \times 5, 2 \times 7, \\ h = 2&: \quad 2^2 p, p \text{ prime}, \ 5 \leq p \leq 19, \\ h = 3&: \quad 2^3 p, p \text{ prime}, \ 7 \leq p \leq 43. \end{aligned}$$

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The SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Technische Universität Graz from July 14 to July 19, 1996. This conference will be sponsored jointly by the Fibonacci Association and Technische Universität Graz.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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SHORT PERIODS OF CONTINUED FRACTION CONVERGENTS MODULO M: A GENERALIZATION OF THE FIBONACCI CASE

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1. INTRODUCTION

The period length of the continued fraction convergents modulo m of reduced quadratic irrationals α was studied in [1]. Of course, for $\alpha = (1+\sqrt{5})/2$, this is just the period length of the Fibonacci sequence modulo m, a well studied problem (see [2] and [6]). The period of the convergents of α modulo m is bounded above by linear expressions in m. These linear bounds on the period are achieved with some frequency, yet there are many moduli m with much smaller periods. However, all the periods are at least $c\log(m)$, where c is a constant depending on α [1]. Work classifying some of the short periods in the special case of the Fibonacci sequence has been done (see [3] and [5]). This paper classifies many m having short periods for the convergents of general reduced quadratic irrationals. They are specified in parametric form by particular polynomials whose values generate moduli giving rise to short periods. The periods are short in the sense that the period lengths grow linearly while the moduli grow exponentially in the families generated by these polynomials.

Consider the following example. Continued fraction convergents are computed via the recursions $p_{-1}=0$, $p_0=1$, $p_n = a_n p_{n-1} + p_{n-2}$, and $q_{-1}=1$, $q_0=0$, $q_n = a_n q_{n-1} + q_{n-2}$. Consider the convergents of $\alpha = [\overline{1,1,2}] = (2+\sqrt{10})/3$ modulo 13 shown below.

n			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a_n			1	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2
p_n (mod 13)	0	1	1	2	5	7	12	5	4	9	9	5	1	7	8	2	12	1	0	1
q_n (mod 13)	1	0	1	1	3	4	7	5	12	4	7	11	5	8	0	5	3	11	1	0

The block $\binom{0\ 1}{1\ 0}$ is repeated after 18 steps and hence the continued fraction convergents are repeated thereafter. We designate the period length of the convergents of α modulo m by $k(\alpha, m)$ or just $k(m)$. In this example, $k(\alpha, 13) = 18$. The period is always well defined for α with purely periodic continued fraction expansions.

Many properties of these periods are known [1]. In particular, if we let t denote the period length of α and d be the discriminant associated with α defined in Section 2, then it is known that, for odd primes p, the period $k(p)$ divides $(p-1)t, 4pt$, or $2(p+1)t$ depending on whether the Legendre symbol $(\frac{d}{p})$ is 1, 0, or -1, respectively. Moreover, a factor of 2 can be removed from the second two bounds if t is even. In Table 1, the period of $\alpha = [\overline{1, 1, 2}]$ is given for the primes less than or equal to 1000, and the quotient of that period with the bounds mentioned above are given by $Q(p)$. Notice that the quotient is 1 for 111 of the 167 primes given; however, the quotient is sometimes quite large. For example, $Q(859) = 43$. While there is not an obvious pattern, we can explain, up to a factor of 2, all of the quotients over 1 appearing in Table 1. The explanation will be given in terms of the families of moduli with short periods that we will construct in Section 4.

2. FUNDAMENTAL MATRICES AND THE $\mathscr{L}_n$-SEQUENCE

The first four theorems below give a matrix reformulation of the process used to find the periods of the convergents, following [1]. Let $\alpha = [\overline{a_1, a_2, \ldots, a_t}]$. **Note:** We will use "t" throughout this paper to designate the length of the period of the purely periodic continued fraction. The convergents at the end of one t-period can be used to compute the convergents at the end of the subsequent t-periods and this information can be used to find the period of the continued fraction sequence modulo m.

Theorem 1: Let $W = \begin{pmatrix} q_{t-1} & q_t \\ p_{t-1} & p_t \end{pmatrix}$. Then $W^n = \begin{pmatrix} q_{nt-1} & q_{nt} \\ p_{nt-1} & p_{nt} \end{pmatrix}$.

The matrix W is called the *fundamental matrix* for α.

The period of α is preserved mod m means that the period of α does not change when the partial quotients are reduced mod m. For example, the convergents of $\alpha = [\overline{1, 2, 3, 4}]$ are the same as those for $[\overline{1, 2}]$ mod 2; hence, the period of α is not preserved mod 2.

Theorem 2:

(i) If $W^n \equiv I \bmod m$, then $k(m) | nt$.

(ii) If the period of α is preserved mod m, then c is the smallest integer such that $W^c \equiv I \bmod m$ if and only if $k(m) = ct$.

As an example of Theorem 2, consider the fundamental matrix for $\alpha = [\overline{1, 1, 2}]$ and its powers modulo 13.

$$W = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}, \qquad W^2 \equiv \begin{pmatrix} 7 & 5 \\ 12 & 5 \end{pmatrix}, \qquad \text{and } W^3 \equiv \begin{pmatrix} 4 & 7 \\ 9 & 9 \end{pmatrix},$$

$$W^4 \equiv \begin{pmatrix} 5 & 8 \\ 1 & 7 \end{pmatrix}, \qquad W^5 \equiv \begin{pmatrix} 8 & 3 \\ 2 & 12 \end{pmatrix}, \qquad \text{and } W^6 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that the sixth power is the first power congruent to the identity; by Theorem 2, $k(13) = 3 \cdot 6 = 18$, as we saw previously.

TABLE 1

The Periods of the Convergents of $\alpha = [\overline{1, 1, 2}]$ Modulo Small Primes

p	$k(p)$	$Q(p)$	p	$k(p)$	$Q(p)$	p	$k(p)$	$Q(p)$
3	6	1	271	90	9	619	3720	1
5	60	1	277	276	3	631	630	3
7	48	1	281	60	14	641	960	2
11	72	1	283	282	3	643	1926	1
13	18	2	293	876	1	647	432	9
17	108	1	307	918	1	653	978	2
19	24	5	311	930	1	659	1320	3
23	144	1	313	1884	1	661	3972	1
29	180	1	317	474	2	673	4044	1
31	90	1	331	1992	1	677	1014	2
37	36	3	337	2028	1	683	2046	1
41	120	1	347	1038	1	691	4152	1
43	126	1	349	2100	1	701	4212	1
47	288	1	353	2124	1	709	4260	1
53	78	2	359	1074	1	719	2154	1
59	120	3	367	2208	1	727	4368	1
61	372	1	373	1116	1	733	732	3
67	66	3	379	2280	1	739	4440	1
71	210	1	383	2304	1	743	4464	1
73	444	1	389	2340	1	751	2250	1
79	234	1	397	594	2	757	2268	1
83	246	1	401	1200	1	761	1140	2
89	264	1	409	1224	1	769	2304	1
97	588	1	419	360	7	773	1158	2
101	612	1	421	2532	1	787	2358	1
103	48	13	431	258	5	797	1194	2
107	318	1	433	2604	1	809	1212	2
109	660	1	439	438	3	811	4872	1
113	684	1	443	1326	1	821	1644	3
127	768	1	449	168	8	823	4944	1
131	72	11	457	2748	1	827	354	7
137	276	3	461	924	3	829	996	5
139	840	1	463	2784	1	839	2514	1
149	900	1	467	1398	1	853	1278	2
151	450	1	479	1434	1	857	156	33
157	468	1	487	2928	1	859	120	43
163	486	1	491	984	3	863	5184	1
167	336	3	499	3000	1	877	2628	1
173	516	1	503	432	7	881	528	5
179	360	3	509	3060	1	883	294	9
181	1092	1	521	390	4	887	5328	1
191	114	5	523	1566	1	907	2718	1
193	1164	1	541	3252	1	911	546	5
197	294	2	547	1638	1	919	2754	1
199	594	1	557	1668	1	929	1392	2
211	1272	1	563	1686	1	937	804	7
223	1344	1	569	1704	1	941	5652	1
227	678	1	571	312	11	947	2838	1
229	1380	1	577	3468	1	953	5724	1
233	468	3	587	1758	1	967	5808	1
239	714	1	593	3564	1	971	5832	1
241	90	8	599	1794	1	977	5868	1
251	1512	1	601	900	2	983	5904	1
257	1548	1	607	3648	1	991	2970	1
263	528	3	613	918	2	997	1494	2
269	1620	1	617	3708	1			

Define

$$C_j = \begin{pmatrix} q_{j-1} & q_j \\ p_{j-1} & p_j \end{pmatrix}.$$

Note that $C_j \equiv I \pmod{m}$ for $j < k(m)$ is possible if j is not a multiple of t. It is not difficult to show that the set of j for which $C_j \equiv I \pmod{m}$ is a union of $\le t$ arithmetic progressions with the difference between consecutive terms in each arithmetic progression equal to $k(m)$.

Next, the general fundamental matrix W has eigenvalues

$$\lambda_1 = \frac{1}{2}((p_t + q_{t-1}) + \sqrt{d}) \quad \text{and} \quad \lambda_2 = \frac{1}{2}((p_t + q_{t-1}) - \sqrt{d})$$

where

$$d = (p_t + q_{t-1})^2 + 4(-1)^{t-1}.$$

It follows immediately that the norm and trace of W are given by

$$\lambda_1\lambda_2 = (-1)^t \quad \text{and} \quad \lambda_1 + \lambda_2 = p_t + q_{t-1}.$$

Theorem 3: Define

$$\mathcal{L}_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{d}}.$$

Then $\mathcal{L}_0 = 0$, $\mathcal{L}_1 = 1$, and $\mathcal{L}_{n+1} = (p_t + q_{t-1})\mathcal{L}_n + (-1)^{t-1}\mathcal{L}_{n-1}$.

One consequence of this theorem is that $\mathcal{L}_n$ is an integer.

Theorem 4: Let W be the fundamental matrix for α. Then

$$W^n = \begin{pmatrix} q_{t-1}\mathcal{L}_n + (-1)^{t-1}\mathcal{L}_{n-1} & q_t\mathcal{L}_n \\ p_{t-1}\mathcal{L}_n & p_t\mathcal{L}_n + (-1)^{t-1}\mathcal{L}_{n-1} \end{pmatrix} = \mathcal{L}_n W + (-1)^{t-1}\mathcal{L}_{n-1} I.$$

Proof: This theorem is proved in [1] except that there the (2, 2) entry of the right-hand side is $\mathcal{L}_{n+1} - q_{t-1}\mathcal{L}_n$. Applying Theorem 3 gives the desired result. □

Theorem 5:

(i) Suppose m is a modulus so that $\mathcal{L}_{n-1} \equiv 1$ and $\mathcal{L}_n \equiv 0 \bmod m$. Then $k(m) | 2nt$ if t is even and $k(m) | nt$ is *t is odd.*

(ii) Suppose the period of α is preserved modulo m, $\gcd(q_t, m) = 1$, and that c is the smallest integer so that $\mathcal{L}_{c-1} \equiv 1$ and $\mathcal{L}_c \equiv 0$. Then $k(m) = ct$ if t is odd and $k(m) = 2ct$ if t is even.

Proof:

(i) Applying the congruences to Theorem 4 gives $W^n \equiv (-1)^{t-1}I$ modulo m. If t is odd, $W^n \equiv I$ and Theorem 2 gives the desired result; otherwise, square both sides to get $W^{2n} \equiv I$ mod m and the case for even t follows.

(ii) Suppose α, m, and c are as described. Again we see $W^c \equiv (-1)^{t-1}I$ and we claim c is the smallest such integer. If not, there is an n with $n < c$ and such that $W^n \equiv (-1)^{t-1}I$. Then, looking at $W_{1,2}^n$ in Theorem 4, we see $q_t\mathcal{L}_n \equiv 0$ so $\mathcal{L}_n \equiv 0$ since $\gcd(q_t, m) = 1$. Then, looking at $W_{1,1}^n$, we see $(-1)^{t-1}\mathcal{L}_{n-1} + q_{t-1}\mathcal{L}_n \equiv (-1)^{t-1}$, which implies $\mathcal{L}_{n-1} \equiv 1$, and these contradict the minimal choice of c. Thus, c is the smallest integer such that $W^c \equiv (-1)^{t-1} I$. If t is odd, Theorem 2(ii) gives $k(m) = ct$. If t is even, we know that $W^{2c} \equiv I$. We claim $2c$ is the smallest power of W giving the identity matrix mod m. If not, say $W^n \equiv I$ with $n < 2c$ is the smallest such power. Consider two cases: $n \geq c$ and $n < c$. If $n \geq c$, we use the Euclidian algorithm to write $n = qc + r$ with $0 \leq r < c$. So $I \equiv W^n = (W^c)^q W^r$. If q is even, this means $W^r \equiv I$, contradicting the minimality of n unless $r = 0$ or $q = 0$. However, $q = 0$ is impossible since $n \geq c$. In the case with $r = 0$, we get $n = cq \geq 2c$, contradicting $n < 2c$. If q is odd, $W^r \equiv -I$, contradicting the minimal choice of c. Next, consider the case $n < c$. The Euclidean algorithm gives $c = qn + r$ with $0 \leq r < n$. So $-I \equiv W^c = (W^n)^q W^r \equiv W^r$. But $r < n < c$, contradicting the minimality of c unless $r = 0$, which is impossible. Thus, $2c$ is the smallest power of W giving the identity, by Theorem 2, $k(m) = 2ct$. □

For example, consider $\alpha = [\overline{1, 1, 2}]$; the trace is 6, so $\mathcal{L}_0 = 0$, $\mathcal{L}_1 = 1$, and $\mathcal{L}_n = 6\mathcal{L}_{n-1} + \mathcal{L}_{n-2}$. Modulo 13, we get

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{L}_n \pmod{13}$	0	1	6	11	7	1	0	1	6	11	7	1	0

Notice that $\mathcal{L}_{n-1} \equiv 1$, $\mathcal{L}_n \equiv 0$ for $n = 6$, and this is the smallest such n. Also $\gcd(3, 13) = 1$ and the period of α is preserved mod 13, so $k(13) = 6 \cdot 3 = 18$ as we have seen.

We now turn to a matrix formulation that can be used to compute the $\mathcal{L}_n$-sequence. In particular, it will allow us to compute reduction formulas for $\mathcal{L}_{in}$ and $\mathcal{L}_{in-1}$ in terms of $\mathcal{L}_n$ and $\mathcal{L}_{n-1}$.

Theorem 6: Let

$$T = \begin{pmatrix} 0 & 1 \\ (-1)^{t-1} & p_t + q_{t-1} \end{pmatrix} = \begin{pmatrix} (-1)^{t-1}\mathcal{L}_0 & \mathcal{L}_1 \\ (-1)^{t-1}\mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix}.$$

Then

$$T^n = \begin{pmatrix} (-1)^{t-1}\mathcal{L}_{n-1} & \mathcal{L}_n \\ (-1)^{t-1}\mathcal{L}_n & \mathcal{L}_{n+1} \end{pmatrix}.$$

Proof: For $n = 1$, T has the desired form. Now suppose the theorem is true for n. Then

$$T^{n+1} = TT^n = \begin{pmatrix} 0 & 1 \\ (-1)^{t-1} & p_t + q_{t-1} \end{pmatrix} \begin{pmatrix} (-1)^{t-1}\mathcal{L}_{n-1} & \mathcal{L}_n \\ (-1)^{t-1}\mathcal{L}_n & \mathcal{L}_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{t-1}\mathcal{L}_n & \mathcal{L}_{n+1} \\ (-1)^{2(t-1)}\mathcal{L}_{n-1} + (-1)^{t-1}(p_t + q_{t-1})\mathcal{L}_n & (-1)^{t-1}\mathcal{L}_n + (p_t + q_{t-1})\mathcal{L}_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{t-1}\mathcal{L}_n & \mathcal{L}_{n+1} \\ (-1)^{t-1}\mathcal{L}_{n+1} & \mathcal{L}_{n+2} \end{pmatrix}$$

as desired. □

Corollary 7: Successive entries in the $\mathcal{L}_n$-sequence satisfy a quadratic identity:

$$\mathcal{L}_{n-1}^2 = -(-1)^{-(t-1)}(p_t + q_{t-1})\mathcal{L}_{n-1}\mathcal{L}_n + (-1)^{-(t-1)}\mathcal{L}_n^2 + (-1)^{tn-2(t-1)}.$$

Proof: Taking the determinant of T^n in Theorem 6 and using the recursion for $\mathcal{L}_{n+1}$ yields

$$(-1)^{tn}=(-1)^{t-1}\mathcal{L}_{n-1}((p_t+q_{t-1})\mathcal{L}_n+(-1)^{t-1}\mathcal{L}_{n-1})-(-1)^{t-1}\mathcal{L}_n^2.$$

The desired formula results from distributing and solving for $\mathcal{L}_{n-1}^2$. □

One can use Theorem 6 to compute reduction formulas for the $\mathcal{L}_n$-sequence. For example,

$$T^{2n}=\begin{pmatrix}(-1)^{t-1}\mathcal{L}_{2n-1} & \mathcal{L}_{2n}\\ (-1)^{t-1}\mathcal{L}_{2n} & \mathcal{L}_{2n+1}\end{pmatrix}$$

$$=(T^n)^2=\begin{pmatrix}(-1)^{2(t-1)}\mathcal{L}_{n-1}^2+(-1)^{t-1}\mathcal{L}_n^2 & (-1)^{t-1}\mathcal{L}_{n-1}\mathcal{L}_n+\mathcal{L}_n\mathcal{L}_{n+1}\\ (-1)^{2(t-1)}\mathcal{L}_{n-1}\mathcal{L}_n+(-1)^{t-1}\mathcal{L}_n\mathcal{L}_{n+1} & (-1)^{t-1}\mathcal{L}_n^2+\mathcal{L}_{n+1}^2\end{pmatrix}$$

Now considering the (1, 1) entries of those, we see that

$$\mathcal{L}_{2n-1}=(-1)^{t-1}\mathcal{L}_{n-1}^2+\mathcal{L}_n^2=-(p_t+q_{t-1})\mathcal{L}_{n-1}\mathcal{L}_n+2\mathcal{L}_n^2+(-1)^{tn-(t-1)}$$

using Corollary 7 for the second equality. Notice that using Corollary 7 removes the appearances of $\mathcal{L}_{n-1}^2$. Likewise,

$$\mathcal{L}_{2n}=-2(-1)^t\mathcal{L}_{n-1}\mathcal{L}_n+(p_t+q_{t-1})\mathcal{L}_n^2.$$

In general, $\mathcal{L}_{in-1}$ and $\mathcal{L}_{in}$ can be described in terms of $\mathcal{L}_{n-1}$ and $\mathcal{L}_n$ in that way. We formalize the idea of eliminating square powers of $\mathcal{L}_{n-1}$ as follows. We define the matrix:

$$U=\begin{pmatrix}(-1)^{t-1}a & b\\ (-1)^{t-1}b & (-1)^{t-1}a+(p_t+q_{t-1})b\end{pmatrix}.$$

U captures the symmetry of T^n. In fact, if $a=\mathcal{L}_{n-1}$ and $b=\mathcal{L}_n$, then U is T^n. We will call a polynomial in a and b a-simplified when the identity

$$a^2=-(-1)^{-(t-1)}(p_t+q_{t-1})ab+(-1)^{-(t-1)}b^2+(-1)^{tn-2(t-1)}$$

has been used to eliminate all appearances of a^2 and other powers of a higher than 1. Our definition generalizes the definition used in [4] and [5]. The next section gives a canonical form for the a-simplified powers of U. This canonical form allows us to identify moduli, m, which generate very short periods for the convergents of continued fractions modulo m; see Section 4.

3. PARAMETRIZING THE a-SIMPLIFIED REDUCTION FORMULAS

We define polynomials R_{2j} and S_{2j}, generalizing polynomials defined in [5], using intertwined recursions. These will be used to parametrize the reduction formulas for the $\mathcal{L}_n$-sequence. Let

$$\begin{cases}R_0=0,\ R_2=1,\ \text{and}\ R_{2j}=S_{2j-2}+(-1)^{tn}R_{2j-4},\\ S_0=2,\ S_2=1,\ \text{and}\ S_{2j}=b^2dR_{2j-2}+(-1)^{tn}S_{2j-4}.\end{cases}$$

Table 2 below gives the values of R_{2j} for small j when n is even. Notice that the power of b is always twice the power of d and that the degree increases at every other term.

The table also suggests the conjecture that if $i \mid j$ then $R_{2i} \mid R_{2j}$.

TABLE 2. R_{2j} for Small j and Even n

$$R_0 = 0$$
$$R_2 = 1$$
$$R_4 = 1$$
$$R_6 = 3 + b^2 d$$
$$R_8 = 2 + b^2 d$$
$$R_{10} = 5 + 5b^2 d + b^4 d^2$$
$$R_{12} = 3 + 4b^2 d + b^4 d^2 = (1 + b^2 d)(3 + b^2 d)$$
$$R_{14} = 7 + 14b^2 d + 7b^4 d^2 + b^6 d^3$$
$$R_{16} = 4 + 10b^2 d + 6b^4 d^2 + b^6 d^3 = (2 + b^2 d)(2 + 4b^2 d + b^4 d^2)$$
$$R_{18} = 9 + 30b^2 d + 27b^4 d^2 + 9b^6 d^3 + b^8 d^4 = (3 + b^2 d)(3 + 9b^2 d + 6b^4 d^2 + b^6 d^3)$$
$$R_{20} = 5 + 20b^2 d + 21b^4 d^2 + 8b^6 d^3 + b^8 d^4 = (1 + 3b^2 d + b^4 d^2)(5 + 5b^2 d + b^4 d^2)$$
$$R_{22} = 11 + 55b^2 d + 77b^4 d^2 + 44b^6 d^3 + 11b^8 d^4 + b^{10} d^5$$
$$R_{24} = 6 + 35b^2 d + 56b^4 d^2 + 36b^6 d^3 + 10b^8 d^4 + b^{10} d^5 = (1 + b^2 d)(2 + b^2 d)(3 + b^2 d)(1 + 4b^2 d + b^4 d^2)$$

Lemma 8:

(i) The polynomials R_{2j} and S_{2j}, with variable b, only include even degree terms.

(ii) $\deg(R_{4j-2}) = \deg(S_{4j-2}) = 2j-2$, $\deg(R_{4j}) = 2j-2$, $\deg(S_{4j}) = 2j$.

(iii) The polynomials R_{2j} and S_{2j} have positive coefficients when tn is even and is identical when tn is odd except that every other coefficient, beginning with the second highest, is the opposite of the corresponding coefficient of R_{2j} or S_{2j}.

Proof:

(i) This follows because the base cases are constants and the general recursions only involve b as b^2.

(ii) $\deg(R_{4j+2}) = \deg(S_{4j} + (-1)^{tn} R_{4j-2}) = \max(2j, 2j-2) = 2j$. Notice that the highest order term is not $(-1)^{tn}$ so there is no possibility of cancellation. The other polynomials can be checked in a similar manner.

(iii) First, we claim that R_{2j} and S_{2j} are homogeneous in the expressions b^2 and $(-1)^{tn}$. The claim is true when $j = 0$ and $j = 1$. Since $\deg(R_{2j}) = \deg(S_{2j-2})$ and $\deg(S_{2j}) = 2 + \deg(R_{2j-2})$, this homogeneity is preserved by the recursive definitions. Hence, the claim is true. Since the highest terms of R_{2j} and S_{2j} do not involve $(-1)^{tn}$, each term with lower powers of b^2 will have complementary powers of $(-1)^{tn}$. Hence, there is an alternation of sign. □

Next, we give a result which shows that certain combinations of these polynomials are 1.

Lemma 9: For $j \geq 1$,

(i) $R_{2j+2} S_{2j-2} - R_{2j} S_{2j} = (-1)^{(j-1)tn}$,

(ii) $R_{2j-2} S_{2j+2} - R_{2j} S_{2j} = -(-1)^{(j-1)tn}$.

