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ON LUCAS v-TRIANGLES

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1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$. Let A and B be fixed nonzero integers with $(A, B) = 1$, and write $\Delta = A^2 - 4B$. We will assume $\Delta \neq 0$, which excludes degenerate cases including $|A| = 2$ and $B = 1$. Define $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ as follows:

$$u_0 = 0, \ u_1 = 1 \text{ and } u_{n+1} = Au_n - Bu_{n-1} \text{ for } n \in \mathbb{Z}^+; \tag{1.1}$$

$$v_0 = 2, \ v_1 = A \text{ and } v_{n+1} = Av_n - Bv_{n-1} \text{ for } n \in \mathbb{Z}^+. \tag{1.2}$$

They are called Lucas sequences. The addition formulas

$$u_{m+n} = \frac{u_m v_n + u_n v_m}{2} \text{ and } v_{m+n} = \frac{v_m v_n + \Delta u_m u_n}{2} \text{ for } m, n \in \mathbb{N} \tag{1.3}$$

are well known. A list of such basic identities can be found in [3].

If $A \neq \pm 1$ or $B \neq 1$, then $u_1, u_2, \ldots$ are nonzero by [1], and so are $v_1 = u_2 / u_1$, $v_2 = u_4 / u_2$, ... In the case $A^2 = B = 1$, we noted in [1] that $u_n = 0 \Leftrightarrow 3 | n$. If $v_n = 0$, then $u_{2n} = u_n v_n = 0$; hence, $3 | n$ and $u_n = 0$, which is impossible since $v_n^2 - \Delta u_n^2 = 4B^n$ (cf. [3]). Thus, $v_0, v_1, v_2, \ldots$ are all nonzero.

We set $v_n! = \prod_{0 < k \leq n} v_k$ for $n \in \mathbb{N}$, and regard an empty product as value 1. For $n, k \in \mathbb{N}$ with $n \geq k$, we define the *Lucas v-triangle* $\left\{{n \atop k}\right\}$ as follows:

$$\left\{{n \atop k}\right\} = \frac{v_n!}{v_k! \, v_{n-k}!}. \tag{1.4}$$

(This definition is not new in the case $A = 1$ and $B = -1$; the reader may consult Wells [5].) Similarly, in the case $A \neq \pm 1$ or $B \neq 1$, Lucas u-triangles can be defined in terms of the sequence $\{u_n\}_{n \in \mathbb{N}}$ (cf. [1]).

Let q be a positive integer. Clearly, $v_q \equiv A^q \pmod{B}$ and hence $(B, v_q) = 1$. Let v_q^* denote the largest divisor of v_q prime to $v_0, \ldots, v_{q-1}$. Then v_q^* is odd since $v_0 = 2$. It is known that $(v_m, v_n) \in \{1, 2, |v_{(m,n)}|\}$ for $m, n \in \mathbb{N}$ (cf. [3] or (2.21) of [4]). If $q \nmid n$, then $(v_{(q,n)}, v_q^*) = 1$ and so $(v_q^*, v_n) = (v_q^*, (v_q, v_n)) = 1$.

For $m \in \mathbb{Z}$, we let $D(m)$ denote the ring of rationals in the form a / b with $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$, and $(b, m) = 1$. When $r \in D(m)$, by $x \equiv r \pmod{m}$ we mean that x can be written as $r + my$ with $y \in D(m)$. For a positive integer q, if $0 \leq k \leq n < q$ then $\left\{{n \atop k}\right\}$ lies in $D(v_q^*)$.

Let p be a prime. A famous theorem of Lucas concerning Pascal's triangles (i.e., binomial coefficients) states that

$$\binom{mp + s}{np + t} \equiv \binom{m}{n}\binom{s}{t} \pmod{p}$$

if m, n, s, t are nonnegative integers with $s, t < p$. An analogy to Lucas u-triangles was obtained by Kimball and Webb [2], by Wilson [6] in some special cases, and by Hu and Sun [1] for the general case. In this paper we aim to establish a similar result for Lucas v-triangles. Recall that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is strong divisible, i.e., $(u_m, u_n) = |u_{(m,n)}|$ for all $m, n \in \mathbb{N}$, while $\{v_n\}_{n \in \mathbb{N}}$ is not in general. This makes our goal more challenging.