Proof: We prove both parts simultaneously by induction. For $j = 1$,

$$R_4S_0 - R_2S_2 = 2\cdot 1 - 1\cdot 1 = 1 = (-1)^{0tn},$$
$$R_0S_4 - R_2S_2 = 0\cdot S_4 - 1\cdot 1 = -1 = -(-1)^{0tn}.$$

Assuming now that parts (i) and (ii) hold for j, consider $j + 1$ in part (i):

$$\begin{aligned} R_{2j+4}S_{2j} - R_{2j+2}S_{2j+2} &= (S_{2j+2} + (-1)^{tn}R_{2j})S_{2j} - (S_{2j} + (-1)^{tn}R_{2j-2})S_{2j+2} \\ &= (-1)^{tn}(R_{2j}S_{2j} - R_{2j-2}S_{2j+2}) \\ &= (-1)^{tn}(-1)^{(j-1)tn} \text{ using the induction hypothesis from (ii)} \\ &= (-1)^{jtn}. \end{aligned}$$

The induction step for part (ii) is similar. □

Theorem 10: The first row of U^{2j} after a-simplification is given by

$$\begin{aligned} v(j) = ((-1)^{tjn} + bR_{2j}((-1)^t a(p_t + q_{t-1})S_{2j} + b(-1)^{tn}dR_{2j-2} \\ + 2b(-1)^{t-1}S_{2j}),\ b(b(p_t + q_{t-1}) + 2(-1)^{t-1}a)R_{2j}S_{2j}). \end{aligned}$$

Proof: By induction on j. The first row of U^2 after a-simplification is

$$((-1)^{nt} - 2(-1)^t b^2 + (-1)^t ab(p_t + q_{t-1}),\ -2(-1)^t ab + b^2(p_t + q_{t-1})),$$

which is the same as $v(1)$. Next, we need to show that $v(j+1)$ equals the a-simplified form of $v(j)U^2$. We begin with the second components. The a-simplified form of $v(j)$ times the second column of U^2 is

$$\begin{aligned} &v(j)(2(-1)^{2t}ab - (-1)^t b^2(p_t + q_{t-1}),\ (-1)^t a(p_t + q_{t-1}) + b^2(p_t + q_{t-1})^2) \\ &= b(2(-1)^{t-1}a + b(p_t + q_{t-1}))\cdot((-1)^{jnt} + (-1)^{nt}b^2dR_{2j}R_{2j-2} + (-1)^{nt}R_{2j}S_{2j} + b^2dR_{2j}S_{2j}), \end{aligned}$$

where we have used the definition of d to simplify. Using Lemma 9(i), we can replace $(-1)^{jnt}$ by R and S polynomials. Thus, the third factor of the above is

$$\begin{aligned} &(-1)^{nt}R_{2j+2}S_{2j-2} - (-1)^{nt}R_{2j}S_{2j} + (-1)^{nt}b^2dR_{2j}R_{2j-2} + (-1)^{nt}R_{2j}S_{2j} + b^2dR_{2j}S_{2j} \\ &\quad = b^2dR_{2j}(S_{2j} + (-1)^{nt}R_{2j-2}) + (-1)^{nt}R_{2j+2}S_{2j-2} \\ &\quad = b^2dR_{2j}R_{2j+2} + (-1)^{nt}R_{2j+2}S_{2j-2} = R_{2j+2}S_{2j+2}. \end{aligned}$$

Thus, the second component of $v(j)$ times the second column of U^2 is

$$b(2(-1)^{t-1}a + b(p_t + q_{t-1}))R_{2j+2}S_{2j+2}.$$

On the other hand, the second component of $v(j+1)$ is the same thing, which checks the induction step for the second component.

The first component can be checked in a similar, but more tedious, manner. □

Consider when $j = 3$ and $\alpha = [\overline{1, 1, 2}]$, for example. Then, by Theorem 10, the first row of U^6 after a-simplification is

$$((-1)^{9n}+bR_6((-1)^3a(p_3+q_2)S_6+b((-1)^{3n}dR_4+2(-1)^2S_6)),\ b(b(p_3+q_2)+2(-1)^2a)R_6S_6),$$

where $R_6=3+b^2d$, $R_4=1$, and $S_6=1+b^2d$. Now, letting $a=\mathscr{L}_{n-1}$, $b=\mathscr{L}_n$, $d=40$, and $p_3+q_2=6$, we get reduction formulas for $\mathscr{L}_{6n-1}$ and $\mathscr{L}_{6n}$ in terms of $\mathscr{L}_{n-1}$ and $\mathscr{L}_n$:

$$\mathscr{L}_{6n-1}=(-1)^{9n}+\mathscr{L}_n(3+40\mathscr{L}_n^2)((-1)6\mathscr{L}_{n-1}(1+40\mathscr{L}_n^2)+\mathscr{L}_n(40(-1)^{3n}+2(1+40\mathscr{L}_n^2)))$$

and

$$\mathscr{L}_{6n}=\mathscr{L}_n(6\mathscr{L}_n+2\mathscr{L}_{n-1})(3+40\mathscr{L}_n^2)(1+40\mathscr{L}_n^2).$$

In particular, let $n=4$, then $\mathscr{L}_3=37$, $\mathscr{L}_4=288$, so

$$\begin{aligned}\mathscr{L}_{23}&=1+228(3+40(228)2)((-6)(37)(1+40(228)^2)+228(40+2(1+40(228)^2)))\\&=230684837784645817\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_{24}&=228(6(228)+2(37))(3+40(228)^2)(1+40(228)^2)\\&=1421544022419889368,\end{aligned}$$

which are straightforward and unpleasant to check.

Corollary 11: Let $j\ge 1$. The first row of U^{2j+1} after a-simplification is given by

$$\begin{aligned}&((-1)^t(-(-1)^{jnt}a-bR_{2j}(abdR_{2j+2}+S_{2j}(-1)^{(n-1)t}(p_t+q_{t-1}))),\\&\qquad b((-1)^{jnt}+R_{2j}(b^2dR_{2j+2}+2S_{2j}(-1)^{nt}))).\end{aligned}$$

Proof: Multiplying out $v(j)U$ and a-simplifying yields

$$\begin{aligned}&((-1)^t(-(-1)^{jnt}a+bR_{2j}(-(-1)^{tn}abdR_{2j-2}+S_{2j}(4(-1)^tab-(-1)^{(n-1)t}(p_t+q_{t-1})-ab(p_t+q_{t-1})^2))),\\&\qquad b((-1)^{jnt}+R_{2j}((-1)^{nt}b^2dR_{2j-2}+S_{2j}(2(-1)^{nt}-4(-1)^tb^2+b^2(p_t+q_{t-1})^2)))).\end{aligned}$$

The recursive definition for R_{2j+2} and the definition for d simplifies this into the desired result. □

Consider when $j=4$ and $\alpha=[\overline{1,1,2}]$, for example. Then, by Corollary 11, the first row of U^9 after a-simplification is

$$((-1)^3(-(-1)^{12n}a-bR_8(abdR_{10}+S_8(-1)^{3(n-1)}(p_3+q_2))),\ b((-1)^{12n}+R_8(b^2dR_{10}+2S_8(-1)^{3n}))),$$

where $R_8=2+b^2d$, $S_8=2+4b^2d+b^4d^2$ and $R_{10}=5+5b^2d+b^4d^2$, as seen in Table 2 and from the recursive definition of S_{2j}. Letting $a=\mathscr{L}_n$, $b=\mathscr{L}_n$, $d=40$, and $p_3+q_2=6$, we get reduction formulas for $\mathscr{L}_{9n-1}$ and $\mathscr{L}_{9n}$ in terms of $\mathscr{L}_{n-1}$ and $\mathscr{L}_n$:

$$\begin{aligned}\mathscr{L}_{9n-1}&=-(-\mathscr{L}_{n-1}-\mathscr{L}_n(2+40\mathscr{L}_n^2)(40\mathscr{L}_{n-1}\mathscr{L}_n(5+200\mathscr{L}_n^2+40^2\mathscr{L}_n^4)\\&\quad+6(-1)^{3(n-1)}(2+160\mathscr{L}_n^2+40^2\mathscr{L}_n^2)))\end{aligned}$$

and

$$\mathscr{L}_{9n}=\mathscr{L}_n(1+(2+40\mathscr{L}_n^2)((40\mathscr{L}_n^2)(5+200\mathscr{L}_n^2+40^2\mathscr{L}_n^2)+2(-1)^{nt}(2+160\mathscr{L}_n^2+40^2\mathscr{L}_n^4))).$$

Let $n = 4$, then $\mathscr{L}_3 = 37$, $\mathscr{L}_4 = 228$, so

$$\begin{aligned}\mathscr{L}_{35} &= -(-37 - 228(2+40(228)^2)(40(37)(228)(5+200(228)^2+(40)^2(228)^4)\\ &\qquad - 6(2+160(228)^2+(40)^2(228)^4)))\\ &= 69169431328219666912786016\,5\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_{36} &= 228(1+(2+40(228)^2)(40(228)^2(5+200(228)^2+(40)^2(228)^4)\\ &\qquad + 2(2+160(228)^2+(40)^2(228)^4)))\\ &= 426241241440438883631091405 2,\end{aligned}$$

which are correct.

4. SHORT PERIODS

The following theorem is our main result. It gives families of moduli with short periods. These families are given by divisors of the polynomials $bR_{2j}(b)$ evaluated at numbers in the $\mathscr{L}_n$-sequence.

Theorem 12: Let m divide $\mathscr{L}_n R_{2j}(\mathscr{L}_n)$.

(i) If t is even, then $k(m)|4jnt$.

(ii) If t is odd but jnt is even, then $k(m)|2jnt$.

(iii) If jnt is odd, then $k(m)|4jnt$.

Proof: First, consider (i) and (ii), where we have jnt is even. Let $a = \mathscr{L}_{n-1}$ and $b = \mathscr{L}_n$ in Theorem 10 and note that all the terms of $v(j)$ are divisible by m except the $(-1)^{jnt}$. With this substitution, $v(j)$ gives the a-simplification of the first row of T^{2jn}. Also using Theorem 6, we see that $v(j) \equiv (1, 0) \equiv ((-1)^{t-1}\mathscr{L}_{2jn-1}, \mathscr{L}_{2jn})$. Thus, $\mathscr{L}_{2jn-1} \equiv (-1)^{t-1} \bmod m$ and $\mathscr{L}_{2jn} \equiv 0 \bmod m$. Hence, by Theorem 5, $k(m)|4jnt$ if t is even and $k(m)|2jnt$ if t is odd.

Now, in part (iii), jnt is odd. The same idea as above works except that $v(j) \equiv (-1, 0) \equiv ((-1)^{t-1}\mathscr{L}_{2jn-1}, \mathscr{L}_{2jn})$. Thus, $\mathscr{L}_{2jn-1} \equiv -1 \bmod m$ and $\mathscr{L}_{2jn} \equiv 0 \bmod m$. Now the identities for $\mathscr{L}_{2n-1}$ and $\mathscr{L}_{2n}$ given after Corollary 7 allow us to see that $\mathscr{L}_{4jn-1} \equiv 1$ (remember t is odd), and $\mathscr{L}_{4jn} \equiv 0$. Now Theorem 5 gives $k(m)|4jnt$ as desired. □

Notice that the bounds for k in all cases are linear in n, while $\mathscr{L}_n$, and hence the modulus, is exponential in n. Thus, we can construct families of moduli having periods logarithmic in the modulus. In Table 3, the example of $\alpha = [\overline{1, 1, 2}]$ with $m = R_6(\mathscr{L}_n)$ is considered. Notice the large m.

Now we can explain, up to a factor of 2, all of the periods less than the linear upper bounds given for the primes in Table 1. Table 4 gives a list of $R_{2j}(\mathscr{L}_n)$ that are divisible by those primes. The upper bounds for the periods given in Theorem 12 fully explain the periods that actually occur in this table, except those marked with an asterisk where the upper bound is twice the actual period.

TABLE 3

Logarithmic Bounds on Periods for a Family of Moduli for $\alpha = [\overline{1, 1, 2}]$

n	$\mathcal{L}_n$	$m = R_6(\mathcal{L}_n)$	$2jnt$
2	6	1443	36
4	228	2079363	72
6	8658	2998438563	108
8	328776	4323746327043	144
10	12484830	6234839205156003	180
12	474094764	8990633810088627843	216
14	18003116202	12964487719308596192163	252

n	$\mathcal{L}_n$	$m = R_6(\mathcal{L}_n)$	$4jnt$
1	1	37	36
3	37	54757	108
5	1405	78960997	180
7	53353	113861704357	252
9	2026009	164188498723237	324
11	76934989	236759701297204837	396
13	2921503573	341407325082070653157	468
15	110940200785	492309126008644584648997	540

TABLE 4

Values of $R_{2j}(\mathcal{L}_n)$ Explaining Short Periods

*13	$R_6(\mathcal{L}_2)$	281	$R_{10}(\mathcal{L}_2)$	*677	$R_{26}(\mathcal{L}_{26})$
19	$R_8(\mathcal{L}_1)$	*317	$R_{158}(\mathcal{L}_2)$	733	$R_{122}(\mathcal{L}_2)$
37	$R_6(\mathcal{L}_1)$	*397	$R_{18}(\mathcal{L}_{22})$	761	$R_{10}(\mathcal{L}_{38})$
*53	$R_{26}(\mathcal{L}_2)$	419	$R_6(\mathcal{L}_{20})$	*773	$R_{386}(\mathcal{L}_2)$
59	$R_8(\mathcal{L}_5)$	*431	$R_{86}(\mathcal{L}_2)$	*797	$R_{398}(\mathcal{L}_2)$
*67	$R_{22}(\mathcal{L}_2)$	*439	$R_{146}(\mathcal{L}_2)$	809	$R_{202}(\mathcal{L}_2)$
103	$R_8(\mathcal{L}_2)$	449	$R_8(\mathcal{L}_7)$	821	$R_{274}(\mathcal{L}_2)$
131	$R_6(\mathcal{L}_4)$	461	$R_{14}(\mathcal{L}_{11})$	*827	$R_{118}(\mathcal{L}_2)$
137	$R_{46}(\mathcal{L}_2)$	491	$R_{82}(\mathcal{L}_4)$	829	$R_{166}(\mathcal{L}_2)$
167	$R_8(\mathcal{L}_{14})$	503	$R_6(\mathcal{L}_{24})$	*853	$R_{142}(\mathcal{L}_6)$
179	$R_6(\mathcal{L}_{20})$	*521	$R_{26}(\mathcal{L}_{10})$	857	$R_{26}(\mathcal{L}_2)$
*191	$R_{38}(\mathcal{L}_2)$	571	$R_8(\mathcal{L}_{13})$	859	$R_8(\mathcal{L}_5)$
*197	$R_{14}(\mathcal{L}_{14})$	601	$R_{10}(\mathcal{L}_{15})$	881	$R_8(\mathcal{L}_{22})$
233	$R_6(\mathcal{L}_{26})$	*613	$R_{14}(\mathcal{L}_{18})$	*883	$R_{14}(\mathcal{L}_{14})$
*241	$R_6(\mathcal{L}_{10})$	*631	$R_{28}(\mathcal{L}_{15})$	*911	$R_{26}(\mathcal{L}_{14})$
*263	$R_{16}(\mathcal{L}_{22})$	647	$R_6(\mathcal{L}_{24})$	929	$R_{16}(\mathcal{L}_{29})$
*271	$R_6(\mathcal{L}_{10})$	*653	$R_{326}(\mathcal{L}_2)$	937	$R_{134}(\mathcal{L}_2)$
277	$R_{46}(\mathcal{L}_2)$	659	$R_{22}(\mathcal{L}_{20})$	*997	$R_{166}(\mathcal{L}_6)$

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THE ORDER OF THE FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

In this paper $v_p(r)$ denotes the exponent of the highest power of a prime p which divides r and is referred to as the p-adic order of r. We characterize the p-adic orders $v_p(F_n)$ and $v_p(L_n)$, i.e., the exponents of a prime p in the prime power decomposition of F_n and L_n, respectively.

The characterization of the divisibility properties of combinatorial quantities has always been a popular area of research. In particular, finding the highest powers of primes which divide these numbers (e.g., factorials, binomial coefficients [14], Stirling numbers [2], [1], [10], [9]) has attracted considerable attention. The analysis of the periodicity *modulo* any integer (e.g., [3], [11], [14], [8]) of these numbers helps exploring their divisibility properties (e.g., [9]). The periodic property of the Fibonacci and Lucas numbers has been extensively studied (e.g., [16], [13], [17], [12]). Here we use some of these properties and methods to find $v_p(F_n)$ and $v_p(L_n)$. An application of the results to the Stirling numbers of the second kind is discussed at the end of the paper.

We note that Halton [5] obtained similar results on the p-adic order of the Fibonacci numbers, and additional references on earlier developments can be found in Robinson [13] and Vinson [15]. The approach presented here is based on a refined analysis of the periodic structure of the Fibonacci numbers by exploring its properties, in particular, around the points where $F_n \equiv 0 \pmod p$. [The smallest n such that $F_n \equiv 0 \pmod p$ is called the rank of apparition of prime p and is denoted by $n(p)$.] This technique is based on that of Wilcox [17] and provides a simple and self-contained analysis of properties related to divisibility. For instance, we obtain another characterization of the ratio of the period to the rank of apparition [15] in terms of $F_{n(p)-1} \pmod p$ for any prime p.

Knuth and Wilf [7] generalized Kummer's result on the highest power of a prime that divides the binomial coefficient. Kummer proved that the p-adic order of a binomial coefficient $\binom{n}{m}$ is the number of "carries" that occur when the integers m and $n-m$ are added in p-ary notation. Knuth and Wilf extended the use of counting "carries" to a broad class of generalized binomial coefficients which includes the Fibonacci numbers (Theorem 2 in [7]). Their method is derived for ***regularly divisible sequences*** [7]; however, it can be modified to include the Lucas numbers, too. We note that $L_{2n} = L_n^2 - 2(-1)^n$; therefore, (L_{2n}, L_n) is either 1 or 2, which illustrates that the Lucas numbers are not regularly divisible.

If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime-decomposition of m, then $v_m(N) = \min_{1\le i\le k}\lfloor v_{p_i}(N)/\alpha_i\rfloor$. Therefore, without loss of generality, we will focus on the characterization of $v_p(F_n)$ and $v_p(L_n)$ where p is a prime.

2. THE 2- AND 5-ADIC ORDERS

It turns out that the 5-adic order of the Fibonacci and Lucas numbers can be computed easily. For the Fibonacci numbers, we use the well-known identity [16]

$$2^{n-1}F_n = \sum_{k=0} \binom{n}{2k+1} 5^k, \; n \geq 1, \tag{1}$$

and obtain

Lemma 1: For all $n \geq 0$, we have $\nu_5(F_n) = \nu_5(n)$. On the other hand, L_n is not divisible by 5 for any n.

Proof: Observe that

$$\nu_5\left(\binom{n}{2k+1}5^k\right) = \nu_5(n) - \nu_5(2k+1) + \nu_5\left(\binom{n-1}{2k}5^k\right) \geq \nu_5(n) - \nu_5(2k+1) + k > \nu_5(n),$$

except for $k = 0$ when

$$\nu_5\left(\binom{n}{2k+1}5^k\right) = \nu_5(n).$$

Identity (1) implies $\nu_5(F_n) = \nu_5(n)$.

For the Lucas numbers, the period of the sequence $\{L_n \bmod 5\}$ is 4 with the cycle $\{1, 3, 4, 2\}$; therefore, 5 can never be a divisor of L_n. □

To derive the 2-adic orders of F_n and L_n, we use congruences proved by Jacobson [6].

Lemma A (Lemma 2 in [6]): Let $k \geq 5$ and $s \geq 1$. Then $F_{2^{k-3}3s} \equiv s2^{k-1} \pmod{2^k}$.

Lemma B (Lemma 4 in [6]): Let $k \geq 5$ and $n \geq 0$ and assume $n \equiv 0 \pmod 6$. Then $F_{n+2^{k-3}3} \equiv F_n + 2^{k-1} \pmod{2^k}$.

Lemma C (Lemma 5 in [6]): Let $n \geq 0$ and assume $n \equiv 3 \pmod 6$. Then $F_n \equiv 2 \pmod{32}$.

We assume that $n \geq 1$ from now on. If $n \equiv 1$ or $2 \pmod 3$, then we know that $F_n \equiv 1 \pmod 2$; thus, $\nu_2(F_n) = 0$ for $n \equiv 1, 2 \pmod 3$. Lemma A yields $\nu_2(F_{12n}) = \nu_2(n) + 4$. By Lemma C, we get $\nu_2(F_n) = 1$ if $n \equiv 3 \pmod 6$, and Lemma B [in the more convenient form $F_n \equiv F_{n+12} + 16 \pmod{32}$] implies that $F_6 = 8 \equiv F_{18} + 16 \equiv F_{30} \equiv F_{42} + 16 \equiv \cdots \pmod{32}$, and in general, $F_{12n+6} \equiv -8$ or $8 \pmod{32}$; therefore, $\nu_2(F_{12n+6}) = 3$.

Similarly, $L_n \equiv 1 \pmod 2$ for $n \not\equiv 0 \pmod 3$. By the duplication formula, $F_{2n} = F_n L_n$, it follows that $\nu_2(L_n) = \nu_2(F_{2n}) - \nu_2(F_n)$. Therefore, $\nu_2(L_{6n+3}) = 2$ and $\nu_2(L_{6n}) = 1$, for it turns out that $\nu_2(L_{12n}) = \nu_2(L_{12n+6}) = 1$.

In summary,

Lemma 2:

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \mod 3, \\ 1, & \text{if } n \equiv 3 \mod 6, \\ 3, & \text{if } n \equiv 6 \mod 12, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \mod 12, \end{cases}$$

and

$$\nu_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \mod 3, \\ 2, & \text{if } n \equiv 3 \mod 6, \\ 1, & \text{if } n \equiv 0 \mod 6. \end{cases}$$

3. p-ADIC ORDERS

In this section we assume that p is a prime different from 2 and 5. It is well known that either F_{p-1} or F_{p+1} is divisible by p for every prime p.

Let $n = n(m)$ be the first positive index for which $F_n \equiv 0 \pmod{m}$. This index is often called the *rank of apparition (appearance)* or *Fibonacci entry-point* of m. The order of p in $F_{n(p)}$ will be denoted by $e = e(p)$, i.e., $e = e(p) = v_p(F_{n(p)}) \geq$, $F_{n(p)} \equiv 0 \pmod{p^e}$ and $F_{n(p)} \not\equiv 0 \pmod{p^{e+1}}$. In this paper $k(m)$ denotes the period *modulo m* of the Fibonacci series.

We shall need

Theorem A (Theorem 3 in [16]): The terms for which $F_n \equiv 0 \pmod{m}$ have subscripts that form a simple arithmetic progression. That is, $n = x \cdot d$ for $x = 0, 1, 2, \ldots,$ and some positive integer $d = d(m)$, gives all n with $F_n \equiv 0 \pmod{m}$.

Note that $d(m)$ is exactly $n(m)$, and $d(p^i) = d(p) = n(p)$ for all $1 \leq i \leq e(p)$. It also follows that $F_m \not\equiv 0 \pmod{p}$ unless m is a multiple of $n(p)$. Clearly, $(p, n(p)) = 1$. From now on we will focus on indices of the form $cn(p)p^\alpha$ where $c \geq 1$ and $\alpha \geq 0$ integers, and $(c, p) = 1$.

We prove

Theorem: For $p \neq 2$ and 5,

$$v_p(F_n) = \begin{cases} v_p(n) + e(p) & \text{if } n \equiv 0 \pmod{n(p)}, \\ 0, & \text{if } n \not\equiv 0 \pmod{n(p)}, \end{cases} \tag{2}$$

and

$$v_p(L_n) = \begin{cases} v_p(n) + e(p), & \text{if } k(p) \neq 4n(p) \text{ and } n \equiv \frac{n(p)}{2} \pmod{n(p)}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Proof: The basic idea of the proof is based on the identity [16]

$$F_{an} = 2^{1-a} F_n (KF_n^2 + aL_n^{a-1}), \tag{4}$$

where K is an integer. We set $a = p, \alpha \geq 1$, and $n = cn(p)p^{\alpha-1}$ such that $(c, p) = 1$. Identity (4) and Theorem A imply that

$$F_{cn(p)p^\alpha} = 2^{1-p} F_{cn(p)p^{\alpha-1}} (K'p^2 + pL^{p-1}_{cn(p)p^{\alpha-1}}),$$

with some integer K'; therefore,

$$v_p(F_{cn(p)p^\alpha}) = v_p(F_{cn(p)p^{\alpha-1}}) + 1,$$

for (F_n, L_n) is either 1 or 2, and inductively,

$$v_p(F_{cn(p)p^\alpha}) = v_p(F_{cn(p)}) + \alpha. \tag{5}$$

We now prove $v_p(F_{cn(p)}) = v_p(F_{n(p)})$. The multiplication identity [4]

$$F_{kn} \equiv kF_n F_{n+1}^{k-1} \pmod{F_n^2} \tag{6}$$

yields $F_{cn(p)} \equiv cF_{n(p)}F_{n(p)+1}^{c-1} \pmod{p^{2e}}$ by setting $n = n(p), k = c,$ and $e = e(p)$. We show that $(F_{n(p)+1}, p) = 1$ by deriving the congruences

$$F_{n(p)+1}^2 \equiv F_{n(p)-1}^2 \equiv \begin{cases} -1 \bmod p, & \text{if } k(p) = 4n(p), \\ +1 \bmod p, & \text{otherwise,} \end{cases} \tag{7}$$

which prove that $v_p(F_{cn(p)}) = v_p(F_{n(p)})$, for $(c, p) = 1$, and $v_p(F_{n(p)}) = e < 2e$. Identity (5) implies $v_p(F_{cn(p)p^\alpha}) = v_p(F_{n(p)}) + \alpha = e(p) + \alpha$ and identity (2).