Our main result is as follows.

Theorem: Let q be a positive integer. For $m, n \in 2\mathbb{N} = \{0, 2, 4, \ldots\}$ with $m \geq n$, and $s, t \in \mathbb{N}$ with $q > s \geq t$, we have

$$\binom{m/2}{n/2}\left\{\begin{matrix} mq+s \\ nq+t \end{matrix}\right\} \equiv \binom{m}{n}\left\{\begin{matrix} s \\ t \end{matrix}\right\}(-B^q)^{\frac{m-n}{2}(nq+t)+\frac{n}{2}(s-t)} \pmod{v_q^*}. \tag{1.5}$$

A proof of the theorem will be presented in Section 3; it depends on several lemmas given in the next section. Our method is different from that of [5] and [6].

2. THREE LEMMAS

As usual, for a real number x, we use $\lfloor x \rfloor$ to denote the greatest integer not exceeding x.

Lemma 2.1: Let $k \in \mathbb{Z}^+$ and $q \in \mathbb{N}$. Then

$$u_{kq} = u_q \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-i-1}{i} v_q^{k-1-2i}(-B^q)^i \tag{2.1}$$

and

$$v_{kq} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k}{k-i}\binom{k-i}{i} v_q^{k-2i}(-B^q)^i, \tag{2.2}$$

where

$$\frac{k}{k-i}\binom{k-i}{i} \in \mathbb{Z} \quad \text{for } i = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor.$$

This known result was included in [3].

From Lemma 2.1, we can deduce

Lemma 2.2: Let $k, q, r \in \mathbb{N}$. Then

$$2v_{kq+r} \equiv \begin{cases} 2v_r(-B^q)^{k/2} + \frac{k}{2}(-B^q)^{k/2-1}\Delta u_q u_r v_q \pmod{v_q^2} & \text{if } 2 \mid k, \\ \Delta u_q u_r(-B^q)^{(k-1)/2} + k(-B^q)^{(k-1)/2} v_r v_q \pmod{v_q^2} & \text{if } 2 \nmid k. \end{cases} \tag{2.3}$$

Moreover, providing $2 \nmid k$, we have

$$\frac{v_{kq}}{k} \equiv (-B^q)^{(k-1)/2} v_q \pmod{v_q^2}. \tag{2.4}$$

Proof: The case $k = 0$ is trivial. Below we let $k \in \mathbb{Z}^+$. Obviously,

$$\binom{k-1-\lfloor \frac{k-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor} v_q^{k-1-2\lfloor \frac{k-1}{2} \rfloor}(-B^q)^{\lfloor \frac{k-1}{2} \rfloor} = \begin{cases} \frac{k}{2}(-B^q)^{k/2-1} v_q & \text{if } 2 \mid k, \\ (-B^q)^{(k-1)/2} & \text{if } 2 \nmid k. \end{cases}$$

So, by (2.1), we have

$$u_{kq} \equiv u_q \times \begin{cases} \frac{k}{2}(-B^q)^{k/2-1}v_q \pmod{v_q^2} & \text{if } 2|k, \\ (-B^q)^{(k-1)/2} \pmod{v_q^2} & \text{if } 2 \nmid k. \end{cases}$$

Similarly, (2.2) implies that

$$v_{kq} \equiv \frac{k}{k-\lfloor\frac{k}{2}\rfloor}\binom{k-\lfloor\frac{k}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor}v_q^{k-2\lfloor\frac{k}{2}\rfloor}(-B^q)^{\lfloor\frac{k}{2}\rfloor}$$

$$\equiv \begin{cases} 2(-B^q)^{k/2} \pmod{v_q^2} & \text{if } 2|k, \\ k(-B^q)^{(k-1)/2}v_q \pmod{v_q^2} & \text{if } 2 \nmid k. \end{cases}$$

As $2v_{kq+r} = v_{kq}v_r + \Delta u_{kq}u_r$, (2.3) follows from the above.