In order to prove identity (7), we set

$$F_{n(p)-1} \equiv x \pmod{p}, \tag{8}$$

and observe that the Fibonacci series around the term $F_{n(p)} \equiv 0 \pmod{p}$ must have the form $\cdots, -8x, 5x, -3x, 2x, -x, x, 0, x, x, 2x, 3x, 5x, 8x, \cdots$. This sequence can be continued backward until we reach the term $F_1 = 1$, i.e., $(-1)^{n(p)}F_{n(p)-1}x \equiv 1 \pmod{p}$. The forward continuation yields $F_{2n(p)-1} \equiv F_{n(p)-1}x \pmod{p}$. If $n(p)$ is even, then

$$F_{n(p)-1}x \equiv 1 \pmod{p} \tag{9}$$

and, by identity (8), $x^2 \equiv 1 \pmod{p}$ follows, i.e., $F_{n(p)-1} \equiv x \equiv \pm 1 \pmod{p}$. On the other hand, $F_{n(p)-1}x \equiv 1 \pmod{p}$ implies that if $x \equiv 1 \pmod{p}$ then $k(p) = n(p)$, and $n(p)/2$ is odd (see [17], Theorem 1, case (iv)). It follows that $k(p)$ is not a multiple of 4, thus $p \equiv \pm 1 \pmod{10}$ (see [16], Corollary, p. 529). On the other hand, if $x \equiv -1 \pmod{p}$ then $F_{n(p)-1} \equiv -1$, $F_{2n(p)-1} \equiv F_{n(p)-1}x \equiv 1 \pmod{p}$, therefore $k(p) = 2n(p)$.

If $n(p)$ is odd, then $F_{n(p)-1}x \equiv -1 \pmod{p}$, and similarly to identity (8) we set $F_{2n(p)-1} \equiv y \pmod{p}$ and repeat the previous argument by substituting the even $2n(p)$ for $n(p)$ and y for x. Here we have $F_{2n(p)-1}y \equiv 1$ and $y^2 \equiv 1 \pmod{p}$ with $y \equiv F_{2n(p)-1} \equiv F_{n(p)-1}x \equiv -1 \pmod{p}$. By identity (8), we obtain that $x^2 \equiv -1 \pmod{p}$. We know from [16] that $k(p)$ must be even and a multiple of $n(p)$, therefore $k(p) = 4n(p)$ must hold. This case occurs, for example, if p is 13, 17, or 61.

To prove identity (3), we apply the duplication formula $L_n = \frac{F_{2n}}{F_n}$, from which we can easily deduce $v_p(L_n)$. We have three cases: either $n \not\equiv 0 \pmod{n(p)}$ and $2n \not\equiv 0 \pmod{n(p)}$, or $n \not\equiv 0 \pmod{n(p)}$ but $2n \equiv 0 \pmod{n(p)}$, or $n \equiv 0 \pmod{n(p)}$.

In the first case, $v_p(F_{2n}) = v_p(F_n) = 0$ implies that $v_p(L_n) = 0$. Similarly, the third case yields $v_p(F_{2n}) = v_p(F_n) = v_p(n) + e(p)$ and $v_p(L_n) = 0$. The second case can never happen if $n(p)$ is odd, that is, $k(p) = 4n(p)$. Otherwise, $n = d \cdot \frac{n(p)}{2}$ must hold with some odd integer d; therefore, $v_p(F_{2n}) = v_p(F_{dn(p)}) = v_p(d) + e(p)$ while $v_p(F_n) = 0$ for n is not a multiple of $n(p)$. The p-adic order of L_n is now $v_p(n) + e(p)$. □

In passing, we note that we fully characterized $\frac{k(p)}{n(p)}$ in terms of $x \equiv F_{n(p)-1} \pmod{p}$ and we found

Lemma 3:

$$\begin{aligned} k(p) &= n(p), && \text{iff } x \equiv 1 && \bmod p, \\ k(p) &= 2n(p), && \text{iff } x \equiv -1 && \bmod p, \\ k(p) &= 4n(p), && \text{iff } x^2 \equiv -1 && \bmod p. \end{aligned}$$

In the first case, p must have the form $10\ell \pm 1$ while the third case requires that $p = 4\ell + 1$.

We note that identities (6) and (7) actually imply

Lemma 4: For every even c and p such that $(c, p) = 1$,

$$F_{cn(p)} \equiv \begin{cases} (-1)^{\frac{c-2}{2}} cF_{n(p)}F_{n(p)+1} & (\bmod p^2), \text{ if } k(p) = 4n(p), \\ cF_{n(p)}F_{n(p)+1} & (\bmod p^2), \text{ otherwise.} \end{cases}$$

For every odd c and p such that $(c, p) = 1$,

$$F_{cn(p)} \equiv \begin{cases} (-1)^{\frac{c-1}{2}} cF_{n(p)} & (\bmod p^2), \text{ if } k(p) = 4n(p), \\ cF_{n(p)} & (\bmod p^2), \text{ otherwise.} \end{cases}$$

The theorem yields $v_p(F_{cn(p)p^\alpha}) = \alpha + 1$ if $e(p) = v_p(F_{n(p)}) = 1$. We note that a prime p is called a *primitive prime factor* of F_n if $p|F_n$, but p does not divide any preceding number in the sequence. According to our notation, p is a primitive prime factor of $F_{n(p)}$. We can consider the *primitive part* F_n' of F_n for which $F_n = F_n' \cdot F_n''$ such that $(F_n', F_n'') = 1$, and p divides F_n' if and only if p is a primitive prime factor of F_n. If we let $m = n(p)$, then F_m' is square-free exactly if $e(p') = 1$ for every primitive prime factor p' of F_m, e.g., for $p' = p$. [Clearly, $m = n(p')$ for all these prime factors.] It appears, however, that saying anything about F_n' being square-free is a difficult problem ([12], p. 49). The interested reader will find a lively discussion on the primitive prime factors of the generalized Lucas sequences in [12].

4. AN APPLICATION

It turns out that the 5-adic analysis of the series F_n and L_n plays a major role in determining $v_5(k!S(n, k))$ where $S(n, k)$ denotes the Stirling numbers of the second kind and $n = a \cdot 5^q$, $k = 2b \cdot 5^z$, a, b, and q are positive integers such that $(a, 5) = (b, 5) = 1$, and $4|a$, while z is a nonnegative integer. For instance, if q is sufficiently large and $z > 0$, then we can derive the identities

$$k!S(n, k) \equiv -2 \cdot 5^{\frac{b}{2}5^z - 1} L_{b \cdot 5^z} \ (\bmod 5^{q+1}), \text{ if } b \text{ is even,}$$

and

$$k!S(n, k) \equiv 2 \cdot 5^{\frac{b \cdot 5^z - 1}{2}} F_{b \cdot 5^z} \ (\bmod 5^{q+1}), \text{ if } b \text{ is odd.}$$

In general, for even k, we obtain

$$v_5(k!S(n, k)) = \begin{cases} \frac{k}{4} - 1. & \text{if } k \equiv 0, 4, 8, 12, 16 \ (\bmod 20), \\ \frac{k-2}{4}, & \text{if } k \equiv 2, 6, 14 \ (\bmod 20), \\ \frac{k-2}{4} + v_5(k), & \text{if } k \equiv 10 \ (\bmod 20), \\ \frac{k-2}{4} + v_5(k+2), & \text{if } k \equiv 18 \ (\bmod 20). \end{cases}$$

Notice that for $n = a \cdot 5^q$, $4|a$, $(a, 5) = 1$, and q sufficiently large, $\nu_5(k!S(n, k))$ can depend on n only if k is odd. Actually, it does depend on n if and only if $k/5$ is an odd integer. The proof will appear in a forthcoming paper. We note that the above identities are generalizations of the identity $\nu_2(k!S(n, k)) = k - 1$, where $n = a \cdot 2^q$, a is odd, and q is sufficiently large (see [9]).

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CONCERNING THE RECURSIVE SEQUENCE $A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^{\alpha_i}$

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1. MAIN RESULT

In [1] H. T. Freitag has raised a conjecture that for the sequence $\{A_n\}$, defined by $A_{n+2} = \sqrt{A_{n+1}} + \sqrt{A_n}$ for all $n \geq 1$, $\lim_{n\to\infty} A_n = 4$ regardless of the choice of $A_1, A_2 > 0$. In this note we will give a positive answer to this conjecture by proving the following more general theorem.

Theorem 1: If $-1 < \alpha_i < 1$, $1 \leq i \leq k$ and $A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^{\alpha_i}$, $n \geq 1$, then

$$\lim_{n\to\infty} A_n = L,$$

the unique root of the equation $\sum_{i=1}^{k} a_i x^{\alpha_i - 1} - 1 = 0$ in the interval $(0, \infty)$, regardless of the choice of $A_i > 0$, $1 \leq i \leq k$, where $a_i \geq 0$, $1 \leq i \leq k$, and $\sum_{i=1}^{k} a_i > 0$.

In particular, if $k = 2$, $a_i = a_2 = 1$, and $\alpha_i = \alpha_2 = 1/2$, we have

$$\lim_{n\to\infty} A_n = 4.$$

This coincides with Freitag's conjecture.

Proof: Let $A_n = Lx_n$. Then

$$x_{n+k} = \sum_{i=1}^{k} \beta_i x_{n+i-1}^{\alpha_i},$$

where $\beta_i = a_i L^{\alpha_i - 1}$, and therefore

$$\sum_{i=1}^{k} \beta_i = 1. \tag{1}$$

Obviously, we only need to prove that

$$\lim_{n\to\infty} x_n = 1. \tag{2}$$

To this end, set $M = \max\{x_i, x_i^{-1};\ 1 \leq i \leq k\}$ and $\alpha = \max\{|\alpha_1|, \ldots, |\alpha_k|\}$. It is obvious that $M \geq 1$, $0 \leq \alpha < 1$, and

$$M \geq x_i \geq M^{-1},\ 1 \leq i \leq k. \tag{3}$$

We will use induction to prove that

$$M^{\alpha^n} \geq x_{kn+i} \geq M^{-\alpha^n},\ 1 \leq i \leq k, \tag{4}$$

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holds for all $n \ge 0$. In fact, from (3), (4) holds when $n = 0$. We assume that (4) holds if $n \le \ell - 1$. For $n = \ell$, from the induction assumption and the definition of M, it follows that

$$M^{\alpha^\ell} \ge M^{|\alpha_i|\alpha^{\ell-1}}, \quad 1 \le i \le k, \tag{5}$$

and

$$M^{-|\alpha_i|\alpha^{\ell-1}} \le x_{(\ell-1)k+i}^{\alpha_i} \le M^{|\alpha_i|\alpha^{\ell-1}}, \quad 1 \le i \le k. \tag{6}$$

Therefore, from (5) and (6), we have

$$x_{k\ell+1} = \sum_{i=1}^{k} \beta_i x_{(\ell-1)k+i}^{\alpha_i} \le \sum_{i=1}^{k} \beta_i M^{|\alpha_i|\alpha^{\ell-1}} \le M^{\alpha^\ell},$$

and, furthermore, we have

$$x_{k\ell+2} = \sum_{i=1}^{k} \beta_i x_{(\ell-1)k+i+1}^{\alpha_i} \le \sum_{i=1}^{k-1} \beta_i M^{|\alpha_i|\alpha^{\ell-1}} + \beta_k M^{|\alpha_k|\alpha^\ell} \le M^{\alpha^\ell}.$$

In the last step we have used the fact that $M^{|\alpha_k|\alpha^\ell} \le M^{\alpha^\ell}$. Similarly, the left-hand inequality of (4) holds for $n = \ell$ and other indices i, $3 \le i \le k$. The right-hand inequality of (4) can be justified in a similar way. Noting that $0 \le \alpha < 1$, we obtain

$$\lim_{n\to\infty} M^{-\alpha^n} = \lim_{n\to\infty} M^{\alpha^n} = 1.$$

By (4), this implies that (2) holds. □

Corollary 1: If $-1 < \alpha_1 = \cdots \alpha_k = \alpha < 1$ and $a_1 = \cdots a_k = 1$, then

$$\lim_{n\to\infty} A_n = k^{\frac{1}{1-\alpha}},$$

independent of the choice of $A_1, A_2, \ldots, A_k > 0$, where $\{A_n\}_1^\infty$ is as defined in Theorem 1.

Corollary 2: If $-1 < \alpha_i < 1$, $a_i \ge 0$, and $\sum_{i=1}^{k} a_i = 1$, then

$$\lim_{n\to\infty} A_n = 1,$$

independent of the choice of $A_1, A_2, \ldots, A_k > 0$, where $\{A_n\}_1^\infty$ is also as defined in Theorem 1. Corollary 2 follows from the fact that $L = 1$ is the only root of the equation $\sum_{i=1}^{k} a_i x^{\alpha_i - 1} - 1 = 0$ in the interval $(0, \infty)$.

2. FURTHER RESULTS

In this section we consider a *linear* recursive sequence, that is, when we choose $\alpha_i = 1$, $1 \le i \le k$, in the recursive sequence considered above.

Theorem 2: Let the complex sequence $\{A_n\}_1^\infty$ satisfy

$$A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}.$$

Then, if $a_i > 0$, $1 \le i \le k$, and $\sum_{i=1}^{k} a_i = 1$, the sequence $\{A_n\}_1^\infty$ converges to a limit which depends on the values of A_i, $1 \le i \le k$.

Proof: We will prove that $x = 1$ is a single root of the eigenpolynomial,

$$p(x) := x^k - \sum_{i=1}^{k} a_i x^{i-1} = 0, \tag{7}$$

of the recursive sequence

$$A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1},$$

and the moduli of all other roots of (7) are less than 1.

In fact, since $\sum_{i=1}^{k} a_i = 1$, we have $p(1) = 0$. This means that $x = 1$ is a root of $p(x)$. From

$$p'(1) = k - \sum_{i=1}^{k} (i-1)a_i \ge 1,$$

it follows that $x = 1$ is a single root of $p(x)$. On the other hand, for $x = re^{i\theta}$, $r \ge 1$, and $0 \le \theta < 2\pi$, we have

$$\left| p(re^{i\theta}) \right| \ge r^k - \left| \sum_{j=1}^{k} a_j r^{j-1} e^{(j-1)\theta i} \right| \ge \left(r - \sum_{i=1}^{k} a_i \right) r^{k-1} \ge 0.$$

It is easy to see that the above inequalities become equalities if and only if $r = 1$ and $\theta = 0$. Therefore, if $x = x_0$ is a zero of $p(x)$, then $|x_0| \le 1$ and $x_0 = 1$ when $|x_0| = 1$. Set

$$p(x) = (x-1)(x-x_1)^{r_i} \cdots (x-x_m)^{r_m}, \tag{8}$$

where $1 + r_1 + \cdots r_m = k, |x_j| < 1$, $1 \le j \le m$, and $x_i \ne x_j$ when $i \ne j$. It is well known that $\{A_n\}_1^\infty$ has the general solution

$$A_n = c + \sum_{i=1}^{m} \sum_{j=0}^{r_i - 1} c_{i,j} n^j x_i^n. \tag{9}$$

From (9), we deduce that

$$\lim_{n \to \infty} A_n = c.$$

The value of c depends on the choice of A_j, $1 \le j \le k$. This completes the proof of Theorem 2. □

Note: Theorem 1 and Theorem 2 can be generalized easily to discuss sequences of functions. To state this precisely, we have

Theorem 3: Let $a_i = a_i(x)$ and $\alpha_i = \alpha_i(x)$, $1 \le i \le k$, be functions defined on a point set $I \subset R^m$, a Euclidean space, and let the function sequence $\{A_n(x)\}_1^\infty$ be defined as

$$A_{n+k}(x) = \sum_{i=1}^{k} a_i A_{n+i-1}^{\alpha_i}(x), \; n \ge 1.$$

Then we have:

(1) If $a_i(x) \geq 0$ and $-1 < \alpha_i(x) < 1$ hold for an $x \in I$, $\{A_n(x)\}_1^\infty$ converges at the point x to $L = L(x)$, the unique root of $\sum_{i=1}^{k} a_i y^{\alpha_i - 1} = 1$ if $a_i(x)$, $1 \leq i \leq k$, are not all zeros and the sequence converges pointwise to zero if $a_i(x) = 0$ for all i, $1 \leq i \leq k$, regardless of the choice of $A_i(x) > 0$, $1 \leq i \leq k$;

(2) If $a_i(x) \geq 0$, $\sum_{i=1}^{k} a_i(x) = 1$, and $\alpha_i(x) = 1$, $1 \leq i \leq k$, hold for an $x \in I$, $\{A_n(x)\}_1^\infty$ converges at the point x. In particular, for case *(1)*, $\{A_n(x)\}_1^\infty$ converges uniformly if there are constants α, $0 \leq \alpha < 1$, $a > 0$, and M such that $|\alpha_i(x)| \leq \alpha$, $1 \leq i \leq k$, $0 < \sum_{i=1}^{k} a_i(x) \leq a$, $x \in I$, and $\sup_{x \in I} \{A_i(x), A_i^{-1}(x) | 1 \leq i \leq k\} \leq M$ hold, respectively.

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ON TRIANGULAR RECTANGULAR NUMBERS

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1. INTRODUCTION

A positive integer n is a **triangular** number if there is another positive integer k such that $n = \frac{1}{2}k(k+1)$. n is a **square** number if there is a positive integer ℓ such that $n = \ell^2$, and n is a **nearly square** number if there is a positive integer ℓ such that $n = \ell(\ell+1)$ (see [1], [4]). More generally, let σ be any nonnegative integer; a positive integer n will be called a **σ-rectangular** number if there is a positive integer ℓ such that $n = \ell(\ell+\sigma)$. Using this definition, a square number is a 0-rectangular number, and a nearly square number is a 1-rectangular number. It is not difficult to show that any positive integer n is an $(n-1)$-rectangular number, and an integer can be a σ-rectangular number for two different values of σ. We consider here the following problem: for a given nonnegative integer σ, generate all the triangular σ-rectangular numbers.

2. A PELLIAN EQUATION

Let n be a triangular σ-rectangular number, then

$$n = \frac{1}{2}k(k+1) = \ell(\ell+\sigma). \tag{1}$$

Since $8n+1 = (2k+1)^2$ and $4n+\sigma^2 = (2\ell+\sigma)^2$, it follows that $r^2 - 2s^2 = 1 - 2\sigma^2$ for $r = 2k+1$ and $s = 2\ell+\sigma$. Hence, we have the following result.

Theorem 1 Let $\sigma \geq 0$, $r \geq 1$, and $s \geq 0$ be three integers such that

$$r^2 - 2s^2 = 1 - 2\sigma^2, \tag{2}$$

and let

$$\begin{bmatrix} k \\ \ell \end{bmatrix} = \frac{1}{2}\begin{bmatrix} r-1 \\ s-\sigma \end{bmatrix}. \tag{3}$$

Then $\frac{1}{2}k(k+1) = \ell(\ell+\sigma)$. Furthermore, any triangular σ-rectangular number can be obtained in this way. □

By direct substitution, if (r, s) is any solution of (2), and we let

$$\begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} \quad \left(\text{or } \begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\right), \tag{4}$$

then (r', s') is also a solution of (2). It follows directly that if (k, ℓ) are such that $\frac{1}{2}k(k+1) = \ell(\ell+\sigma)$, and if we let

$$\begin{bmatrix} k' \\ \ell' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k \\ \ell \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}, \tag{5}$$

then $\frac{1}{2}k'(k'+1) = \ell'(\ell'+\sigma)$.

3. THE CASE $\sigma = 0$: TRIANGULAR SQUARE NUMBERS

It is known that the "smallest" solution (or the fundamental solution) of (2) for $\sigma = 0$ is $r_0 = 1$ and $s_0 = 0$ (see [2], [3], [5]). Furthermore, all the solutions of (2) are generated by the following recursive scheme: $r_0 = 1$, $s_0 = 0$, and

$$\begin{bmatrix} r_{i+1} \\ s_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} r_i \\ s_i \end{bmatrix} \quad (i = 0, 1, 2, \ldots). \tag{6}$$

Hence, any triangular square number can be obtained from the recursive scheme: $k_0 = 0$, $\ell_0 = 0$, and

$$\begin{bmatrix} k_{i+1} \\ \ell_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_i \\ \ell_i \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (i = 0, 1, 2, \ldots). \tag{7}$$

TABLE 1. Triangular Square Numbers

	$n_i = \frac{1}{2}k_i(k_i+1) = \ell_i^2$		
i	k_i	ℓ_i	n_i
0	0	0	0
1	1	1	1
2	8	6	36
3	49	35	1225
4	288	204	41616
5	1681	1189	1413721

4. THE CASE $\sigma > 0$

Let us observe that $r = 1$ and $s = \sigma$ is always a solution of (2). With this initial value we can generate infinitely many solutions of (2) using (6). But it happens that this sequence of solutions does not contain all the solutions of (2) for some values of σ. We are led to the problem of finding all the "smallest" (or fundamental) solutions of (2). This problem is addressed elsewhere for more general pellian equations ([2], [3], [5]). We present here a simple proof for equation (2) using Fermat's descent method ([3], [5]). The method is based on the next two lemmas.

Lemma 2: Let $\sigma > 0$. If (r, s) is any solution of (2), then r is odd, $|s| \geq \sigma$, and $|s| >$ (=, <, resp.) $\sqrt{2\sigma^2 - 1}$ if any only if $|r| >$ (=, <, resp.) $|s|$.

Proof: Equation (2) is equivalent to $2(s^2 - \sigma^2) = r^2 - 1$ and $r^2 - s^2 = s^2 - (2\sigma^2 - 1)$. □

Lemma 3: Let $\sigma > 0$. Assume that (r, s) and $(\tilde{r}, \tilde{s})$ are two solutions of (2) such that

$$\begin{bmatrix} \tilde{r} \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}.$$

(a) If $r \geq 0$ and $s > \sqrt{2\sigma^2 - 1}$, then $0 < \tilde{s} < s$, $-\tilde{s} < \tilde{r} < r$, and it follows that if

$$\begin{aligned} \tilde{s} > \sqrt{2\sigma^2-1} &\quad \text{then} \quad \tilde{r} > \tilde{s}, \\ \tilde{s} = \sqrt{2\sigma^2-1} &\quad \text{then} \quad \tilde{r} = \tilde{s}, \\ \tilde{s} < \sqrt{2\sigma^2-1} &\quad \text{then} \quad |\tilde{r}| < \tilde{s}. \end{aligned}$$

(b) If $|r| < s$ and $0 < s < \sqrt{2\sigma^2 - 1}$, then $\tilde{s} > s$, $|\tilde{r}| > \tilde{s}$, and it follows that $\tilde{s} > \sqrt{2\sigma^2 - 1}$.

Proof: ***(a)*** From Lemma 2, $r > s > \sqrt{2\sigma^2 - 1}$. Then $\tilde{s} = s - 2(r - s) < s$ and $\tilde{s} = r - 3(r - s) = (s(r-s) + 2(2\sigma^2 - 1)) / (r + s) > 0$ because $(r - s)(r + s) = s^2 - (2\sigma^2 - 1)$. Also, if $\tilde{r} + \tilde{s} = r - s > 0$, then $\tilde{r} > -\tilde{s}$ and $\tilde{r} < r$. We complete the proof using Lemma 2.

(b) Since $\tilde{s} = s + 2(s - r)$ and $\tilde{r} = -\tilde{s} + (r - s)$, we have $\tilde{s} > s > 0$ and $\tilde{r} < -\tilde{s}$. Then $|\tilde{r}| > \tilde{s}$ and hence $\tilde{s} > \sqrt{2\sigma^2 - 1}$. □

Definition 4: Let $\sigma > 0$. A fundamental solution for (2) is a solution (r, s) of (2) such that

$$\sigma \leq s \leq \sqrt{2\sigma^2 - 1} \quad \text{and} \quad -s < r \leq s. \ \square$$

Finally, using Fermat's descent method, we have the following result.

Theorem 5: Let $\sigma > 0$. For any positive solution (r, s) of (2), there exists a unique fundamental solution (r_0, s_0) of (2) and a nonnegative integer i such that

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^i \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}. \ \square$$

To find all the fundamental solutions of (2) for a given σ, we can consider a systematic method based on the following facts:

(i) $(1, \sigma)$ is always a fundamental solution,
(ii) r is always odd,
(iii) s and σ have the same parity.

Hence, for a given σ we can consider s with the parity of σ in the interval $[\sigma, \sqrt{2\sigma^2 - 1}]$ for which $r = \sqrt{1 - 2\sigma^2 + 2s^2}$ is an integer. Table 2 presents the fundamental solutions of (2) for σ = 1, ..., 30. Let us remark that if $2\sigma^2 - 1$ is a prime number, (2) has no fundamental solution but $(\pm 1, \sigma)$ (see [2], Theorem 110).

Finally, to generate the triangular σ-rectangular numbers, we consider the fundamental solutions (r_0, s_0) of (2) and

(i) if $r_0 > 0$, then $\begin{bmatrix} k_0 \\ \ell_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} r_0 - 1 \\ s_0 - \sigma \end{bmatrix}$,

(ii) if $r_0 < 0$, then $\begin{bmatrix} k_0 \\ \ell_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}$, and we use (5). We associate a class of triangular σ-rectangular numbers to each fundamental solution of (2), and the classes are distinct.

TABLE 2. Fundamental Solutions of (2) for $\sigma = 1, \ldots, 30$

σ	$2\sigma^2-1$	(r, s)	σ	$2\sigma^2-1$	(r, s)
1	1^*	$(1, 1)$	16	511	$(\pm1, 16), (\pm17, 20)$
2	$7^\bullet$	$(\pm1, 2)$	17	$577^\bullet$	$(\pm1, 17)$
3	$17^\bullet$	$(\pm1, 3)$	18	$647^\bullet$	$(\pm1, 18)$
4	$31^\bullet$	$(\pm1, 4)$	19	721	$(\pm1, 19), (\pm23, 25)$
5	49^*	$(\pm1, 5), (7, 7)$	20	799	$(\pm1, 20), (\pm13, 22)$
6	$71^\bullet$	$(\pm1, 6)$	21	$881^\bullet$	$(\pm1, 21)$
7	$97^\bullet$	$(\pm1, 7)$	22	$967^\bullet$	$(\pm1, 22)$
8	$127^\bullet$	$(\pm1, 8)$	23	1057	$(\pm1, 23), (\pm25, 29)$
9	161	$(\pm1, 9), (\pm9, 11)$	24	$1151^\bullet$	$(\pm1, 24)$
10	$199^\bullet$	$(\pm1, 10)$	25	$1249^\bullet$	$(\pm1, 25)$
11	$241^\bullet$	$(\pm1, 11)$	26	1351	$(\pm1, 26), (\pm31, 34)$
12	287	$(\pm1, 12), (\pm15, 16)$	27	1457	$(\pm1, 27), (\pm15, 29)$
13	$337^\bullet$	$(\pm1, 13)$	28	$1567^\bullet$	$(\pm1. 28)$
14	391	$(\pm1, 14), (\pm11, 16)$	29	1681^*	$(\pm1, 29), (\pm41, 41)$
15	$449^\bullet$	$(\pm1, 15)$	30	1799	$(\pm1, 30), (\pm33, 38)$

* a square number; $^\bullet$ a prime number

Example 6: Consider $\sigma = 12$. Using (5), we have

$$\begin{bmatrix} k_{i+1} \\ \ell_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_i \\ \ell_i \end{bmatrix} + \begin{bmatrix} 25 \\ 13 \end{bmatrix} \quad (i = 0, 1, 2, \ldots),$$

where the (k_0, ℓ_0) are as given in Table 3. In this case, there exist four different classes of triangular 12-rectangular numbers. □

TABLE 3. Initial Values (k_0, ℓ_0)

$\sigma = 12$		$\sigma = 29$	
(r_0, s_0)	(k_0, ℓ_0)	(r_0, s_0)	(k_0, ℓ_0)
$(1, 12)$	$(0, 0)$	$(1, 29)$	$(0, 0)$
$(-1, 12)$	$(22, 11)$	$(-1, 29)$	$(56, 28)$
$(15, 16)$	$(7, 2)$	$(41, 41)$	$(20, 6)$
$(-15, 16)$	$(9, 3)$		

Example 7: Consider $\sigma = 29$. Using (5), we have

$$\begin{bmatrix} k_{i+1} \\ \ell_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_i \\ \ell_i \end{bmatrix} + \begin{bmatrix} 59 \\ 30 \end{bmatrix} \quad (i = 0, 1, 2, \ldots),$$

where the (k_0, ℓ_0) are as given in Table 3 above. In this case, there exist three different classes of triangular 29-rectangular numbers. □

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REMARK ON A NEW DIRECTION FOR A GENERALIZATION OF THE FIBONACCI SEQUENCE

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In [5] Peter Hope gives the idea for a Fibonacci-type sequence with the form

$$x_0 = 0,\ x_1 = 1,\ x_{n+2} = a_n x_{n+1} + b_n x_n \quad (n \geq 0),$$

where $\{a_n\}$ and $\{b_n\}$ are given sequences with positive numbers.

Combining this idea with the ideas for a generalization of the Fibonacci sequence from [1], [3], and [6], we shall introduce the new direction for a generalization of the Fibonacci sequence. At the moment, all generalizations of this sequence are "linear." The one proposed here has a "multiplicative" form. The analog of the standard Fibonacci sequence in this form will be

$$x_0 = a,\ x_1 =,\ x_{n+2} = x_{n+1} x_n \quad (n \geq 0),$$

where a and b are real numbers. Directly, it can be seen that, for $n \geq 1$,

$$x_n = a^{f_{n-1}} b^{f_n}.$$

In the case of two (or more) sequences, by analogy with [1], [3], and [6], we shall define the following four schemes.

Scheme I:
$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1}\beta_n, \quad (n \geq 0) \\ \beta_{n+2} = \alpha_{n+1}\alpha_n. \end{cases}$$

Scheme II:
$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1}\beta_n, \quad (n \geq 0) \\ \beta_{n+2} = \beta_{n+1}\alpha_n. \end{cases}$$

Scheme III:
$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1}\alpha_n, \quad (n \geq 0) \\ \beta_{n+2} = \alpha_{n+1}\beta_n. \end{cases}$$

Scheme IV:
$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1}\alpha_n, \quad (n \geq 0) \\ \beta_{n+2} = \beta_{n+1}\beta_n. \end{cases}$$

Scheme I is analogous to the scheme from [3]; Scheme II is analogous to the scheme from [1] and [6]; Scheme III is analogous to the third scheme from [3]; Scheme IV is a trivial scheme because it contains two Fibonacci sequences in the above-defined "multiplicative form.

Let $\{F_n\}$ be the classical Fibonacci sequence.

The n^{th} terms of these schemes are determined, e.g., as shown in [1], [3], and [6]. We shall give the formulas of the n^{th} terms using the notation from [1] and [3]. These terms are as follows.

Scheme I:

$$\begin{cases} \alpha_{n+2} = a^{\frac{1}{2}\left(F_{n+1}+3\left[\frac{n+2}{3}\right]-n-1\right)} b^{\frac{1}{2}\left(F_{n+1}-3\left[\frac{n+2}{3}\right]+n+1\right)} c^{\frac{1}{2}\left(F_{n+2}-3\left[\frac{n}{3}\right]+n-1\right)} d^{\frac{1}{2}\left(F_{n+2}+3\left[\frac{n}{3}\right]-n+1\right)}, \\ \beta_{n+2} = a^{\frac{1}{2}\left(F_{n+1}-3\left[\frac{n+2}{3}\right]+n+1\right)} b^{\frac{1}{2}\left(F_{n+1}+3\left[\frac{n+2}{3}\right]-n-1\right)} c^{\frac{1}{2}\left(F_{n+2}+3\left[\frac{n}{3}\right]-n+1\right)} d^{\frac{1}{2}\left(F_{n+2}-3\left[\frac{n}{3}\right]+n-1\right)}. \end{cases}$$

Scheme II:

$$\begin{cases} \alpha_{n+2} = a^{\frac{1}{2}(F_{n+1}+\psi(n+2))} b^{\frac{1}{2}(F_{n+1}+\psi(n+5))} c^{\frac{1}{2}(F_{n+2}+\psi(n))} d^{\frac{1}{2}(F_{n+2}+\psi(n+3))}, \\ \beta_{n+2} = a^{\frac{1}{2}(F_{n+1}+\psi(n+5))} b^{\frac{1}{2}(F_{n+1}+\psi(n+2))} c^{\frac{1}{2}(F_{n+2}+\psi(n+3))} d^{\frac{1}{2}(F_{n+2}+\psi(n))}, \end{cases}$$

where ψ is an integer function defined for every $k \geq 0$ by

$$\begin{array}{c|cccccc} m & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \psi(6k+m) & 1 & 0 & -1 & -1 & 0 & 1 \end{array} \quad \text{(see [1]).}$$

Scheme III:

$$\begin{cases} \alpha_{n+2} = a^{\sigma(n)F_{n+1}} b^{\sigma(n+1)F_{n+1}} c^{\sigma(n+1)F_{n+2}} d^{\sigma(n)F_{n+2}}, \\ \beta_{n+2} = a^{\sigma(n+1)F_{n+1}} b^{\sigma(n)F_{n+1}} c^{\sigma(n)F_{n+2}} d^{\sigma(n+1)F_{n+2}}, \end{cases}$$

where σ is an integer function defined for every $k \geq 0$ by

$$\begin{array}{c|cc} m & 0 & 1 \\ \hline \psi(2k+m) & 1 & 0 \end{array}.$$

Scheme IV:

$$\begin{cases} \alpha_{n+2} = a^{F_{n+1}} c^{F_{n+2}}, \\ \beta_{n+2} = b^{F_{n+1}} d^{F_{n+2}}. \end{cases}$$

The research from [2], [4], [6], [7], and [8] can also be transformed here.

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SOME BINOMIAL FIBONACCI IDENTITIES

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1. INTRODUCTION

Fibonacci identities are equalities of expressions made up from elements of the Fibonacci (F_n) and/or Lucas (L_n) sequences which are valid for all values of the subscript n. Fibonacci identities involving binomial coefficients are commonly referred to as *binomial Fibonacci identities* (e.g., see [1]). Perhaps the simplest and most widely used among them is

$$\sum_{h=0}^{n}\binom{n}{h}F_h = F_{2n}. \tag{1.1}$$

The aim of this paper is to use certain combinatorial identities to derive some unusual binomial Fibonacci identities.

In Section 2 we shall apply the so-called "XY-transform" [1] to the well-known (e.g., see [2]) Waring's formula

$$X^n + Y^n = \sum_{h=0}^{\lfloor n/2\rfloor}(-1)^h D_n(h)(XY)^h(X+Y)^{n-2h} \quad (n \ge 1) \tag{1.2}$$

where

$$D_n(h) = \frac{n}{n-h}\binom{n-h}{h} \quad \left(= \binom{n-h}{h} + \binom{n-h-1}{h-1}, \text{ see [9], p. } 64\right) \tag{1.3}$$

and the symbol $\lfloor\cdot\rfloor$ denotes the greatest integer function.

In Section 3 the same technique will be applied to the combinatorial identities

$$(X+Y)^n = \sum_{h=1}^{n}\binom{2n-1-h}{n-1}(X^h+Y^h)\left(\frac{XY}{X+Y}\right)^{n-h} \quad (n \ge 1) \tag{1.4}$$

and

$$(X+Y)^n = \sum_{h=0}^{n-1}C_n(h)[X^{n-h}+(-1)^hY^{n-h}]\left(\frac{XY}{X-Y}\right)^{h} \quad (n \ge 1), \tag{1.5}$$

where

$$C_n(h) = \sum_{j=0}^{h}(-1)^j\binom{h-1}{j}\binom{n}{h-j}, \tag{1.6}$$

which have been proved by L. Toscano in [10]. It has to be pointed out that (1.4)-(1.6) are quoted in [10] as due to J. G. Van de Corput.

Finally, in Section 4 we shall use the properties of the combinatorial sum

$$S_n(k, x) \stackrel{\text{def}}{=} \sum_{h=0}^{\infty}\binom{n}{kh}x^h, \tag{1.7}$$

where n and k are arbitrary positive integers, and x is an arbitrary real quantity, to obtain certain binomial Fibonacci identities, some of which involve trigonometrical expressions. Observe that the upper range indicator of the sum (1.7) has been put equal to infinity only for the sake of convenience. In fact, if n is finite, then this sum is finite as well because the binomial coefficient vanishes when $h > n/k$. Several properties of $S_n(k, x)$ have been presented by the author at the XIII Österreichischer Mathematikerkongress [3]. The detailed proofs of these properties are available in [4].

Throughout the paper $\alpha = 1-\beta = (1+\sqrt{5})/2$ denotes the golden section. The Binet forms $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ are used without specific reference.

2. USE OF WARING'S FORMULA

Following Dresel [1], first let us put $X = \alpha^k$ and $Y = \beta^k$ [whence $XY = (-1)^k$ and $X+Y = L_k$] in (1.2), thus getting the expression of L_{nk} as a polynomial in L_k:

$$L_{nk} = \sum_{h=0}^{\lfloor n/2 \rfloor} (-1)^{(k+1)h} D_n(h) L_k^{n-2h} \quad (n \geq 1). \tag{2.1}$$

Then let us put $X = \alpha^k$ and $y = -\beta^k$ [whence $XY = (-1)^{k+1}$ and $X+Y = \sqrt{5}F_k$] in (1.2), thus obtaining

$$L_{nk} = \sum_{h=0}^{n/2} (-1)^{hk} D_n(h) 5^{n/2-h} F_k^{n-2h} \quad (n \geq 2 \text{ even}), \tag{2.2}$$

and

$$F_{nk} = \sum_{h=0}^{(n-1)/2} (-1)^{hk} D_n(h) 5^{(n-1)/2-h} F_k^{n-2h} \quad (n \geq 1 \text{ odd}). \tag{2.3}$$

Observe that the identity (2.3), the proof of which is nothing but a trivial replacement, is an equivalent form of the statement of Theorem 1 of [8].

3. USE OF TOSCANO'S FORMULAS

We shall confine ourselves to give two simple examples of application of formulas (1.4) and (1.5).

Letting $X = \alpha^k$ and $Y = \beta^k$ in (1.4) yields

$$L_k^n = \sum_{h=1}^{n} (-1)^{k(n-h)} \binom{2n-1-h}{n-1} \frac{L_{hk}}{L_k^{n-k}} \quad (n \geq 1) \tag{3.1}$$

whence, after multiplying both sides by L_k^n, we obtain

$$L_k^{2n} = (-1)^{kn} \sum_{h=1}^{n} \binom{2n-1-h}{n-1} L_{hk} L_k^h \quad (n \geq 1). \tag{3.2}$$

Observe that, for $k = 1$, the identity (3.2) reduces to

$$\sum_{h=1}^{n} (-1)^{n+h} \binom{2n-1-h}{n-1} L_h = 1 \quad (n \geq 1). \tag{3.3}$$

Letting $X = \alpha^k$ and $y = -\beta^k$ in (1.5) yields

$$(\sqrt{5}F_k)^n = \sum_{h=0}^{n-1} C_n(h)\left[\alpha^{k(n-h)} + (-1)^h(-1)^{k(n-h)}\right]\frac{(-1)^{(k+1)h}}{L_k^h} \quad (n \geq 1) \tag{3.4}$$

whence

$$F_k^n = \frac{1}{5^{\lfloor n/2 \rfloor}} \sum_{h=0}^{n-1} (-1)^{(k+1)h} C_n(h) \frac{A_{k(n-h)}}{L_k^h} \quad (n \geq 1), \tag{3.5}$$

where A stands for L (F) when n is even (odd). Observe that, for $k = 1$, the identity (3.5) reduces to

$$\sum_{h=0}^{n-1} C_n(h) A_{n-h} = 5^{\lfloor n/2 \rfloor} \quad (n \geq 1). \tag{3.6}$$

4. USE OF $S_n(k, x)$

First, we recall some of the results established in [3] and [4], then we use them to obtain some binomial Fibonacci identities. The following notation is used throughout this section:

(i) $i = \sqrt{-1}$: the imaginary unit,

(ii) $r_s(k) = e^{i2\pi(s-1)/k}$: the s^{th} $(s = 1, 2, \ldots, k)$ of the k distinct k^{th} roots of 1,

(iii) $x^{1/k}$ (or $\sqrt{x}$ in the case $k = 2$): the principal value of the k^{th} root of x,

(iv) $\operatorname{atan} x$: the principal branch of the function $\tan^{-1} x$ $(-\pi/2 < \operatorname{atan} x < \pi/2)$.

We point out that $\operatorname{atan} x$ is the value of $\tan^{-1} x$ one obtains by means of the common pocket calculators. The results obtained in this section can then be readily checked.

4.1 Some Results Concerning $S_n(k, x)$ and Certain Related Sums

The following identity is perhaps the main result established in [3] and [4]:

$$S_n(k, x) = \frac{1}{k}\sum_{s=1}^{k}\left[1 + r_s(k)x^{1/k}\right]^n. \tag{4.1}$$

For the convenience of the reader, we report the short and elegant proof of (4.1) that has been suggested by the referee.

Proof of (4.1):

$$\begin{aligned}\sum_{s=1}^{k}\left[1 + x^{1/k}e^{i2\pi(s-1)/k}\right]^n &= \sum_{s=1}^{k}\sum_{h=0}^{n}\binom{n}{h}\left[xe^{i2\pi(s-1)}\right]^{h/k} \\ &= \sum_{h=0}^{n}\binom{n}{h}x^{h/k}\sum_{s=1}^{k}\left[e^{i2\pi h/k}\right]^{s-1} \qquad (4.2)\\ &= k\sum_{h=0}^{\infty}\binom{n}{kh}x^h,\end{aligned}$$

since the summation (4.2) is just the sum of a geometric progression equal to k if $k \mid h$, and zero otherwise. Q.E.D.

Observe that, for k = 2 and 3, the identity (4.1) is reported on pages 123 and 135 of [9], respectively. The extension of (4.1) to negative values of n (e.g., see [9], p. 1) yields

$$\begin{aligned} S_{-n}(k,x) &= \sum_{h=0}^{\infty}\binom{-n}{kh}x^h = \sum_{h=0}^{\infty}(-1)^{kh}\binom{n-1+kh}{kh}x^h \\ &= \frac{1}{k}\sum_{s=1}^{k}\left[1+r_s(k)x^{1/k}\right]^{-n} \quad (\text{if } |x|<1). \end{aligned} \tag{4.3}$$

Moreover, we consider the combinatorial sums

$$S_n^{(1)}(k,x) \stackrel{\text{def}}{=} \sum_{h=0}^{\infty} h\binom{n}{kh}x^h = \frac{nx^{1/k}}{k^2}\sum_{s=1}^{k} r_s(k)\left[1+r_s(k)x^{1/k}\right]^{n-1}, \tag{4.4}$$

$$R_n(x) \stackrel{\text{def}}{=} \sum_{h=0}^{\infty}\binom{n}{2h+1}x^h = \left[(1+\sqrt{x})^n-(1-\sqrt{x})^n\right]/(2\sqrt{x}) \quad (x\neq 0). \tag{4.5}$$

Special cases of (4.1) and (4.3)-(4.5) which interest us are:

$$S_n(2,2) = Q_n/2 \quad (Q_n\text{, the } n^{\text{th}} \text{ Pell-Lucas number [7]}); \tag{4.6}$$

$$S_n(2,5) = 2^{n-1}L_n; \tag{4.7}$$

$$S_n^{(1)}(2,5) = 5n2^{n-3}F_{n-1}; \tag{4.8}$$

$$R_n(5) = 2^{n-1}F_n. \tag{4.9}$$

Using the identity (4.1) along with the polar form for complex numbers yields the following trigonometrical expressions for $S_n(k,x)$ ($k = 2, 3$, and 4).

$$S_n(2,x) = (1-x)^{n/2}\cos(n\,\text{atan}\sqrt{-x}) \quad (x<0), \tag{4.10}$$

$$\begin{aligned} S_n(3,x) &= \frac{1}{3}\left[(1+x^{1/3})^n + 2X^{n/2}\cos\left(n\,\text{atan}\frac{\sqrt{3}x^{1/3}}{2-x^{1/3}}\right)\right] \quad (x<8) \\ &\qquad (\text{where } X = x^{2/3}-x^{1/3}+1), \end{aligned} \tag{4.11}$$

$$S_n(4,x) = \frac{1}{4}\left[(1+x^{1/4})^n + (1-x^{1/4})^n + 2(1+\sqrt{x})^{n/2}\cos(n\,\text{atan}\,x^{1/4})\right] \quad (x\geq 0). \tag{4.12}$$

Some special cases of (4.10)-(4.12) are:

$$S_n(2,-1) = 2^{n/2}\cos(n\pi/4); \tag{4.13}$$

$$S_n(3,1) = [2^n + 2\cos(n\pi/3)]/3 \quad (\text{cf. } 0.152\text{-}1 \text{ of [5]}); \tag{4.14}$$

$$S_n(3,-1) = [3^{n/2}2\cos(n\pi/6)]/3; \tag{4.15}$$

$$S_n(4,1) = [2^{n-1} + 2^{n/2}\cos(n\pi/4)]/2 \quad (\text{cf. } 0.153\text{-}1 \text{ of [5]}); \tag{4.16}$$

$$S_n(4,25) = 2^{n-2}L_n + \frac{6^{n/2}}{2}\cos(n\,\text{atan}\sqrt{5}). \tag{4.17}$$

4.2 Some Fibonacci Identities

By using (4.1), (4.3), and (4.4) along with the Binet forms for Fibonacci and Lucas numbers, and some usual identities available in [6], pages 52-60, a great variety of somewhat unusual Fibonacci identities can be obtained, a small sample of which is reported in the sequel. To save space, only one of them will be proved in full detail. Particular emphasis is given to the use of $S_n(2, x)$. It can be noted that the special cases (4.7)-(4.9) and (4.17) are themselves Fibonacci identities.