Now suppose that k is odd. By Lemma 2.1,

$$\frac{v_{kq}}{k} = \sum_{i=0}^{\frac{k-1}{2}}\frac{1}{k-i}\binom{k-i}{k-2i}v_q^{k-2i}(-B^q)^i$$

$$= v_q(-B^q)^{(k-1)/2} + v_q^2\sum_{0\le i\le\frac{k-3}{2}}\frac{v_q^{k-2i-2}}{k-2i}\binom{k-i-1}{k-2i-1}(-B^q)^i.$$

For any prime p, clearly $p^{3-2}/3 \in D(p)$, and for $n = 4, 5, \ldots$ we also have $p^{n-2}/n \in D(p)$ because

$$(1+p-1)^{n-2} \ge 1+\binom{n-2}{1}(p-1)+(p-1)^{n-2} \ge 2+(n-2)(p-1) \ge n.$$

When $0 \le i \le (k-3)/2$, by the above, $v_q^{k-2i-2}/(k-2i) \in D(p)$ for any prime p dividing v_q, so $v_q^{k-2i-2}/(k-2i) \in D(v_q)$. Thus, we have the desired (2.4).

Lemma 2.3: Let q be any positive integer, and let m, n be even integers with $m \ge n \ge 0$. Then

$$\binom{m/2}{n/2}\left\{\begin{matrix}mq\\nq\end{matrix}\right\} \equiv \binom{m}{n}(-B^q)^{\frac{m-n}{2}nq} \pmod{v_q^*}. \tag{2.5}$$

Proof: Recall that $(v_q^*, 2B) = 1$. In view of (2.4), for $i = 1, 3, 5, \ldots$ we have

$$\frac{v_{iq}}{i} \equiv (-B^q)^{\frac{i-1}{2}}v_q \pmod{v_q^2}.$$

Observe that

$$\binom{m/2}{n/2}\prod_{\substack{0\le k<n\\2\nmid k}}\frac{v_{(m-k)q}}{v_{(n-k)q}} = \prod_{0\le j<n/2}\frac{m/2-j}{n/2-j}\cdot\prod_{\substack{0\le k<n\\2\nmid k}}\frac{m-k}{n-k}\cdot\prod_{\substack{0\le k<n\\2\nmid k}}\frac{v_{(m-k)q}/(m-k)}{v_{(n-k)q}/(n-k)}$$

$$= \prod_{0\le k<n}\frac{m-k}{n-k}\cdot\prod_{\substack{0\le k<n\\2\nmid k}}\frac{v_{(m-k)q}/((m-k)v_q)}{v_{(n-k)q}/((n-k)v_q)}$$

$$\equiv \binom{m}{n}\prod_{\substack{0\le k<n\\2\nmid k}}\frac{(-B^q)^{(m-k-1)/2}}{(-B^q)^{(n-k-1)/2}} = \binom{m}{n}(-B^q)^{\frac{m-n}{2}\cdot\frac{n}{2}} \pmod{v_q^*}.$$

By (2.2), for $i = 2, 4, 6, \ldots$ we have $v_{iq} \equiv 2(-B^q)^{i/2} \pmod{v_q^2}$, and hence $(v_{iq}, v_q^*) = 1$.