4.2.1 Results

$$\sum_{h=0}^{\infty} h\binom{n}{h} F_h = nF_{2n-1}. \tag{4.18}$$

$$\sum_{h=0}^{\infty} (-1)^h \binom{n-1+h}{h} \frac{F_h}{2^h} = -2^n B_n / 5^{\lfloor (n+1)/2 \rfloor}, \tag{4.19}$$

where B stands for F (L) when n is even (odd). Observe that letting $n = 1$ and 2 in (4.19), and combining the results yield the remarkable equality

$$\sum_{h=0}^{\infty} (-1)^h F_h / 2^h = \sum_{h=0}^{\infty} (-1)^h h F_h / 2^h \ \ (= -2/5), \tag{4.20}$$

which shows how the sum on the left-hand side is unconcerned at the introduction of the factor h.

$$\sum_{h=0}^{\infty} \binom{n}{2h} F_{2h} = (F_{2n} - F_n)/2. \tag{4.21}$$

$$\sum_{h=0}^{\infty} h\binom{n}{2h} F_{2h} = n(F_{2n-1} - F_{n-2})/4. \tag{4.22}$$

$$\sum_{h=0}^{\infty} h\binom{n}{2h} F_{2h}^2 = \begin{cases} n[5^{n/2} F_{n+1} + L_{n+1} - 2^n]/20 & (n \text{ even}), \\ n[(5^{(n-1)/2} - 1) L_{n+1} - 2^n]/20 & (n \text{ odd}). \end{cases} \tag{4.23}$$

$$\sum_{h=0}^{\infty} \binom{n}{2h} F_h^2 = \frac{1}{10}\left[L_{2n} + L_n - 2^{(n+4)/2} \cos\frac{n\pi}{4}\right]. \tag{4.24}$$

$$\sum_{h=0}^{\infty} h\binom{n}{2h} F_h^2 = \frac{n}{20}\left[L_{2n-1} + L_{n-2} + 2^{(n+3)/2} \sin\frac{(n-1)\pi}{4}\right]. \tag{4.25}$$

$$\sum_{h=0}^{\infty} \binom{n}{3h} F_{3h} = \frac{F_{2n}}{3} + \frac{2^{(n+4)/2}}{3\sqrt{5}} \sin\left(\frac{n\pi}{6}\right) \sin\left[\frac{n(\operatorname{atan}\sqrt{15} - \pi)}{2}\right]. \tag{4.26}$$

Observe that the right-hand side of (4.26) reduces to $F_{2n}/3$ whenever $n \equiv 0 \pmod 6$.

$$\sum_{h=0}^{\infty} \binom{n}{4h} F_{4h} = \frac{F_{2n} - F_n}{4} + \frac{5^{n/4}}{2} F_{n/2} \cos(n \operatorname{atan} \alpha) \ \ (n \text{ even}). \tag{4.27}$$

4.2.2 A Proof

Proof of (4.27): By (4.12) and taking into account that the principal value of $(\beta^4)^{1/4}$ is $-\beta = (\sqrt{5}-1)/2$, let us rewrite the left-hand side of (4.27) as

$$\begin{aligned}
X_n &= \frac{1}{\sqrt{5}}[S_n(4,\alpha^4) - S_n(4,\beta^4)] \\
&= \frac{1}{\sqrt{5}}\bigg(\frac{1}{4}[(1+\alpha)^n + (1-\alpha)^n + 2(1+\alpha^2)^{n/2}\cos(n\operatorname{atan}\alpha)] \\
&\quad - \frac{1}{4}\{(1-\beta)^n + (1+\beta)^n + 2(1+\beta^2)^{n/2}\cos[n\operatorname{atan}(-\beta)]\}\bigg) \\
&= \frac{F_{2n}-F_n}{4} + \frac{1}{2\sqrt{5}}\{(1+\alpha^2)^{n/2}\cos(n\operatorname{atan}\alpha) - (1+\beta^2)^{n/2}\cos[n\operatorname{atan}(-\beta)]\} \\
&= \frac{F_{2n}-F_n}{4} + \frac{1}{2\sqrt{5}}\{(\sqrt{5}\alpha)^{n/2}\cos(n\operatorname{atan}\alpha) - (-\sqrt{5}\beta)^{n/2}\cos[n\operatorname{atan}(-\beta)]\} \\
&= \frac{F_{2n}-F_n}{4} + \frac{5^{(n-2)/4}}{2}\{\alpha^{n/2}\cos(n\operatorname{atan}\alpha) - (-\beta)^{n/2}\cos[n\operatorname{atan}(-\beta)]\}.
\end{aligned} \tag{4.28}$$

The identity (4.28) is valid for all positive n. For n even, (4.28) simplifies remarkably. Consider the trigonometrical identity

$$\operatorname{atan} x + \operatorname{atan}(1/x) = \pi/2, \tag{4.29}$$

whence

$$\operatorname{atan}(-\beta) = \pi/2 - \operatorname{atan}\alpha, \tag{4.30}$$

and replace (4.30) in (4.28), thus getting the identity

$$X_n = \frac{F_{2n}-F_n}{4} + \frac{5^{(n-2)/4}}{2}\left[\alpha^{n/2}\cos(n\operatorname{atan}\alpha) - (-\beta)^{n/2}\cos\left(\frac{n\pi}{2} - n\operatorname{atan}\alpha\right)\right]. \tag{4.31}$$

Recalling that

$$\cos\left(\frac{n\pi}{2} - x\right) = \begin{cases} \cos x, & \text{if } n \equiv 0 \pmod 4, \\ -\cos x, & \text{if } n \equiv 2 \pmod 4, \end{cases} \tag{4.32}$$

the identity (4.31) becomes

$$X_n = \frac{F_{2n}-F_n}{4} + \frac{5^{(n-2)/4}}{2}(\alpha^{n/2} - \beta^{n/2})\cos(n\operatorname{atan}\alpha) \quad (n \text{ even}), \tag{4.33}$$

whence (4.27) is immediately obtained. Q.E.D.

5. CONCLUDING REMARKS

Several binomial Fibonacci identities, most of which we believe to be new, have been established in this paper. In our eyes, the most interesting among them are those derived from the specialization of the combinatorial sum $S_n(k, x)$ to the cases $1 \le k \le 4$.

These identities represent only a small sample of the possibilities available to us. In fact, apart from the extension of the study to values of k greater than 4, the results obtained in Section 4 can apply *mutatis mutandis* to second-order recurring sequences other than the Fibonacci sequence. For instance, the analog of (4.27) for Pell numbers P_n is

$$\sum_{h=0}^{\infty}\binom{n}{4h}P_{4h}=2^{(n-4)/2}P_n+(-1)^{n/2}2^{(3n-4)/4}P_{n/2}\cos\frac{n\pi}{8}\quad(n\text{ even}),\tag{5.1}$$

which reduces to $2^{(n-4)/2}P_n$ when $n\equiv 4 \pmod 8$.

Moreover, the sequences $S_n(k,m,x)$ defined as

$$S_n(k,m,x)\stackrel{\text{def}}{=}\sum_{h=0}^{\infty}\binom{n}{kh}S_h(m,x)\tag{5.2}$$

seem to be worthy of thorough investigation. This will be the goal of a future work. As a minor illustration, we leave the interested reader the proof of the identity

$$S_n(2,2,1)=(2+Q_n)/4\tag{5.3}$$

which involves the Pell-Lucas numbers.

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ON THE SYSTEM OF CONGRUENCES $\prod_{j \neq i} n_j \equiv 1 \bmod n_i$

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We seek integers $n_1, \ldots, n_k$, all ≥ 2, for which

$$\prod_{j \neq i} n_j \equiv 1 \bmod n_i \tag{1}$$

for all i. Problems of this sort arise, for instance, in connection with the Chinese remainder theorem and structure theory for finite Abelian groups. Curiously, this system has received little attention compared to the system

$$\prod_{j \neq i} n_j \equiv -1 \bmod n_i \tag{2}$$

(see [3], [5], [6], [7], [11]). System (2) has attracted interest because it is equivalent to the unit fraction equation

$$\sum_{i=1}^{k} 1/n_i + 1/\prod_{i=1}^{k} n_i = m, \text{ an integer.} \tag{3}$$

Especially for $m = 1$ this problem is not only interesting in its own right in the field of Egyptian fractions, but also has proved to have application to the topology of singular points of algebraic surfaces [4]. In this paper we will apply what is known about system (2) to derive a large number of solutions to system (1). All solutions to (1) with 7 or fewer terms are given in the appendices, together with techniques for producing some 398 solutions with 8 terms and 1411 with 9 terms.

Lemma 1: Let $n_1, \ldots, n_k$ be positive integers, relatively prime in pairs. Put

$$X = \prod_{i=1}^{k} n_i, \quad Y = \sum_{i=1}^{k} \prod_{j \neq i} n_j,$$

and let D be the smallest positive residue of $-Y \bmod X$.

(a) If $X \equiv 1$ (resp. -1) mod D, then $n_1, \ldots, n_k, n_{k+1}$ satisfy (1) [resp. (2)] for $n_{k+1} = (X-1)/D$ [resp. $(X+1)/D$].

(b) If $X^2 - D$ admits a factor $P \equiv -X \bmod D$, then $n_1, \ldots, n_k, n_{k+1}, n_{k+2}$ satisfy (1) for $n_{k+1} = (X+P)/D$ and $n_{k+2} = (X+Q)/D$, where $Q = (X^2 - D)/P$.

Proof: For example, see [4], Proposition 12. (a) is immediate. For (b) we have

(i) $(\prod_{i=1}^{k} n_i) n_{k+1} = P n_{k+2} + 1$,

(ii) $(\prod_{i=1}^{k} n_i) n_{k+2} = Q n_{k+1} + 1$,

while for $i \leq k$, computing modulo n_i gives

(iii) $(\prod_{j \neq i} n_j) n_{k+1} n_{k+2} \equiv YPQD^{-2} \equiv (-D)(-D)D^{-2} \equiv 1$,

where D^{-1} is well defined mod n_i since D and X are relatively prime.

As a special case, if $n_1, \ldots, n_k$ satisfy (2), then $D = 1$. Thus,

Corollary 2: Let $n_1, \ldots, n_k$ satisfy (2). Then

(a) $n_1, \ldots, n_k, n_{k+1}$ also satisfy (2) for $n_{k+1} = \prod_{i=1}^k n_i + 1$,

(b) $n_1, \ldots, n_k, n_{k+1}$ satisfy (1) for $n_{k+1} = \prod_{i=1}^k n_i - 1$, and

(c) if $P \big| \prod_{i=1}^k n_i^2 - 1$, then $n_1, \ldots, n_k, n_{k+1}, n_{k+2}$ satisfy (1) for $n_{k+1} = \prod_{i=1}^k n_i + P$, $n_{k+2} = \prod_{i=1}^k n_i + Q$, where $Q = (\prod_{i=1}^k n_i^2 - 1)/P$.

Since all solutions to (1) are known for $k \leq 7$ (see [4]), as well as some 500 independent infinite sequences of solutions for increasingly large k (see [1]), part (b) gives a rich family of solutions to the congruences (1) obtained in this trivial way. To make use of part (c), we must be able to find factors of numbers of the form $\prod_{i=1}^k n_i^2 - 1$. Immediately we have the factors $\prod_{i=1}^k n_i - 1$ and $\prod_{i=1}^k n_i + 1$; hence, the following corollary.

Corollary 3: Let $n_1, \ldots, n_k$ satisfy (2). Then $n_1, \ldots, n_k, n_{k+1}, n_{k+2}$ satisfy (1) for $n_{k+1} = 2\prod_{i=1}^k n_i - 1$, $n_{k+2} = 2\prod_{i=1}^k n_i + 1$ (as well as for $n_{k+1} = \prod_{i=1}^k n_i + 1$, $n_{k+2} = \prod_{i=1}^k n_i^2 + \prod_{i=1}^k n_i - 1$).

By finding further factors of $\prod_{i=1}^k n_i - 1$ and $\prod_{i=1}^k n_i + 1$ for fixed $n_1, \ldots, n_k$ satisfying (2), we can find further solutions to (1) (see Appendix 2 below). But a more fruitful approach has proven to be as follows (cf. [12]). Choose a prime P, then try to find a solution $n_1, \ldots, n_k$ to (2) for which P divides $\prod_{i=1}^k n_i - 1$ or $\prod_{i=1}^k n_i + 1$.

For P a positive integer, consider the relation "succeeds mod P" defined on the set Z_p of integers mod P by y succeeds x mod P if $y = x^2 + x$. We will write $x \prec y$ if there is a finite sequence $x_0 = x, x_1, \ldots, x_\ell = y$, $\ell \geq 1$, such that x_i succeeds x_{i-1} for $i = 1, \ldots, \ell$ ($x \prec x$ is permissible), and we will write $x \preccurlyeq y$ if $x \prec y$ or $x = y$. Some properties of this relation are worked out in [1] in connection with equation (3). To give a particular example, which will be referred to later, for $P = 19$ the relation "succeeds" is represented by the following directed graph.

$$\begin{tikzcd}
\mathbf{9} \arrow[d] & & \mathbf{17} \arrow[d] & \mathbf{16} \arrow[d] & \mathbf{8} \arrow[r] & \mathbf{15} \arrow[d] & \mathbf{10} \arrow[l] & & \mathbf{13} \arrow[d] & \mathbf{7} \arrow[d] & \\
\mathbf{14} \arrow[r] & \mathbf{1} \arrow[r] & \mathbf{2} \arrow[r] & \mathbf{6} \arrow[r] & \mathbf{4} \arrow[lll, bend left] & \mathbf{12} \arrow[l] & \mathbf{3} \arrow[l] & \mathbf{5} \arrow[r] & \mathbf{11} \arrow[r] & \mathbf{-1} \arrow[r] & \mathbf{0} \arrow[loop right]
\end{tikzcd}$$

Proposition 4: Let $n_1, \ldots, n_k$ satisfy (2), let P be a positive integer, and suppose that $\prod_{i=1}^k n_i \preccurlyeq \pm 1$ mod P. Put $n_{k+1} = \prod_{i=1}^k n_i + 1$, and for $\ell = 2, 3, \ldots$, put $n_{k+\ell} = n_{k+\ell-1}^2 - n_{k+\ell-1} + 1$. Then, for some $\ell \geq 1$, $n_1, \ldots, n_{k+\ell-1}, n_{k+\ell} + P - 1, n_{k+\ell} + Q - 1$ satisfy (1), for appropriate choice of Q.

Proof: First we note that $\forall \ell\; n_{k+\ell} = \prod_{i<k+\ell} n_i + 1$. Thus $n_1, \ldots, n_{k+\ell}$ satisfy (2) $\forall \ell$. Furthermore, the products $\prod_{i\leq k+\ell} n_i = n_{k+\ell+1} - 1$ satisfy the relation

$$\prod_{i\leq k+\ell} n_i = \left(\prod_{i\leq k+\ell-1} n_i\right)\left(\prod_{i\leq k+\ell-1} n_i + 1\right),$$

that is, $\prod_{i\leq k+\ell} n_i$ succeeds $\prod_{i\leq k+\ell-1} n_i$ mod P. Since $\prod_{i=1}^k n_i \preccurlyeq \pm 1$, it follows that P divides

$\prod_{i\le k+\ell}^{k} n_i \mp 1$ for some ℓ. By Lemma 1(c), then, $n_1, \ldots, n_{k+\ell-1}, n_{k+\ell}+P-1$, $n_{k+\ell}+Q-1$ satisfy (1) for this choice of ℓ and for $Q = ((n_{k+\ell}-1)^2-1)/P$.

Remark: For a few small primes P, $x \preccurlyeq \pm 1 \bmod P$ for **every** integer $x \bmod P$ except $x=0$. $P = 2, 3, 5, 7$, and 19 (see graph above), for instance, have this property. Thus, we have

Corollary 5: Let $P = 2, 3, 5, 7$, or 19. Let $n_1, \ldots, n_k$ satisfy (2), where $P \nmid n_i \forall i$. Then $\prod_{i=1}^{k} n_i \preccurlyeq \pm 1$ and we obtain a solution to (1) as in Proposition 4.

Note: In connection with the prime $P = 2$, it should be mentioned that no solution to (1) or (2) is known with each n_i odd. For $P = 3$, the shortest solution to (2) with each $n_i \not\equiv 0 \bmod 3$ is (2, 5, 7, 11, 17, 157, 961, 4398619). This leads to the solution (2, 5, 7, 11, 17, 157, 961, 4398619, 8687184244716671, 75467170101653548887992820605569) to (1), where no term is divisible by 3. Indeed, applying Corollary 2(c) to appropriate factors of

$$(2\cdot 5\cdot 7\cdot 11\cdot 17\cdot 157\cdot 961\cdot 4398619)^2 - 1 = 3\cdot 719\cdot 2287\cdot 466201\cdot 2715929\cdot 12082314665809$$

gives sixteen distinct solutions to (1) with 10 terms, none $\equiv 0 \bmod 3$. However, there may be a shorter solution to (1) with this feature.

We also observe that for $P = 5$ and $P = 19$, $1 \prec 1$. Thus, $P \mid \prod_{i=1}^{k+\ell} n_i - 1$ for infinitely many ℓ, and we have an infinite sequence of solutions to (1) based on these primes. In general,

Corollary 6: Let $n_1, \ldots, n_k$ satisfy (2) and let P be an integer such that $\prod_{i=1}^{k} n_i \preccurlyeq 1$ and $1 \prec 1$. Then the procedure of Proposition 4 gives infinitely many solutions to (1).

Proof: Let ℓ_0 be the smallest of the indices for which $\prod_{i=1}^{k+\ell} n_i \equiv 1 \bmod P$, and let m_0 be the smallest positive integer for which we have a chain of successors $1 \to x_i \to x_2 \to \cdots \to x_{m_0-1} \to 1 \bmod P$. Then $\prod_{i=1}^{k+\ell_0+jm_0} n_i \equiv 1 \bmod P \ \forall j = 1, 2, \ldots$, each of which gives a solution to (1) by Lemma 2(c).

Primes $P < 1000$ for which $1 \prec 1$ are 5, 19, 31, 41, 89, 409, 431, 461, 569, and 661.

PRIMALITY TESTING AND FIBONACCI NUMBERS

The methods of the previous section show that when $\prod_{i=1}^{k} n_i \pm 1$ have many factors for various solutions $n_1, \ldots, n_k$ to (2), then we obtain many solutions to (1). It is equally interesting to inquire whether these numbers are prime. For instance, $2\cdot 3 \pm 1 = \{5, 7\}$, $2\cdot 3\cdot 7 \pm 1 = \{41, 43\}$, $2\cdot 3\cdot 7\cdot 43\cdot 1807 \pm 1 = \{3263441, 3263443\}$, and $2\cdot 3\cdot 11\cdot 23\cdot 31 \pm 1 = \{47057, 47059\}$ are four pairs of twin primes, where the indicated factors are solutions to (2). In the case of $N = \prod n_i + 1$, primality tests of Fermat type are especially appropriate because we know many factors of $N-1$. Indeed, if there is an integer y for which $y^{N-1} \equiv 1 \bmod N$ but $y^{\prod_{j\neq i} n_j} \not\equiv 1 \bmod N \ \forall i$, then N is "very probably prime" and we need only find the factors of each n_i to complete the test. Some solutions to (2) for which $\prod_{i=1}^{k} n_i + 1$ is prime are (2), (2, 3), (2, 3, 7), (2, 3, 11, 23, 31), (2, 3, 7, 43, 1807), (2, 3, 7, 47, 395), (2, 3, 7, 47, 403, 19403), (2, 3, 7, 47, 415, 8111), (2, 3, 7, 55, 179, 24323), (2, 3, 7, 43, 3263, 4051, 2558951), (2, 3, 7, 55, 179, 24323, 10057317271), (2, 3, 11, 23, 31, 47423, 6114059), and (2, 3, 11, 25, 29, 1097, 2753). These are all such examples with $k \le 7$.

For $\Pi n_i - 1$ we will focus our attention on the sequence $2, 3, 7, 43, \ldots, y_k, \ldots$, where $y_k = \Pi_{i<k} y_i + 1$. By Corollary 2(a), $\forall k$ the first k terms of this sequence satisfy (2). Put $x_k = \Pi_{i\leq k} y_i = y_{k+1} - 1$. Then $x_k = x_{k-1}^2 + x_{k-1}$ and we have the succession relation $1 \to 2 \to 6 \to \cdots \to x_{k-1} \to 1$ mod P for any divisor P of $x_k - 1$.

Lemma 7:

(a) If $m|k$ then $(x_m - 1)|(x_k - 1)$.

(b) (i) $(x_{k-1}+2)|(x_k - 2)$ and (ii) if $\ell|(k-1)$ then $(x_\ell - 1)|(x_k - 2)$.

Proof:

(a) If $m|k$, say $k = md$. Then mod $(x_m - 1)$ we have the sequence of successions $1 \to 2 \to 6 \to \cdots \to x_{m-1} \to 1$, and after d repetitions of this loop we obtain $x_k \equiv 1$ mod $(x_m - 1)$ and $(x_m - 1)|(x_k - 1)$.

(b) From $x_k = x_{k-1}^2 + x_{k-1}$, we have $x_k - 2 = (x_{k-1}+2)(x_{k-1} - 1)$, hence assertion (i). Now assertion (ii) follows from (a).

Corollary 8 [of (a)]: If k is composite, then so is $x_k - 1$.

If k is prime, then $x_k - 1$ may be prime and, again, since we know several factors of $x_k - 2$ by (b) above, primality tests of Fermat type are available. A variation on this theme is to apply a Lucas-type test using the Fibonacci numbers. As a historical sidelight, in connection with the unit fraction equation (3), Fibonacci was the first to prove, in 1202, that if $m, n_1, \ldots, n_k$ is any collection of positive integers with $\sum_{i=1}^{k} 1/n_i < m$, then there exist $\ell, n_{k+1}, \ldots, n_{k+\ell}$ such that $\sum_{i=1}^{k+\ell} 1/n_i = m$ (but not necessarily with $n_{k+\ell} = \Pi_{i<k+\ell} n_i$).

Lemma 9: Let $\{x_\ell\}$ denote the sequence of positive integers defined by $x_0 = 1$, $x_\ell = x_{\ell-1}^2 + x_{\ell-1}$ for $\ell \geq 1$, and let k be an odd prime. Put $y = x_{k-1} + 1$. Then $\forall i = 1, 2, \ldots$,

$$y^i \equiv F_i y + F_{i-1} \bmod (x_k - 1), \tag{4}$$

where $\{F_i\}$ denotes the Fibonacci numbers, beginning with $F_0 = 0, F_1 = 1$. Furthermore, both y and $2y - 1$ are invertible in the ring $Z_{(x_k - 1)}$ of integers mod $(x_k - 1)$ and $\forall i$

$$F_i \equiv (y^i - (-y)^{-i})(2y-1)^{-1} \bmod (x_k - 1). \tag{5}$$

Proof: For the first assertion we use induction on i. If $i = 1$ the claim is just that

$$y \equiv F_1 y + F_0 = 1y + 0.$$

Now let $i > 1$ and assume the claim to be true for all smaller indices—in particular, that $y^{i-1} \equiv F_{i-1} y + F_{i-2}$ mod $(x_k - 1)$. From $y = x_{k-1} + 1$ and $x_{k-1}^2 + x_{k-1} = x_k$ we have

$$y^2 = x_{k-1}^2 + x_{k-1} + x_{k-1} + 1 = x_k + y \equiv y + 1 \bmod (x_k - 1). \tag{6}$$

Thus, modulo $(x_k - 1)$,

$$\begin{aligned} y^i = y(y^{i-1}) &\equiv y(F_{i-1}y + F_{i-2}) \equiv F_{i-1}y^2 + F_{i-2}y \\ &\equiv F_{i-1}(y+1) + F_{i-2}y \equiv (F_{i-1} + F_{i-2})y + F_{i-1} \equiv F_i y + F_{i-1} \end{aligned}$$

as required.

As for invertibility of y, note that

$$yx_{k-1} = (x_{i-1}+1)x_{k-1} = x_{k-1}^2 + x_{k-1} = x_k \equiv 1 \bmod (x_k - 1),$$

so that y^{-i} exists in $Z_{(x_k-1)}$ and is equal to x_{k-1}. Furthermore, we have

$$(-y^{-1})^2 = x_{k-1}^2 \equiv -x_{k-1} + 1 = (-y^{-1}) + 1 \bmod (x_k - 1).$$

Since this is equation (6) above with $-y^{-1}$ in place of y, the same inductive proof as above shows that also

$$(-y)^{-i} \equiv F_i(-y^{-1}) + F_{i-1} \bmod (x_k - 1). \tag{7}$$

Subtracting (7) from (6) now gives

$$y^i - (-y)^{-i} \equiv F_i(y + y^{-1}) \equiv F_i(2y-1) \bmod (x_k - 1).$$

To complete the proof of (5), we must show that $2y-1$ is invertible in $Z_{(x_k-1)}$—that is, that $2y-1$ and $x_k - 1$ have no common factors.

To see this, we compute

$$\begin{aligned}(2y-1)^2 &= (2x_{k-1}+1)^2 = 4x_{k-1}^2 + 4x_{k-1} + 1\\ &= 4x_k + 1 = 4(x_k - 1) + 5,\end{aligned}$$

so any common divisor of $2y-1$ and $x_k - 1$ must also divide 5. But in the sequence $\{x_\ell\}_{\ell=0}^{\infty} = \{1, 2, 6, 42, 1806, \ldots\}$, $x_\ell \equiv 2 \bmod 5$ for all odd ℓ. In particular, $x_k - 1 \equiv 1 \bmod 5$, so 5 does not divide $x_k - 1$ and we conclude that $2y-1$ and $x_k - 1$ are mutually prime as claimed. Thus, $2y-1$ is invertible mod $(x_k - 1)$ and the proof of equation (5) is complete.

Remark: Another way to view this connection between the Fibonacci numbers and the powers of y is to note that y and $(-y^{-1})$ are two solutions modulo$(x_k - 1)$ to the quadratic equation $Y^2 - Y - 1 = 0$. That is, we may regard y as the "golden mean" $y = (1+\sqrt{5})/2$ in $Z_{(x_k-1)}$, where 2 is invertible since $(x_k - 1)$ is odd and where $\sqrt{5}$ exists by quadratic reciprocity. Thus, equation (5) is the equivalent in $Z_{(x_k-1)}$ of the well-studied computational formula

$$F_i = \left[\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right]\Big/\sqrt{5}.$$

Proposition 10: Let $\{x_\ell\}, k, Y$ be as in Lemma 9. Then the sequence of Fibonacci numbers modulo $(x_k - 1)$ repeats with some period λ, where λ divides the order of the multiplicative group $Z^*_{(x_k-1)}$ of invertible elements of $Z_{(x_k-1)}$. If $\lambda = x_k - 2$, then $x_k - 1$ is prime and $Z^*_{(x_k-1)}$ is the cyclic group generated by y.

Proof: In any case, since there are only $(x_k - 1)^2$ pairs of integers mod $(x_k - 1)$, the sequence $\{F_i\}$ in $Z_{(x_k-1)}$ must repeat after at most $(x_k - 1)^2$ terms. Let λ be the smallest positive integer for which $F_{i+\lambda} \equiv F_i$ for all i. By equation (4) of Lemma 9, then, $y^{i+\lambda} \equiv y^i \; \forall i$.

Conversely, if μ is the order of y in the group $Z^*_{(x_k-1)}$, then equation (5) of Lemma 9 shows that

$$F_{i+\mu} \equiv F_i \bmod (x_k - 1) \text{ for all } i.$$

We conclude that $\mu = \lambda$ and that the period of $\{F_i\}$ is the same as the multiplicative order of y in $Z^*_{(x_k-1)}$. Since this order must divide the order of $Z^*_{(x_k-1)}$ by Lagrange's theorem, we have proved the first assertion.

Finally, if $\lambda = x_k - 2$, then $y, y^2, \ldots, y^{x_k-2} = 1$ are all distinct in $Z^*_{(x_k-1)}$, so $|Z^*_{(x_k-1)}| = x_i - 2$ and $x_k - 1$ is coprime to each of $1, 2, \ldots, x_k - 2$. Thus, $x_k - 1$ is prime as claimed, with $Z^*_{(x_k-1)}$ the cyclic group consisting of powers of y.

Remarks: As the proof shows, the condition $\lambda = x_k - 2$ is equivalent to $F_{x_k-2} \equiv 0$ and $F_{x_k-1} \equiv 1 \bmod (x_k - 1)$, but $(F_i, F_{i+1}) \not\equiv (0, 1) \bmod (x_k - 1)\ \forall$ proper divisors i of $x_k - 2$. An example where these computations can be carried out by hand is $k = 3$, $x_k - 1 = 2\cdot 3\cdot 7 - 1 = 41$, $y = 7$. The Fibonacci numbers $(F_{40}, F_{41}) \equiv (0, 1)$ but (F_8, F_9) and $(F_{20}, F_{21}) \not\equiv (0, 1) \bmod 41$, so Z^*_{41} consists of powers of 7. Similarly, $y = 1807$ generates the multiplicative group of integers modulo the prime $x_5 - 1 = 3263441$.

APPLICATION TO ALGEBRAIC SURFACES

Our interest was first attracted to number theoretic problems of this type because of the following considerations from the topology of complex surfaces. Let S be an algebraic surface over C with a normal isolated singular point P. Let $f : \widetilde{S} \to S$ be the minimal normal resolution of singularities with exceptional curve $C = f^{-1}(P) = \bigcup_{i=1}^n C_i$, where each C_i is nonsingular and meets C_j, if at all, transversally in a single point $\forall j \neq i$. C is represented by its dual weighted intersection graph Γ, in which each vertex v_i corresponds to a component C_i, with edges $\{v_i, v_j\}$ whenever C_i meets C_j, and with positive integer weight $w_i = -C_i^2$ assigned to the vertex v_i, where C_i^2 is the self-intersection number (the Chern class of the normal line bundle of the embedding of C_i in $\widetilde{S}$). If each C_i is rational, then Γ completely determines the topology of a neighborhood U of the singular point in S. In particular, if Γ has no cycles then U is the cone on a smooth real three-manifold M whose fundamental group π_1 is generated by $v_1, \ldots, v_n$ with relations $\prod_{j=1}^n v_j^{-(C_i\cdot C_j)} = 1\ \forall i$ and $v_i v_j = v_j v_i$ if C_i meets C_j [9]. From this, it follows that the first homology group of M is the Abelian group with these generators and relations, with order the determinant of the weighted intersection matrix of Γ, written $|\Gamma|$.

This determinant, in turn, can be calculated very quickly using techniques of graph theory in linear algebra [8]. In particular, if Γ is any weighted tree, v_0 a vertex of Γ of weight w_0, we have the following "expansion by a vertex" formula ([2], eq. 2.13). Let $v_1, \ldots, v_k$ be the vertices of Γ that meet v_0, denote by Γ_i the component of $\Gamma - \{v_0\}$ which contains v_i, and put $\Gamma_i' = \Gamma_i - \{v_i\}$. Then

$$|\Gamma| = w_0 \prod_{i=1}^{k} |\Gamma_i| - \sum_{i=1}^{k} |\Gamma_i'| \prod_{j\neq i} |\Gamma_j|. \tag{8}$$

A recurring problem in two-dimensional singularity theory is to classify or to find examples of complex surface singularities whose local fundamental group π_1 satisfies some standard group theoretic criterion, such as being solvable [13] or nilpotent [10]. By the preceding discussion, π_1 is **perfect** (generated by commutators) if and only if Γ is acyclic, each exceptional component C_i is rational, and $|\Gamma| = 1$. The results of this paper give a large family of such "perfect" singularities.

A weighted graph Γ will be called **standard star-shaped** if it consists of linear arms $\Gamma_1, \ldots, \Gamma_k$, each vertex having weight 2, joined at a terminal vertex v_{i1} to a common central vertex v_0 of weight w_0 (see Figure 1).

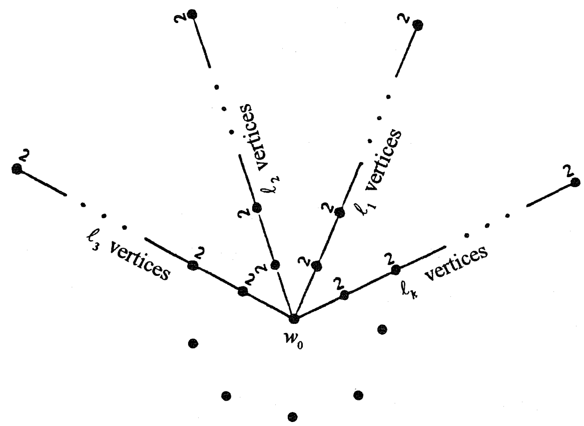

FIGURE 1

Theorem 11: Let $P \in S$ be an isolated complex surface singularity with minimal normal resolution $f : \widetilde{S} \to S$. Suppose that each component of the exceptional curve is rational and that the weighted dual intersection graph Γ of $f^{-1}(P)$ is standard star-shaped with k arms as pictured in Figure 1. For $i = 1, \ldots, k$ put $n_i = \ell_i + 1$, where ℓ_i is the length of the i^{th} arm of Γ. Then the local fundamental group π_1 of P in S is perfect if and only if $n_1, \ldots, n_k$ satisfy the system of congruences (1) with $(\sum_{j=1}^k \prod_{j\neq i} n_j - 1)/\prod_{i=1}^k n_i = k - w_0$.

Proof: The linear graph A_ℓ on ℓ vertices with all weights 2 has determinant $\ell + 1$. Hence, for the graph Γ of Figure 1, formula (8) above becomes

$$|\Gamma| = w_0 \prod_{i=1}^k n_i - \sum_{i=1}^k (n_i - 1) \prod_{j\neq i} n_j = (w_0 - k) \prod_{i=1}^k n_i + \sum_{i=1}^k \prod_{j\neq i} n_j .$$

Thus, $|\Gamma| = 1$ if and only if $\sum_{i=1}^k \prod_{j\neq i} n_j = (k - w_0) \prod_{i=1}^k n_i + 1$.

Remarks: The best-studied example is the rational double point E_8, corresponding to the solution (2, 3, 5), whose local fundamental group is the perfect extension of degree 2 of the alternating group on 5 letters. In general, in connection with the central weight w_0 it should be noted that no solution to (1) is known for which the integer $m = (\sum_{i=1}^k \prod_{j\neq i} n_j - 1)/\prod_{i=1}^k n_i$ is larger than 1.

To aid our understanding of these complex surfaces, we can model their real analogs as follows. Let $(n_1, \ldots, n_k)$ be a solution to the congruence (1). Denote by M_i the "Moebius band with n_i twists," and attach the M_i to a central Moebius band with 1 twist by the technique of plumbing. The surface under study is then the cone on the boundary of this object. The cone is a smooth two-dimensional real manifold with a singular point at the tip of the cone. See Figure 2, where the construction is illustrated for the solution (2, 3, 5).

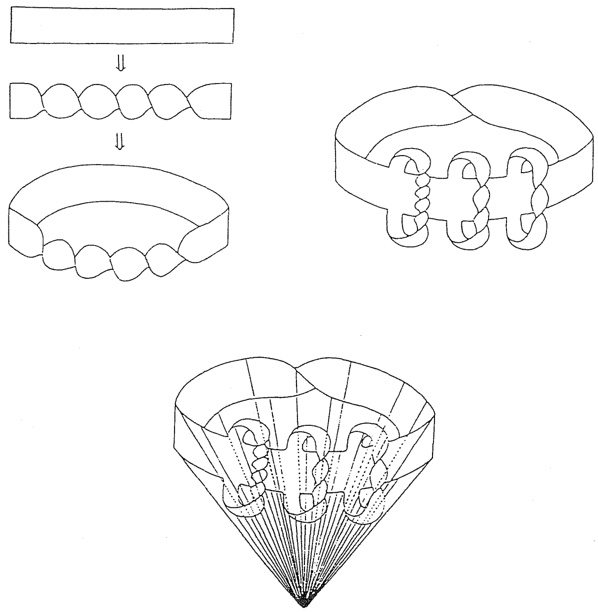

FIGURE 2

Appendix 1: The complete set of solutions to the congruence system $\prod_{j\neq i} n_j \equiv 1 \bmod (n_i)$ with 7 or fewer terms (equivalently, the complete set of solutions to the unit fraction equation $\sum_{i=1}^{k} n_i^{-1} - \prod_{i=1}^{k} n_i^{-1} = 1,\ k \leq 7$):

$k = 3$: 2,3,5

$k = 4$: 2,3,7,41
2,3,11,13

$k = 5$: 2,3,7,43,1805
2,3,7,83,85
2,3,11,17,59

$k = 6$: 2,3,7,43,1807,3263441
2,3,7,43,1811,654133
2,3,7,43,1819,252701
2,3,7,43,1825,173471
2,3,7,43,1871,51985
2,3,7,43,1901,36139
2,3,7,43,1945,25271
2,3,7,43,2053,15011
2,3,7,43,2167,10841
2,3,7,43,2501,6499
2,3,7,43,3041,4447
2,3,7,43,3611,3613
2,3,7,47,395,779729
2,3,7,47,481,2203
2,3,7,53,271,799
2,3,7,71,103,61429
2,3,11,23,31,47057

$k = 7$: 2,3,7,43,1807,3263443,10650056950805
2,3,7,43,1807,6526883,6526885
2,3,7,43,1823,193667,637617223445
2,3,7,43,1907,34165,17766223
2,3,7,43,1907,43115,163073
2,3,7,43,2159,11047,98567401
2,3,7,43,2533,7807,32435
2,3,7,43,3307,3979,642279641
2,3,7,47,395,779731,607979652629
2,3,7,47,395,779819,6832003021
2,3,7,47,395,788491,701757789
2,3,7,47,395,1559459,1559461
2,3,7,47,401,25535,1837531099
2,3,7,47,403,19403,15435513365
2,3,7,47,415,8111,6646612309
2,3,7,47,449,3299,379591
2,3,7,47,583,1223,140479765
2,3,7,55,179,24323,10057317269
2,3,7,59,163,1381,775807
2,3,7,71,103,61441,319853515
2,3,7,71,103,61477,79005919
2,3,7,71,103,61559,29133437
2,3,7,71,103,61955,7238201
2,3,7,71,103,62857,2704339
2,3,7,71,103,67213,713863
2,3,11,23,31,47059,2214502421
2,3,11,23,31,94115,94117

Appendix 2: Prime factorization of $\prod_{i=1}^{k} n_i \pm 1$ for all solutions $n_1, \ldots, n_k$ of the system of congruences (2) $\prod_{j \neq i} n_j \equiv -1 \bmod n_i$ for $k = 6$ and 7. These lists provide 380 solutions to (1) $\prod_{j \neq i} n_j \equiv 1 \bmod n_i$ with 8 terms and 1368 solutions with 9 terms, by applying Corollary 2(c). Together with solutions obtained by applying Corollary 2(b) to known solutions of 92), this gives a total of 398 solutions to (1) for $k = 8$ and 1411 solutions for $k = 9$.

$\underline{k = 6}$

$(n_i, \ldots, n_k)$	$\prod_{i=1}^{k} n_i - 1$	$\prod_{i=1}^{k} n_i + 1$
2,3,7,43,1807,3263443	$5 \cdot 41 \cdot 89 \cdot 5119 \cdot 114031$	$547 \cdot 607 \cdot 1033 \cdot 31051$
2,3,7,43,1823,193667	$5 \cdot 36931 \cdot 3453019$	$37 \cdot 449 \cdot 38380619$
2,3,7,47,395,779731	$31 \cdot 71 \cdot 5939 \cdot 46511$	$13 \cdot 46767665587$
2,3,7,47,403,19403	$5 \cdot 101 \cdot 30565373$	15435513367 (prime)
2,3,7,47,415,8111	$251 \cdot 269 \cdot 98411$	6646612311 (prime)
2,3,7,47,583,1223	$5 \cdot 29 \cdot 241 \cdot 40277$	1407479767 (prime)
2,3,7,55,179,24323	$9181 \cdot 1095449$	$67 \cdot 103 \cdot 1457371$
2,3,11,23,31,47059	$19 \cdot 116552759$	$19 \cdot 116552759$

$\underline{k = 7}$

$(n_1, \ldots, n_k)$	$\prod_{i=1}^{k} n_i - 1$	$\prod_{i=1}^{k} n_i + 1$
2,3,7,43,1807,3263443,10650056950807	$15541 \cdot 38780342479 \cdot 188197244219$	$29881 \cdot 67003 \cdot 9119521 \cdot 6212157481$
2,3,7,43,1807,3263447,213001400915	$17 \cdot 240131 \cdot 5556966386354188067$	$362464859 \cdot 625848207273l7729$
2,3,7,43,1807,3263591,71480133827	$7477 \cdot 2907138253 \cdot 35023852553$	$5 \cdot 1890875263 \cdot 80523769616513$
2,3,7,43,1807,3264187,14298637519	$5 \cdot 519 \cdot 19 \cdot 19267 \cdot 875960006253011$	$596059 \cdot 255538497028486753$
2,3,7,43,1823,193667,637617223447	$5849 \cdot 26926271 \cdot 2581441251359$	$10243 \cdot 32491 \cdot 1221602263409851$
2,3,7,43,3262,4051,2558951	$37 \cdot 59 \cdot 27983710363519$	61088439723561979 (prime)
2,3,7,43,3559,3667,33816127	$5 \cdot 17 \cdot 353 \cdot 26563596744757$	$577 \cdot 36857 \cdot 37478716883$
2,3,7,47,395,779731,607979652631	36963925801270344569529 (prime)	$14479 \cdot 117594511 \cdot 217096324699$
2,3,7,47,395,779831,6020372531	$191 \cdot 4241 \cdot 7621 \cdot 592999740779$	$1 \cdot 33279394787344850632l$
2,3,7,47,403,19403,15435513367	$239 \cdot 419 \cdot 2379196062425981$	$1021 \cdot 2333546257467l9063$
2,3,7,47,415,8111,6646612311	$31 \cdot 31 \cdot 71 \cdot 829 \cdot 15629 \cdot 49942679$	$19 \cdot 409 \cdot 5557 \cdot 1022402698813$
2,3,7,47,583,1223,1407479767	$1831 \cdot 11161 \cdot 96937735031$	$127 \cdot 38977 \cdot 400195490437$
2,3,7,55,179,24323,10057317271	$29 \cdot 2311 \cdot 5881 \cdot 256634582371$	101149630679497570171 (prime)
2,3,7,67,187,283,334651	$733 \cdot 67989255821$	$5 \cdot 139 \cdot 419 \cdot 479 \cdot 357281$
2,3,11,17,101,149,3109	$61819 \cdot 849179$	$13 \cdot 4038107431$
2,3,11,23,31,47059,2214502423	$5 \cdot 4789 \cdot 1970279 \cdot 103946471$	$6961 \cdot 1513457 \cdot 4590859291$
2,3,11,23,31,47063,442938131	$37 \cdot 127 \cdot 208761638439227$	$5 \cdot 5 \cdot 7 \cdot 5605548223005301$
2,3,11,23,31,47095,59897203	$19 \cdot 928771 \cdot 7522333121$	$7 \cdot 7 \cdot 109 \cdot 566857 \cdot 43844863$
2,3,11,23,31,47131,30382063	$43 \cdot 1193 \cdot 2311 \cdot 8429 \cdot 67433$	$5 \cdot 5 \cdot 3083 \cdot 874266518009$
2,3,11,23,31,47243,12017087	$46062647 \cdot 579990991$	$17321 \cdot 23293 \cdot 66217343$
2,3,11,23,31,47423,6114059	$5 \cdot 59 \cdot 178681 \cdot 258852119$	13644326865136507 (prime)
2,3,11,23,31,49759,866923	$5 \cdot 405990274405861$	$331 \cdot 6132783601297$
2,3,11,23,31,60563,211031	$2017 \cdot 298181849369$	$5 \cdot 5 \cdot 7 \cdot 109 \cdot 31529897257$
2,3,11,25,29,1097,2753	$7 \cdot 9601 \cdot 2150207$	144508961851 (prime)
2,3,11,31,35,67,369067	$17 \cdot 23833 \cdot 4370449$	$1553 \cdot 1140203147$
2,3,13,25,29,67,2981	$2113 \cdot 5345273$	$4783 \cdot 2361397$

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REMAINDER FORMULAS INVOLVING GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION

Synthetic division schemes to calculate the linear remainder when a polynomial is divided by a quadratic are used in numerical algorithms, such as Bairstow's method, for finding quadratic factors of polynomials. In this paper formulas for the linear remainder are derived in terms of the coefficients of the polynomial and coefficients of the quadratic divisor. These take a particularly compact form when expressed in terms of generalized Fibonacci and Lucas polynomials. Three different forms of the remainder are considered and second-order linear partial differential equations are introduced which have the linear remainder coefficients as solutions. Ordinary differential equations satisfied by two families of Fibonacci and Lucas polynomials are derived using identities which relate them to the generalized polynomials, and nonpolynomial solutions are deduced from corresponding solutions of the partial differential equations.

For any polynomial it is proved there exists a two-variable potential function with the property that its critical points correspond to coefficients in the quadratic factors of the polynomial. The potential function is defined by the linear remainder coefficients and an explicit formula is obtained in terms of the coefficients of the polynomial and generalized Lucas polynomials.

Hoggatt and Long [6] and Frei [2] have shown the generalized Fibonacci polynomials $F_n(x, y)$ and generalized Lucas polynomials $L_n(x, y)$ satisfy the Binet formulas

$$F_n(x, y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n(x, y) = \alpha^n + \beta^n$$

for $n \geq 0$ where α and β are the zeros of $t^2 - xt - y$ so that

$$\alpha = \tfrac{1}{2}(x + \sqrt{x^2 + 4y}), \quad \beta = \tfrac{1}{2}(x - \sqrt{x^2 + 4y})$$

and

$$\alpha + \beta = x, \quad \alpha\beta = -y, \quad \alpha - \beta = \sqrt{x^2 + 4y}.$$

These formulas are used to obtain many of the results in the following sections.

2. REMAINDER FORMULAS

The remainder coefficients $F(x, y)$ and $G(x, y)$ are defined by

$$P(t) = (t^2 - xt - y)Q(t) + F(x, y)(t - x) + G(x, y), \tag{1}$$

where $Q(t)$ is the quotient when a polynomial $P(t)$ is divided by $t^2 - xt - y$. Let

$$\begin{aligned} P(t) &= a_N t^N + a_{N-1} t^{N-1} + \cdots + a_1 t + a_0, \\ Q(t) &= b_N t^{N-2} + b_{N-1} t^{N-3} + \cdots + b_3 t + b_2, \end{aligned} \quad (N \geq 2)$$

and equate coefficients of powers of t in (1), giving the recurrence relation

$$\begin{aligned} &b_N = a_N, \quad b_{N-1} = a_{N-1} + xb_N, \\ &b_n = a_n + xb_{n+1} + yb_{n+2}, \end{aligned} \qquad (n = N-2, N-3, \ldots, 1, 0),$$

where $F(x, y) = b_1$ and $G(x, y) = b_0$. These form the basis for the synthetic division calculation of the remainder coefficients, described by Mathews [7], for numerical values of x and y.

Although these recurrence relations can be used to generate expressions for b_1 and b_0, it is simpler to obtain explicit formulas for the remainder coefficients by substituting $t = \alpha$ and $t = \beta$ in (1) giving

$$P(\alpha) = -\beta F(x, y) + G(x, y), \quad P(\beta) = -\alpha F(x, y) + G(x, y), \tag{2}$$

respectively. It follows that

$$F(x, y) = \frac{P(\alpha) - P(\beta)}{\alpha - \beta}, \quad G(x, y) = \frac{\alpha P(\alpha) - \beta P(\beta)}{\alpha - \beta},$$

and, using the Binet formulas, these can be further simplified to

$$F(x, y) = \sum_{n=0}^{N} a_n F_n(x, y), \quad G(x, y) = \sum_{n=0}^{N} a_n F_{n+1}(x, y).$$

If the linear remainder in (1) is taken instead as $F(x, y)t + H(x, y)$, which is the form used by Fröberg [3], then $H(x, y) = G(x, y) - xF(x, y)$ and it can be shown that

$$H(x, y) = \frac{\alpha P(\beta) - \beta P(\alpha)}{\alpha - \beta} = a_0 + y\sum_{n=1}^{N} a_n F_{n-1}(x, y).$$

Similarly, by taking the remainder as $F(x, y)(t - \frac{1}{2}x) + L(x, y)$, we have $L(x, y) = G(x, y) - \frac{1}{2}xF(x, y)$ and $L(x, y) = \frac{1}{2}[P(\alpha) + P(\beta)] = \frac{1}{2}\sum_{n=0}^{N} a_n L_n(x, y)$.

3. DIFFERENTIAL EQUATIONS

Consider the linear partial differential equations

$$\frac{\partial^2 \ell}{\partial x^2} - x\frac{\partial^2 \ell}{\partial x \partial y} - y\frac{\partial^2 \ell}{\partial y^2} = \frac{\partial \ell}{\partial y} \tag{3}$$

$$\frac{\partial^2 f}{\partial x^2} - x\frac{\partial^2 f}{\partial x \partial y} - y\frac{\partial^2 f}{\partial y^2} = 2\frac{\partial f}{\partial y}, \tag{4}$$

$$\frac{\partial^2 h}{\partial x^2} - x\frac{\partial^2 h}{\partial x \partial y} - y\frac{\partial^2 h}{\partial y^2} = -\frac{x}{y}\frac{\partial h}{\partial x}. \tag{5}$$

The change of variables from x, y, to α, β in the region $x^2 + 4y > 0$, where (3)-(5) are classified as hyperbolic, transforms them into their canonical form.

It can be shown, using the chain rule, that the canonical form of (3) is

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = 0,$$

from which the general solution is $\ell = \ell_1(\alpha) + \ell_2(\beta)$. Substituting $\ell_1(\alpha) = \alpha^n$ and $\ell_2(\beta) = \beta^n$ gives $\ell = L_n(x, y)$ as a solution of (3), and the principle of superposition means the remainder coefficient $L(x, y) = \sum_{n=0}^{N} a_n L_n(x, y)$ is also a solution. Substituting $\ell_1(\alpha) = \alpha^n$ and $\ell_2(\beta) = -\beta^n$ gives $\ell = \sqrt{x^2 + 4y} F_n(x, y)$; hence, $\sqrt{x^2 + 4y} \sum_{n=0}^{N} c_n F_n(x, y)$ is another solution of (3) for arbitrary constants c_n.

Similarly, the canonical form of (4) can be derived as

$$\frac{\partial^2}{\partial\alpha\partial\beta}[(\alpha - \beta)f] = 0,$$

from which it can be deduced that the remainder coefficients

$$F(x, y) = \sum_{n=0}^{N} a_n F_n(x, y), \quad G(x, y) = \sum_{n=0}^{N} a_n F_{n+1}(x, y), \text{ and also } \sum_{n=0}^{N} c_n L_n(x, y) / \sqrt{x^2 + 4y}$$

are solutions of (4).

The canonical form of (5) is

$$\frac{\partial^2}{\partial\alpha\partial\beta}\left[\frac{\alpha - \beta}{\alpha\beta} h\right] = 0,$$

which can be shown to have as solutions the remainder coefficient

$$H(x, y) = a_0 + y\sum_{n=1}^{N} a_n F_{n-1}(x, y) \quad \text{and} \quad \left[c_0 x + y\sum_{n=1}^{N} c_n L_{n-1}(x, y)\right]\Big/\sqrt{x^2 + 4y}.$$

The above solutions also apply in the region $x^2 + 4y < 0$, where (3)-(5) are classified as elliptic, although it would be appropriate to replace $\sqrt{x^2 + 4y}$ by $\sqrt{-x^2 - 4y}$ in the nonpolynomial solutions.

It is not the purpose of this paper to investigate all solutions of (3)-(5), but it is not difficult to see that definition of $F_n(x, y)$ and $L_n(x, y)$ for $n < 0$ (see [2]) and allowing infinite sums, subject to any convergence conditions being satisfied, would produce further solutions.

The single variable polynomials $F_n(1, z)$ and $L_n(1, z)$, $n \geq 0$, with the properties

$$F_n(x, y) = x^{n-1} F_n(1, z), \tag{6}$$

$$L_n(x, y) = x^n L_n(1, z), \tag{7}$$

where $z = y / x^2$, are referred to as the Fibonacci and Lucas polynomials, respectively, by Doman and Williams [1]. Galvez and Devesa [4] have shown that they satisfy the ordinary differential equations

$$z(1 + 4z)\frac{d^2 F_n}{dz^2} - [(n-1) + 2(2n-5)z]\frac{dF_n}{dz} + (n-1)(n-2)F_n = 0, \tag{8}$$

$$z(1 + 4z)\frac{d^2 L_n}{dz^2} - [(n-1) + 2(2n-3)z]\frac{dL_n}{dz} + n(n-1)L_n = 0, \tag{9}$$

which may also be proved by substituting (6) and (7) into (4) and (3), respectively.

Using the earlier results, it can be shown that a second linearly independent solution of (8) is $L_n(1, z) / \sqrt{|1+4z|}$, and a second linearly independent solution of (9) is $\sqrt{|1+4z|}\, F_n(1, z)$.

The polynomials $F_n(u, 1)$ and $L_n(u, 1)$, also referred to as Fibonacci and Lucas polynomials by Hoggatt and Bicknell [5], are related to the generalized polynomials by

$$F_n(x, y) = y^{(n-1)/2} F_n(u, 1), \quad L_n(x, y) = y^{n/2} L_n(u, 1),$$

where $u = x / \sqrt{y}$. Substitution into (4) and (3) shows they satisfy

$$(4+u^2)\frac{d^2 F_n}{du^2} + 3u\frac{dF_n}{du} - (n^2-1)F_n = 0,$$

$$(4+u^2)\frac{d^2 L_n}{du^2} + u\frac{dL_n}{du} - n^2 L_n = 0,$$

which also have solutions $L_n(u, 1) / \sqrt{u^2+4}$ and $\sqrt{u^2+4}\, F_n(u, 1)$, respectively.

4. A POTENTIAL FUNCTION

Differentiating (1) with respect to x and rearranging gives

$$tQ(t) = (t^2 - xt - y)\frac{\partial Q(t)}{\partial x} + \frac{\partial F}{\partial x}(t-x) + \frac{\partial G}{\partial x} - F, \tag{10}$$

whereas, differentiating (1) with respect to y, multiplying by t and rearranging gives

$$tQ(t) = (t^2 - xt - y)\left[t\frac{\partial Q(t)}{\partial y} + \frac{\partial F}{\partial y}\right] + \frac{\partial G}{\partial y}(t-x) + x\frac{\partial G}{\partial y} + y\frac{\partial F}{\partial y}. \tag{11}$$

Comparing (10) and (11) gives

$$\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}, \tag{12}$$

$$\frac{\partial G}{\partial x} = F + x\frac{\partial G}{\partial y} + y\frac{\partial F}{\partial y}. \tag{13}$$

Equation (12) is the condition for the existence of a $\phi(x, y)$, defined as the potential function of $P(t)$ and also denoted by $\phi[P(t)]$, with the properties

$$\frac{\partial \phi}{\partial x} = G(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = F(x, y). \tag{14}$$

Substituting (14) into (13) proves that ϕ satisfies (3); hence, it may be expressed in the form $\ell_1(\alpha) + \ell_2(\beta)$. It is easily shown, using the chain rule (14) and (2), that

$$\frac{\partial \phi}{\partial \alpha} = P(\alpha) \quad \text{and} \quad \frac{\partial \phi}{\partial \beta} = P(\beta),$$

and therefore,

$$\phi = \sum_{n=0}^{N} \frac{a_n}{n+1}(\alpha^{n+1} + \beta^{n+1}) = \sum_{n=0}^{N} \frac{a_n}{n+1} L_{n+1}(x, y),$$

where $\phi[0]$ is defined to be zero.

If $F(x^*, y^*) = G(x^*, y^*) = 0$, then from (1) and (14) it follows that the polynomial $P(t)$ has a quadratic factor $t^2 - x^*t - y^*$ if and only if (x^*, y^*) is a critical point of its potential function $\phi[P(t)]$. Obviously any linear combination of generalized Lucas polynomials, excluding the constant $L_0(x, y)$, may be considered as the potential function of some polynomial. In the case when this polynomial has N distinct real roots, its potential function has $N(N-1)/2$ critical points all deducible from pairwise multiplication of the linear factors of the polynomial. Then calculation of the roots, say by the Newton-Raphson method, would generally be a computationally efficient way of finding the larger number of critical points of the linear combination of generalized Lucas polynomials when $N \geq 4$.

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A NOTE ON GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

We let F_n represent the n^{th} *Fibonacci number*. In [2] and [3] we find relationships between the Fibonacci numbers and their associated matrices. The purpose of this paper is to develop relationships between the generalized Fibonacci numbers and the permanent of a $(0, 1)$-matrix. The *k-generalized Fibonacci sequence* $\{\mathbf{g}_n^{(k)}\}$ is defined as: $\mathbf{g}_1^{(k)} = \mathbf{g}_2^{(k)} = \cdots = \mathbf{g}_{k-2}^{(k)} = 0$, $\mathbf{g}_{k-1}^{(k)} = \mathbf{g}_k^{(k)} = 1$, and, for $n > k \geq 2$,

$$\mathbf{g}_n^{(k)} = \mathbf{g}_{n-1}^{(k)} + \mathbf{g}_{n-2}^{(k)} + \cdots + \mathbf{g}_{n-k}^{(k)}. \tag{1.1}$$

We call $\mathbf{g}_n^{(k)}$ the n^{th} *k-generalized Fibonacci number.*

For example, if $k = 8$, then $\mathbf{g}_1^{(8)} = \cdots = \mathbf{g}_6^{(8)} = 0$, $\mathbf{g}_7^{(8)} = \mathbf{g}_8^{(8)} = 1$, and the sequence of 8-generalized Fibonacci numbers is given by $0, 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, \ldots$.

When $k = 3$, the fundamental recurrence relation $\mathbf{g}_{n+1}^{(3)} = \mathbf{g}_n^{(3)} + \mathbf{g}_{n-1}^{(3)} + \mathbf{g}_{n-2}^{(3)}$ can also be defined by the vector recurrence relation

$$\begin{pmatrix} \mathbf{g}_{n-1}^{(3)} \\ \mathbf{g}_n^{(3)} \\ \mathbf{g}_{n+1}^{(3)} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{g}_{n-2}^{(3)} \\ \mathbf{g}_{n-1}^{(3)} \\ \mathbf{g}_n^{(3)} \end{pmatrix}. \tag{1.2}$$

Letting

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{1.3}$$

and applying (1.2) n times, we have

$$\begin{pmatrix} \mathbf{g}_{n+1}^{(3)} \\ \mathbf{g}_{n+2}^{(3)} \\ \mathbf{g}_{n+3}^{(3)} \end{pmatrix} = A^n \begin{pmatrix} \mathbf{g}_1^{(3)} \\ \mathbf{g}_2^{(3)} \\ \mathbf{g}_3^{(3)} \end{pmatrix}. \tag{1.4}$$

Similarly, for the k-generalized sequence

$$\mathbf{g}_{n+1}^{(k)} = \mathbf{g}_n^{(k)} + \mathbf{g}_{n-1}^{(k)} + \cdots + \mathbf{g}_{n-k+1}^{(k)}, \tag{1.5}$$

the matrix and the vector recurrence relation are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}_{k \times k},$$

and

$$\begin{pmatrix} \mathbf{g}_{n+1}^{(k)} \\ \mathbf{g}_{n+2}^{(k)} \\ \vdots \\ \mathbf{g}_{n+k}^{(k)} \end{pmatrix} = A^n \begin{pmatrix} \mathbf{g}_1^{(k)} \\ \mathbf{g}_2^{(k)} \\ \vdots \\ \mathbf{g}_k^{(k)} \end{pmatrix}. \tag{1.6}$$

We now consider the relationship between $\mathbf{g}_n^{(k)}$ and the *permanent* of a $(0,1)$-matrix. The *permanent* of an n-square matrix $A=[a_{ij}]$ is defined by

$$\operatorname{per} A = \sum_{\alpha\in S_n}\prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.7}$$

where the summation extends over all permutations σ of the symmetric group S_n. A matrix is said to be a $(0,1)$-matrix if each of its entries is either 0 or 1.

Let $A=[a_{ij}]$ be an $m\times n$ real matrix with row vectors $\alpha_1,\alpha_2,\ldots,\alpha_m$. We say A *is contractible on column* (resp. *row*) k if column (resp. *row*) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik}\neq 0\neq a_{jk}$ and $i\neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called *the contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki}\neq 0\neq a_{kj}$ and $i\neq j$, then the matrix $A_{k:ij}=[A_{ij:k}^T]^T$ is called *the contraction of A on row k relative to columns i and j*.

We say that A *can be contracted to a matrix B* if either $B=A$ or there exist matrices A_0, A_1, $\ldots, A_t$ $(t\geq 1)$ such that $A_0=A$, $A_t=B$, and A_r is a contraction of A_{r-1} for $r=1,\ldots,t$.

2. k-GENERALIZED FIBONACCI NUMBERS

In [1], we find the following result.

Lemma 1: Let A be a nonnegative integral matrix of order $n>1$ and let B be a contraction of A. Then

$$\operatorname{per} A = \operatorname{per} B. \tag{2.1}$$

Furthermore, if we let $\mathcal{F}^{(n,k)}=[f_{ij}]$ be the $n\times n$ $(0,1)$-$(k+1)^{\text{st}}$ (*super diagonal*) *matrix* defined by

$$\mathcal{F}^{(n,k)} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & & & & \ddots & \vdots \\ & & \ddots & \ddots & & & & & \ddots & 0 \\ \vdots & & & \ddots & \ddots & & \ddots & & & 1 \\ \vdots & & & & \ddots & \ddots & & & & \vdots \\ 0 & & & \cdots & \cdots & & 0 & 1 & & 1 \end{bmatrix}, \tag{2.2}$$

where $f_{11}=\cdots=f_{1k}=1$ and $f_{1k+1}=\cdots=f_{1n}=0$, then $\mathcal{F}^{(n,k)}$ is contractible on column 1 relative to rows 1 and 2. In particular, if $k=2$, then $\mathcal{F}^{(n,k)}$ is turned to be the $(0,1)$-tridiagonal (*Toeplitz*) matrix $T^{(n)}$ of order n.

Lemma 2: Let $T_p^{(n)} = [t_{ij}]$ be the p^{th} contraction of the matrix $T^{(n)}$, $1 \le p \le n-2$. Then $t_{11} = F_{p+2}$ and $t_{12} = F_{p+1}$, where F_p is the p^{th} *Fibonacci number*, $p = 1, 2, \ldots, n-2$.

Proof: We use induction on p. Since

$$T_1^{(n)} = \begin{bmatrix} 2 & 1 & & 0 & \\ 1 & 1 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & 0 & & 1 & 1 \end{bmatrix},$$

the case for $p = 1$ is true. Since

$$T_{p-1}^{(n)} = \begin{bmatrix} F_{p+1} & F_p & & & 0 \\ 1 & 1 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & 0 & & 1 & 1 \end{bmatrix},$$

by the induction assumption, $T_{p-1}^{(n)}$ is contractible on column 1 relative to rows 1 and 2. Thus,

$$T_p^{(n)} = \begin{bmatrix} F_{p+1} + F_p & F_{p+1} & & 0 & \\ 1 & 1 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & 0 & & 1 & 1 \end{bmatrix}.$$

However, $F_{p+1} + F_p = F_{p+2}$, $t_{11} = F_{p+2}$ and $t_{12} = F_{p+1}$, so the proof is complete.

Lemma 3: Let $\mathcal{F}_t^{(n,k)} = [f_{ij}]$ be the t^{th} contraction of $\mathcal{F}^{(n,k)}$, $1 \le t \le n-2$. Then, for $k > t+1$,

$$\begin{aligned} f_{11} &= \cdots f_{1k-t} = \mathbf{g}_{k+t}^{(k)}, \\ f_{1j} &= f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}, \end{aligned} \qquad k-t+1 \le j \le n-t,$$

and, for $k \le t+1$,

$$\begin{aligned} f_{11} &= g_{k+t}, \\ f_{1j} &= f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}, \end{aligned} \qquad 2 \le j \le n-t.$$

In any case, if $f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)} < 0$, we let f_{ij} be zero.

Proof: We proceed by induction. The result is easily established for $t = 1$. We now assume the theorem is true for t and consider $\mathcal{F}_{t+1}^{(n,k)}$. We examine two cases.

For the first case, assume $k > t+1$. Let $\mathcal{F}_t^{(n,k)} = [f_{ij}]$. Then $f_{11} = \cdots = f_{1k-t} = \mathbf{g}_{k+t}^{(k)}$ and $f_{1j} = f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}$, $k-t+1 \le j \le n-t$. Let $\mathcal{F}_{t+1}^{(n,k)} = [f_{ij}^{\dagger}]$. By contradiction,

$$\begin{aligned} f_{1q}^{\dagger} &= f_{11} + f_{1p},\ p = 2, \ldots, k = t,\ q = 1, \ldots, k-t-1 \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t-1}^{(k)} + \cdots + \mathbf{g}_{t}^{(k)}. \end{aligned}$$

Since $k > t+1$, $\mathbf{g}_t^{(k)} = 0$. Thus, $f_{1q}^{\dagger} = \mathbf{g}_{k+t+1}^{(k)}$, $q = 1, \ldots, k-(t+1)$, and

$$\begin{aligned} f_{1k+t}^{\dagger} &= f_{11} + f_{1k-t+1} \\ &= f_{11} + f_{1k-t} - \mathbf{g}_{t+k-t+1-2}^{(k)} \\ &= f_{11} + f_{1k-t} - \mathbf{g}_{k-1}^{(k)} \\ &= f_{1k-t-1}^{\dagger} - \mathbf{g}_{(t+1)+(k-t)-2}^{(k)}. \end{aligned}$$

Hence, $f_{1k-t}^{\dagger} = f_{1k-t-1}^{\dagger} - \mathbf{g}_{(t+1)+(k-t)-2}^{(k)}$. So, by the recurrence relation

$$f_{1j}^{\dagger} = f_{1j-1}^{\dagger} - \mathbf{g}_{(t+1)+j-2}^{(k)},\ k-t \le j \le n-(t-1).$$

For the second case, we let $k \le t+1$. If $t = 1$, then $k = 2$ and we are done, by Lemma 2. Let $\mathcal{F}_t^{(n,k)} = [f_{ij}]$. Then $f_{11} = \mathbf{g}_{k+t}^{(k)}$ and $f_{1j} = f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}$, $2 \le j \le n-t$. Let $\mathcal{F}_{t+1}^{(n,k)} = [f_{ij}^{\dagger}]$. Then, by Lemma 1,

$$\begin{aligned} f_{11}^{\dagger} &= f_{11} + f_{12} \\ &= \mathbf{g}_{k+t}^{(k)} + f_{11} - \mathbf{g}_{t+2-2}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t}^{(k)} - \mathbf{g}_{t}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t-1}^{(k)} + \cdots + \mathbf{g}_{t}^{(k)} - \mathbf{g}_{t}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \cdots + \mathbf{g}_{t+1}^{(k)} \\ &= \mathbf{g}_{k+t+1}, \end{aligned}$$

$$\begin{aligned} f_{12}^{\dagger} &= f_{11} + f_{13} \\ &= f_{11} + f_{12} - \mathbf{g}_{t3-2}^{(k)} \\ &= f_{11} + (f_{11} - \mathbf{g}_{t}^{(k)}) - \mathbf{g}_{t+1}^{(k)} \\ &= \mathbf{g}_{k+t+1}^{(k)} - \mathbf{g}_{t+1}^{(k)} \\ &= f_{11}^{\dagger} - \mathbf{g}_{(t+1)+2-2}^{(k)}, \end{aligned}$$

so that $f_{12}^{\dagger} = f_{11}^{\dagger} - \mathbf{g}_{(t+1)+2-2}^{(k)}$. Thus, by the recurrence relation, $f_{ij}^{\dagger} = f_{1j-1}^{\dagger} - \mathbf{g}_{(t+1)+j-2}^{(k)}$ and the proof is completed.

Theorem 1: Let $\mathbf{g}_{n+1}^{(k)}$ be the $(n+1)^{\text{st}}$ k-generalized Fibonacci number, $n \ge k$. Then

$$\operatorname{per} \mathcal{F}^{(n,k)} = \mathbf{g}_{n+k-1}^{(k)}. \tag{2.3}$$

Proof: Since $\mathcal{F}^{(n,k)}$ is contractible, $\mathcal{F}^{(n,k)}$ can be contracted to a 2-square integral matrix B. By Lemma 3,

$$B = \mathcal{F}_{n-2}^{(n,k)} = \begin{bmatrix} \mathbf{g}_{n+k-2}^{(k)} & \mathbf{g}_{n+k-2}^{(k)} - \mathbf{g}_{n-2}^{(k)} \\ 1 & 1 \end{bmatrix},$$

and by Lemma 1,

$$\begin{aligned} \operatorname{per} \mathcal{F}^{(n,k)} = \operatorname{per} B &= \mathbf{g}_{n+k-2}^{(k)} + \mathbf{g}_{n+k-2}^{(k)} - \mathbf{g}_{n-2}^{(k)} \\ &= \mathbf{g}_{n+k-2}^{(k)} + \mathbf{g}_{n+k-3}^{(k)} + \cdots + \mathbf{g}_{n-2}^{(k)} - \mathbf{g}_{n-2}^{(k)} \\ &= \mathbf{g}_{n+k-2}^{(k)} + \mathbf{g}_{n+k-3}^{(k)} + \cdots + \mathbf{g}_{n-1}^{(k)} \\ &= \mathbf{g}_{n+k-1}^{(k)}, \end{aligned}$$

and the proof is completed.

Corollary: The $(n+1)^{\text{st}}$ Fibonacci number is equal to the permanent of the $(0,1)$-tridiagonal matrix of order n.

The next theorem shows that we can find a nontridiagonal matrix whose permanent also equals the $(n+1)^{\text{st}}$ Fibonacci number.

Theorem 2: Let

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 0 & 1 & 1 \end{bmatrix}_{n\times n}.$$

Then

$$\operatorname{per} P^T U P = F_{n+1}. \tag{2.4}$$

for any permutation matrix P.

Proof: The matrix U can be contracted on column 1 so that

$$U_1 = \begin{bmatrix} 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 1 & 1 & 1 \end{bmatrix},$$

where $(U_1)_{11} = 1 = F_2$ and $(U_1)_{12} = 2 = F_3$. Furthermore, the matrix U_1 can be contracted on column 1 so that

$$U_2 = \begin{bmatrix} 2 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 1 & 1 & 1 \end{bmatrix},$$

where $(U_2)_{11} = 2 = F_3$ and $(U_2)_{12} = 3 = F_4$. Continuing this process, we have

$$U_t = \begin{bmatrix} F_{t+1} & F_{t+2} & F_{t+2} & F_{t+2} & \cdots & F_{t+2} \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & & \cdots & 0 & 1 & 1 \end{bmatrix}$$

for $1 \le t \le n-2$. Hence,

$$U_{n+3} = \begin{bmatrix} F_{n+2} & F_{n-1} & F_{n-1} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which, by contraction of U_{n-3} on column 1, gives

$$U_{n-2} = \begin{bmatrix} F_{n-1} & F_{n-1}+F_{n-2} \\ 1 & 1 \end{bmatrix} = U_{n-2} = \begin{bmatrix} F_{n-1} & F_n \\ 1 & 1 \end{bmatrix}.$$

Applying Lemma 1, we have

$$\text{per } U = \text{per } U_t = \text{per } U_{n-2} = F_n + F_{n-1} = F_{n+1}.$$

Since the permanent is permutation similarity invariant, the proof is completed.

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NEW EDITORIAL POLICIES

The Board of Directors of The Fibonacci Association during their last business meeting voted to incorporate the following two editorial policies effective January 1, 1995

1. All articles submitted for publication in *The Fibonacci Quarterly* will be blind refereed.
2. In place of Assistant Editors, *The Fibonacci Quarterly* will change to utilization of an Editorial Board.

ON A SYSTEM OF SEQUENCES DEFINED BY A RECURRENCE RELATION

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1. INTRODUCTION

Sequences defined by recurrence relations have been studied in many papers. Some of these studies treated the system of sequences defined by a recurrence relation. For instance, Lucas [6] studied the second-order case; Shannon and Horadam [8] dealt with the third-order recurrence relations.

The purpose of this note is to summarize some properties of the system of m sequences $\{u_{k,n}\}$ (where $k = 1, 2, \ldots, m$) defined by the recurrence relation

$$u_{k,n} = P_1 u_{k,n-1} + P_2 u_{k,n-2} + \cdots + P_m u_{k,n-m} \tag{1}$$

with initial conditions

$$u_{k,n} = \delta_{k,n+1} \text{ (where } k = 1, 2, \ldots, m;\ n = 0, 1, \ldots, m-1), \tag{2}$$

where the right-hand side stands for Kronecker's delta.

We will first write down the fundamental relations, and then consider the calculation of $u_{k,n}$. Finally, we will deal with some applications.

2. FUNDAMENTAL RELATIONS

A few leading terms for each of these sequences can be found in the following table:

$k \backslash n$	0	1	2	$\cdots$	$\cdots$	$\cdots$	$m-1$	m	$m+1$	$m+2$
1	1	0	0	$\cdots$	$\cdots$	$\cdots$	0	P_m	P_1P_m	$P_m(P_1^2+P_2)$
2	0	1	0	$\cdots$	$\cdots$	$\cdots$	0	P_{m-1}	$P_m+P_1P_{m-1}$	$P_{m-1}(P_1^2+P_2)+P_1P_m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	0	$\cdots$	$\cdots$	$\cdots$	0	1	0	P_2	$P_3+P_1P_2$	$P_2(P_1^2+P_2)+P_1P_3+P_4$
m	0	$\cdots$	$\cdots$	$\cdots$	0	0	1	P_1	$P_2+P_1^2$	$P_1(P_1^2+P_2)+P_1P_2+P_3$

Now, the fundamental relations

$$u_{1,n+1} = P_m u_{m,n},\ u_{k,n+1} = u_{k-1,n} + P_{m-k-1} u_{m,n} \text{ (for } k = 2, \ldots, m) \tag{3}$$

can be established easily by induction.

Using the matrices

$$U_n = \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ \vdots \\ u_{m-1,n} \\ u_{m,n} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & \cdots & 0 & P_m \\ 1 & 0 & \cdots & 0 & P_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & P_2 \\ 0 & 0 & \cdots & 1 & P_1 \end{pmatrix},$$

these relations can be written as

$$U_{n+1} = TU_n \tag{4}$$

which gives us

$$U_n = T^n U_0, \ T^n = (U_n, U_{n+1}, U_{n+2}, \ldots, U_{n+m-1}). \tag{5}$$

The generating functions

$$G_k(x) = u_{k,0} + u_{k,1}x + u_{k,2}x^2 + \cdots + u_{k,n}x^n + \cdots \ (\text{where } k = 1, 2, \ldots, m)$$

for these sequences are given by

$$G_k(x) = x^{k-1}H_k(x) / H_0(x), \tag{6}$$

where $H_k(x) = 1 - P_1x - P_2x^2 - \cdots - P_{m-k}x^{m-k}$ for $k = 0, 1, \ldots, m-1$ and $H_m(x) = 1$.

3. CALCULATION OF TERMS

From the generating function (6), we can easily determine the formula for $u_{m,n}$, which is

$$u_{m,m+n-1} = \sum_{s=1}^{n} \sum_{\substack{r_1+r_2+\cdots+r_s=n \\ 1\le r_i \le m \ (i=1,2,\ldots,s)}} P_{r_1}P_{r_2}\cdots P_{r_s},$$

where the summation runs over all the decompositions of n into the integers not exceeding m.

Following the method of Shannon and Horadam [7], it is easy to see that

$$u_{m,m+n-1} = \sum_{i_m=0}^{[t_m/m]} \cdots \sum_{i_3=0}^{[t_3/3]} \sum_{i_2=0}^{[t_2/2]} \frac{(t_1 + i_2 + \cdots + i_m)!}{t_1!\, i_2! \cdots i_m!} P_1^{t_1} P_2^{i_2} \cdots P_m^{i_m},$$

where $t_m = n$, and $t_k = t_{k+1} - (k+1)i_{k+1}$ for $k = m-1, \ldots, 2, 1$.

If the coefficients $P_1, P_2, \ldots, P_m$ are given numerically, we have an $O(m^2 \log n)$ algorithm for computing U_n by using (4) and (5). To see this, examine Er [2] or Gries and Levin [4].

In the case of $P_m \neq 0$, we can also define $u_{k,n}$ and U_n for negative values of n by using the recurrence relation (1) in the opposite direction. Formulas (4) and (5) are also valid for negative n and, in fact,

$$T^{-1} = \begin{pmatrix} Q_m & 1 & 0 & \cdots & 0 \\ Q_{m-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_2 & 0 & 0 & \cdots & 1 \\ Q_1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $Q_1 = P_m^{-1}$ and $Q_k = -P_{k-1}P_m^{-1}$ for $k = 2, 3, \ldots, m$. Thus, we have a similar algorithm for computing U_n when n is negative.

Following the ideas of Barakat [1] who used 2×2 matrices, we obtain similar formulas for the $m\times m$ matrices.

Theorem 1: Let X be an $m\times m$ matrix that has the characteristic equation

$$\lambda^m - P_1\lambda^{m-1} - \cdots - P_{m-1}\lambda - P_m = 0. \tag{7}$$

Then we have

$$X^n = u_{m,n}X^{m-1} + u_{m-1,n}X^{m-2} + \cdots + u_{2,n}X + u_{1,n}E, \tag{8}$$

where the coefficients are defined by (1) and (2).If X is regular, then $P_m \neq 0$, so that (8) is valid even for negative values of n.

Proof: For $n = 0, 1, \ldots, m-1$, (8) is valid by (2). On the other hand, we have

$$X^m = P_1X^{m-1} + P_2X^{m-2} + \cdots + P_{m-1}X + P_mE \tag{9}$$

by (7). Using this equality, we complete the proof for positive n by induction. To prove (8) for negative n, we use (1) and (9) in the opposite direction.

Next, we consider the evaluation of some series related to the sequences $\{u_{k,n}\}$.

Theorem 2: Let

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots \tag{10}$$

be a function defined by the power series in x that has the radius of convergence R with

$$R > \max_{1\le i\le m} |\lambda_i|, \tag{11}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of the characteristic equation

$$\varphi(\lambda) = \lambda^m - P_1\lambda^{m-1} - \cdots - P_{m-1}\lambda - P_m = 0$$

of the difference equations given in (1). For the sequences $\{u_{k,n}\}$ defined by (1) and (2), we consider the series

$$f_k = \sum_{n=0}^{\infty} c_n u_{k,n} \quad (k = 1, \ldots, m). \tag{12}$$

If all the λ_i's are distinct, f_k has the representation

$$f_k = (-1)^{m-k}\sum_{i=1}^{m} f(\lambda_i)\frac{p_{m-k,i}}{\varphi'(\lambda_i)} \quad (k = 1, 2, \ldots, m), \tag{13}$$

where $p_{0,i} = 1$, and $p_{h,i}$ stands for the elementary symmetric polynomial of degree h in $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m$, for $1 \le h \le m-1$.

Proof: Using (8) for $X = T$ as defined in (4), we have

$$f(T) = \sum_{n=0}^{\infty} c_n \sum_{k=1}^{m} u_{k,n}T^{k-1} = \sum_{k=1}^{m} f_k T^{k-1}.$$

If we compare this formula with Sylvester's formula (see Frazer et al. [3])

$$f(T)=\sum_{i=1}^{m} f(\lambda_i)\frac{(T-\lambda_1E)\cdots(T-\lambda_{i-1}E)(T-\lambda_{i+1}E)\cdots(T-\lambda_mE)}{(\lambda_i-\lambda_1)\cdots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_m)},$$

we have (13), since the minimal polynomial of T is of degree m in this case.

Applying this formula for $m=2$ and $m=3$ to the functions e^x, $\sin x$, $\cos x$, $\sinh x$, and so on, we obtain many of the formulas shown in [1], [7], [8], and [9].

If $\lambda_j=\lambda_i$, the formula for f_k corresponding to (13) will be given by taking the limit as λ_j tends to λ_i in (13).

For the sequence $\{u_n\}$ satisfying the same recurrence as (1) with initial conditions $u_n=a_n$ for $k=0,1,\ldots,m-1$, we have

$$\sum_{n=0}^{\infty} c_n u_n = \sum_{k=1}^{m} a_{k-1} f_k,$$

which can also be calculated directly from the general solution.

J. Z. and J. S. Lee [5] applied this latter method to a function $f(x)$ that had a geometric progression for its coefficients in order to characterize some B-power fractions.

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A NOTE ON MULTIPLICATIVE PARTITIONS OF BIPARTITE NUMBERS

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1. INTRODUCTION

For a positive integer n, let $f(n)$ be the number of essentially different ways of writing n as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. For example, $f(12)=4$, since $12=2\cdot 6=3\cdot 4=2\cdot 2\cdot 3$. This function was introduced by Hughes and Shallit [1], who proved that $f(n)\le 2n^{\sqrt{2}}$ for all n. Mattics and Dodd [2] improved the inequality so that $f(n)\le n/\log n$ for all $n>1, n\neq 144$. Landman and Greenwell [3] generalized the notion of multiplicative partitions to bipartite numbers. For positive integers m and n, $mn>1$, let $f_2(m,n)$ denote the number of essentially different ways of writing the pair (m,n) as a product $\prod_{1\le i\le k}(a_i,b_i)$, where $a_ib_i>1$ for $1\le i\le k$ and where multiplication is done coordinate-wise. Similarly, for positive integers m and n, $mn>1$, let $g(m,n)$ be the number of essentially different ways of writing the pair (m,n) as a product $\prod_{1\le i\le k}(a_i,b_i)$, where $a_i>1$, $b_i>1$ for $1\le i\le k$. Let $g(1,1)=f_2(1,1)$ be 1. For example, $f_2(6,2)=5$, since $(6,2)=(6,1)(1,2)=(3,2)(2,1)=(3,1)(2,2)=(3,1)(1,2)(2,1)$ and $g(6,4)=2$, since $(6,4)=(3,2)(2,2)$. In a recent paper [3], Landman and Greenwell proved that

$$f_2(m,n)<\frac{(mn)^{1.516}}{\log(mn)},$$

and they conjectured that 1.516 can be replaced by 1.251. In this paper we approximate $g(m,n)$ by a completely multiplicative function $h(mn)$. Using this approximation, we prove that

$$f_2(m,n)<(2160)^2(mn)^{1.143}.$$

We also prove that $f_2(m,n)<(mn)^{1.251}/\log(mn)$ for $mn\ge 10^{83}$.

2. NOTATIONS

For convenience, we will define some notations and conventions used in this paper. Let N denote the set of all positive integers and p_i denote the i^{th} prime (i.e., $p_i=2, p_2=3$, etc.). The prime factorizations of $m>1$ and $n>1$ may be considered as $m=\prod_{i=1}^{t}q_i^{\alpha_i}$, $n=\prod_{j=1}^{s}s_i^{\beta_j}$, where $\{q_i\}$ are the distinct prime factors of m, $\{s_j\}$ are the distinct prime factors of n, and $\{\alpha_i\}$, $\{\beta_j\}$ are nonincreasing sequences of positive integers. Let $\hat{m}=\prod_{i=1}^{t}p_i^{\alpha_i}\le m$ and $\hat{n}=\prod_{j=1}^{s}p_j^{\beta_j}\le n$. Then, clearly, $f_2(m,n)=f_2(\hat{m},\hat{n})$. Hence, let $M=\{a\in N\mid a=\prod_{i=1}^{k}p_i^{\theta_i}>1$, where $\{\theta_i\}$ is a nonincreasing sequence of positive integers and $k\in N\}$. The completely multiplicative functions h and T are defined on N as follows:

(a) $T(1)=1$; $T(2)=(7/4)$; $T(3)=(11/4)$; $T(p_r)=(r+7/4)$ for $r\geq 3$; $T(ab)=T(a)T(b)$ for a, $b\in N$

(b) $h(1)=1$; $h(p_i)=r_i$, where $\{r_i\}_{i\geq 1}$ is the sequence of real numbers defined by

$$r_{i+1}=1+\prod_{j=1}^{i}\frac{r_j}{r_j-1}\sqrt{1+\left(\prod_{k=1}^{i}\frac{r_k}{r_k-1}-1\right)^2}\quad \text{for } i\geq 1 \text{ and } r_i=2;$$

$h(ab)=h(a)h(b)$ for $a,b\in N$.

For any positive integer k, the multiplicative function $d^{(k)}$ is defined on N as follows:

$$d^{(k)}(a)=\sum_{\substack{\ell|a\\ p_i\nmid\ell \text{ for all } i\geq k}}1$$

[i.e., $d^{(k)}(p_i^b)=1$ for $i\geq k$; $d^{(k)}(p_i^b)=b+1$ for $i<k$].

3. PROOF OF THE MAIN RESULT

Throughout this paper, all variables represent nonnegative integers, unless otherwise specified. The following lemma will be used frequently in the remainder of our work.

Lemma 1: $r_i+2<r_{i+1}<r_i+2.5$ if $i\geq 7$.

Proof: Fix $i\geq 6$ and let $y=\prod_{j=1}^{i}r_j/(r_j-1)$. Then $y>4$ and

$$\begin{aligned}r_{i+2}&=y\left(1+\frac{1}{y\sqrt{(y-1)^2+1}}\right)\sqrt{\left(y-1+\frac{1}{\sqrt{(y-1)^2+1}}\right)^2+1}+1\\&<\left(y+\frac{1}{\sqrt{(y-1)^2+1}}\right)\left(\sqrt{(y-1)^2+1}+\frac{1}{\sqrt{(y-1)^2+1}}\right)+1\\&=r_{i+1}+\frac{y}{\sqrt{(y-1)^2+1}}+1+\frac{1}{(y-1)^2+1}<r_{i+1}+2.5.\end{aligned}$$

Similarly, one can prove that $r_{i+2}>r_{i+1}+2$. Q.E.D.

Lemma 2: If $m=\prod_{i=1}^{t}p_i^{\alpha_i}\in M$ and $1\leq s\leq t$, then

$$\sum_{\ell|m}\frac{d^{(s)}(\ell)}{h(\ell)}\leq\frac{r_t}{r_t-1}\cdot\prod_{i=1}^{t-1}\left(\frac{r_i}{r_i-1}\right)^2.$$

Proof: From Lemma 1, we know that $r_i>1$ for all $i\geq 1$. Then we have

$$\begin{aligned}\sum_{\ell|m}\frac{d^{(s)}(\ell)}{h(\ell)}&=\prod_{i=1}^{s-1}\left(\sum_{j=0}^{\alpha_i}\frac{j+1}{r_i^j}\right)\cdot\prod_{a=s}^{t}\left(\sum_{k=0}^{\alpha_a}\frac{1}{r_a^k}\right)\leq\prod_{i=1}^{s-1}\left(\sum_{j=0}^{\infty}\frac{j+1}{r_i^j}\right)\cdot\prod_{a=s}^{t}\left(\sum_{k=0}^{\infty}\frac{1}{r_a^k}\right)\\&=\prod_{i=1}^{s-1}\left(\frac{r_i}{r_i-1}\right)^2\cdot\prod_{a=s}^{t}\left(\frac{r_a}{r_a-1}\right).\quad \text{Q.E.D.}\end{aligned}$$

With the aid of Lemma 2, we establish an upper bound on $g(m,n)$.

Proposition 1: The function $g(m,n)$ satisfies the inequality:

$$g(m,n) \le h(m)\cdot h(n) = \left(\prod_{i=1}^{t} r_i^{\alpha_i}\right)\left(\prod_{j=1}^{s} r_j^{\beta_j}\right),$$

where $m = \prod_{i=1}^{t} p_i^{\alpha_i}$ and $n = \prod_{j=1}^{s} p_j^{\beta_j}$.

Proof: It is enough to show that $g(m,n) \le h(mn)$ for $m, n \in M$, since, for any positive integers $a = \prod_{i=1}^{c} q_i^{a_i}$ and $b = \prod_{j=1}^{d} s_j^{b_j}$,

$$g(a,b) = g\left(\prod_{i=1}^{c} p_i^{a_i}, \prod_{j=1}^{d} p_j^{b_j}\right) \quad \text{and} \quad h\left(\prod_{i=1}^{c} p_i^{a_i}\right)h\left(\prod_{j=1}^{d} p_j^{b_j}\right) \le h(a)h(b),$$

where $\{q_i\}$ are the distinct prime factors of a, $\{s_j\}$ are the distinct prime factors of b, and $\{a_i\}$, $\{b_i\}$ are nonincreasing sequences of positive integers. The statement clearly holds for the case $n \le 2$, since $g(m,1) = 0$ for $m > 1$. Hence, without loss of generality, we may assume $m \ge n > 2$. Let $m' = m/p_t$ and $n' = n/p_s$. First, we introduce some sets:

$$S = \{\{(a_i,b_i)\}_{1\le i\le e} \mid (1)\ (m,n) = \textstyle\prod_{1\le i\le e}(a_i,b_i),\ (2)\ a_j, b_j \ge 2 \text{ for all } 1 \le j \le e,$$
$$(3)\ a_j \ge a_{j+1};\ \text{and if } a_j = a_{j+1},\ \text{then } b_j \ge b_{j+1} \text{ for all } 1 \le j \le e-1\};$$

$$A(\ell,k) = \{\{(a_i,b_i)\}_{1\le i\le e} \in S \mid (a_{i_0}, b_{i_0}) = (p_t\ell, p_s k) \text{ for some } 1 \le i_0 \le e\};$$

$$B(\ell,k) = \{\{(a_i,b_i)\}_{1\le i\le e} \in S \mid p_t \nmid a_{i_2},\ p_s \nmid b_{i_1} \text{ and } (a_{i_1}a_{i_2}, b_{i_1}b_{i_2}) = (p_t\ell, p_s k)$$
$$\text{for some } 1 \le i_1, i_2 \le e\};$$

$$C(\ell,k) = \{\{(a_1,b_1),(a_2,b_2)\} \mid (1)\ p_t \nmid a_2, a_2 \ge 2,\ (2)\ p_s \nmid b_1, b_1 \ge 2,$$
$$(3)\ (a_1a_2, b_1b_2) = (p_t\ell, p_s k)\}.$$

Since

$$S = \bigcup_{\substack{\ell \mid m' \\ k \mid n'}} (A(\ell,k) \cup B(\ell,k)),$$

we get the following inequality:

$$g(m,n) = |S| \le \sum_{\substack{\ell \mid m' \\ k \mid n'}} (|A(\ell,k)| + |B(\ell,k)|) \le \sum_{\substack{\ell \mid m' \\ k \mid n'}} (|A(\ell,k)| + |A(\ell,k)|\cdot|C(\ell,k)|)$$

$$\le \sum_{\substack{\ell \mid m' \\ k \mid n'}} g\left(\frac{m'}{\ell}, \frac{n'}{k}\right)\{1 + (d^{(t)}(\ell) - 1)(d^{(s)}(k) - 1)\}.$$

From Lemma 2 and the induction hypothesis, we have

$$g(m,n) \le \sum_{\substack{\ell \mid m' \\ k \mid n'}} \frac{h(m')h(n')}{h(\ell)h(k)}\{(d^{(t)}(\ell) - 1)(d^{(s)}(k) - 1) + 1\} =$$

$$= h(m')h(n')\sum_{\substack{\ell|m' \\ k|n'}} \frac{d^{(t)}(\ell)d^{(s)}(k) - d^{(s)}(k)d^{(1)}(\ell) - d^{(t)}(\ell)d^{(1)}(k) + 2d^{(1)}(\ell)d^{(1)}(k)}{h(\ell)h(k)}$$

$$\leq \frac{h(m)}{r_t - 1}\frac{h(n)}{r_s - 1}(x^2y^2 - xy^2 - yx^2 + 2xy)$$

$$= h(m)h(n)\frac{(x-1)(y-1)+1}{\sqrt{1+(x-1)^2}\sqrt{1+(y-1)^2}} \leq h(m)h(n),$$

where $x = \prod_{i=1}^{t-1} r_i/(r_i - 1)$ and $y = \prod_{j=1}^{s-1} r_j/(r_j - 1)$. Q.E.D.

Lemma 3: If $m \in M$, $\lambda = 1.143$, then $h(m) \leq m^\lambda$.

Proof: From Lemma 1, we know that $r_i \leq 2.5i$ for all $i \geq 1$. Since λ satisfies the following two inequalities,

(a) $$\left(\prod_{i=1}^{s} p_i\right)^\lambda \geq \prod_{i=1}^{s} r_i \text{ for all } 1 \leq s \leq 12,$$

(b) $$p_i^\lambda \geq (i\log(i))^\lambda \geq i \cdot 12^{\lambda-1}(\log(12))^\lambda \geq 2.5i \geq r_i \text{ for all } i \geq 12,$$

we get $h\left(\prod_{i=1}^{t} p_i\right) \leq \left(\prod_{i=1}^{t} p_i\right)^\lambda$ for all $t \geq 1$. (**Note:** $p_i \geq i\log i$ for any positive integer i, see [4].)

From the induction hypothesis on $m \in M$, we have

$$h(m) = h\left(\prod_{i=1}^{t} p_i^{\alpha_i}\right) = h\left(\prod_{i=1}^{t} p_i\right)h\left(\prod_{i=1}^{t} p_i^{\alpha_i - 1}\right) \leq \left(\prod_{i=1}^{t} p_i\right)^\lambda \left(\prod_{i=1}^{t} p_i^{\alpha_i - 1}\right)^\lambda = m^\lambda,$$

where $m = \prod_{i=1}^{t} p_i^{\alpha_i}$. Q.E.D.

The following corollary is an immediate consequence of Proposition 1 and Lemma 3 above.

Corollary 1: $g(m,n) \leq (mn)^{1.143}$.

Lemma 4: For any positive integer t,

$$\prod_{i=1}^{t} \frac{r_i^2}{r_i - u_i} \leq 2160\left(\prod_{i=1}^{t} p_i\right)^\lambda,$$

where $\lambda = 1.143$ and $u_i = T(p_i)$ for $i \geq 1$.

Proof: Direct computation shows the inequality holds for $t \leq 24$. From Lemma 1 and the Appendix, we know that $2i + 7/4 < r_i < 2.5i$ for all $i \geq 25$. Fix $i \geq 25$. Then we have

$$\frac{r_i^2}{r_i - u_i} \leq \frac{(2.5i)^2}{(2i+1.75)-(i+1.75)} = 6.25i \leq 25^{\lambda-1}(\log 25)^\lambda i \leq (i\log i)^\lambda \leq (p_i)^\lambda. \text{ Q.E.D.}$$

In [2], Mattics and Dodd proved that $f_2(a,1) \leq T(a) \leq a$. Using this fact, we prove the following proposition.

Proposition 2: If $m=\prod_{i=1}^{t}p_i^{\alpha_i}$, $n=\prod_{j=1}^{s}p_j^{\beta_j}\in M$, then

$$f_2(m,n)\le\left(\prod_{i=1}^{t}\frac{r_i^{\alpha_i+1}-u_i^{\alpha_i+1}}{r_i-u_i}\right)\left(\prod_{j=1}^{s}\frac{r_j^{\beta_j+1}-u_j^{\beta_j+1}}{r_j-u_j}\right)\le(2160)^2(mn)^{1.143},$$

where $u_k=T(p_k)$ for $k\ge 1$.

Proof: For any factorization $(a_1,b_1)(a_2,b_2)\cdots(a_e,b_e)$ of (m,n), there exist unique integers ℓ and k such that

$$\ell=\prod_{\substack{1\le i\le e\\ b_i=1}}a_i \quad\text{and}\quad k=\prod_{\substack{1\le i\le e\\ a_i=1}}b_i.$$

By Proposition 1, we have

$$f_2(m,n)=\sum_{\substack{\ell|m\\ k|n}}g(m/\ell,n/k)f_2(\ell,1)f_2(1,k)\le\sum_{\substack{\ell|m\\ k|n}}h(m/\ell)h(n/k)T(\ell)T(k)$$

$$=\left(\sum_{\ell|m}h(m/\ell)T(\ell)\right)\left(\sum_{k|n}h(n/k)T(k)\right)=\left(\prod_{i=1}^{t}\sum_{\ell=0}^{\alpha_i}h(p_i^{\ell})T(p_i^{\alpha_i-\ell})\right)\left(\prod_{j=1}^{s}\sum_{k=0}^{\beta_j}h(p_j^{k})T(p_j^{\beta_j-k})\right)$$

$$=\left(\prod_{i=1}^{t}\frac{r_i^{\alpha_i+1}-u_i^{\alpha_i+1}}{r_i-u_i}\right)\left(\prod_{j=1}^{s}\frac{r_j^{\beta_j+1}-u_j^{\beta_j+1}}{r_j-u_j}\right)\le\left(\prod_{i=1}^{t}\frac{r_i^{\alpha_i+1}}{r_i-u_i}\right)\left(\prod_{j=1}^{s}\frac{r_j^{\beta_j+1}}{r_j-u_j}\right).$$

By Lemma 3 and Lemma 4,

$$f_2(m,n)\le\left(\prod_{i=1}^{t}\frac{r_i^2}{r_i-u_i}\right)\left(\prod_{j=1}^{s}\frac{r_j^2}{r_j-u_j}\right)\left(\prod_{i=1}^{t}r_i^{\alpha_i-1}\right)\left(\prod_{j=1}^{s}r_j^{\beta_j-1}\right)\le(2160)^2(mn)^{1.143}.\ \text{Q.E.D.}$$

Theorem 3: For any pair of positive integers m and n,

$$f_2(m,n)<\frac{(mn)^{1.251}}{\log(mn)}$$

for $mn>10^{83}$.

Proof: From Proposition 2, it is enough to show that the following inequality holds for $mn>10^{83}$:

$$(2160)^2(mn)^{1.143}\le\frac{(mn)^{1.251}}{\log(mn)}.$$

Let $f(t)=t^{0.108}-(2160)^2\log(t)$. Then we have $f(t)\ge 0$ for $t\ge 10^{83}$, since $f(10^{83})\ge 0$ and $f'(t)\cdot t=0.108t^{0.108}-(2160)^2>0$ for all $t\ge 10^{83}$. Hence, we have

$$(mn)^{1.251}-(mn)^{1.143}(2160)^2\log(mn)=(mn)^{1.143}f(mn)\ge 0$$

for $mn\ge 10^{83}$. Q.E.D.

APPENDIX

The following table shows the sequence $\{r_i\}_{i\geq 1}$ used in this paper.

n	1	2	3	4	5	6
r_n	2	3.82843	6.35584	8.80023	11.1791	13.5137
n	7	8	9	10	11	12
r_n	15.8164	18.0947	20.3538	22.5992	24.8273	27.0461

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