Whenever $0 \le j < nq$ and $j \not\equiv q \pmod{2q}$, we have $(v_{nq-j}, v_q^*) = 1$. Also,

$$2v_{mq-j} = 2v_{(m-n)q+(nq-j)} \equiv 2v_{nq-j}(-B^q)^{(m-n)/2} \pmod{v_q^*}$$

by (2.3). Thus,

$$\prod_{\substack{0 \le j < nq \\ 2q \nmid j-q}} \frac{v_{mq-j}}{v_{nq-j}} \equiv \prod_{\substack{0 \le j < nq \\ 2q \nmid j-q}} (-B^q)^{\frac{m-n}{2}} = (-B^q)^{\frac{m-n}{2}\left(nq-\frac{n}{2}\right)} \pmod{v_q^*}.$$

Combining the above, we obtain that

$$\binom{m/2}{n/2}\begin{Bmatrix} mq \\ nq \end{Bmatrix} = \binom{m/2}{n/2} \prod_{0 \le j < nq} \frac{v_{mq-j}}{v_{nq-j}}$$

$$= \binom{m/2}{n/2} \prod_{\substack{0 \le k < n \\ 2 \nmid k}} \frac{v_{(m-k)q}}{v_{(n-k)q}} \cdot \prod_{\substack{0 \le j < nq \\ 2q \nmid j-q}} \frac{v_{mq-j}}{v_{nq-j}}$$

$$\equiv \binom{m}{n}(-B^q)^{\frac{m-n}{2}\cdot\frac{n}{2}}(-B^q)^{\frac{m-n}{2}\left(nq-\frac{n}{2}\right)}$$

$$= \binom{m}{n}(-B^q)^{\frac{m-n}{2}nq} \pmod{v_q^*}.$$

This completes the proof of Lemma 2.3.

3. PROOF OF THE THEOREM

Recall that

$$\begin{Bmatrix} s \\ t \end{Bmatrix} \in D(v_q^*)$$

since $s < q$. Clearly,

$$\begin{Bmatrix} mq+s \\ nq+t \end{Bmatrix} = \frac{\prod_{(m-n)q < j \le mq} v_j}{\prod_{0 < j \le nq} v_j} \cdot \frac{\prod_{0 < r \le s}(2v_{mq+r})}{\prod_{0 < r \le t}(2v_{nq+r}) \cdot \prod_{0 < r \le s-t}(2v_{(m-n)q+r})}.$$

Applying Lemmas 2.2 and 2.3, we then get that

$$\binom{m/2}{n/2}\begin{Bmatrix} mq+s \\ nq+t \end{Bmatrix} \equiv \binom{m/2}{n/2}\begin{Bmatrix} mq \\ nq \end{Bmatrix} \frac{\prod_{0 < r \le s}(2v_r(-B^q)^{m/2})}{\prod_{0 < r \le t}(2v_r(-B^q)^{n/2}) \cdot \prod_{0 < r \le s-t}(2v_r(-B^q)^{(m-n)/2})}$$

$$\equiv \binom{m}{n}(-B^q)^{\frac{m-n}{2}nq} \frac{v_s!}{v_t! v_{s-t}!}(-B^q)^{\frac{m}{2}s-\frac{n}{2}t-\frac{m-n}{2}(s-t)}$$

$$\equiv \binom{m}{n}\begin{Bmatrix} s \\ t \end{Bmatrix}(-B^q)^{\frac{m-n}{2}(nq+t)+\frac{n}{2}(s-t)} \pmod{v_q^*}.$$

This completes the proof of the Theorem.

ACKNOWLEDGMENT

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AN OLYMPIAD PROBLEM, EULER'S SEQUENCE, AND STIRLING'S FORMULA

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1. INTRODUCTION

There are several ways of defining the real number e. The most common of them is to define e as the limit of the nondecreasing sequence

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n\geq 1}.$$

Related to this definition is the following problem proposed in 1990 at the Romanian County Olympiad: "Study the convergence of the sequence $\{x_n\}_{n\geq 1}$ defined by

$$\left(1+\frac{1}{n}\right)^{n+x_n} = e."$$

The problem is not hard to solve, but, surprisingly, a different approach to solving it than the one given originally by the proposers yields some interesting applications. The solution given by the proposers used l'Hôpital's rule. For this, we write

$$x_n = \frac{1}{\ln\left(1+\frac{1}{n}\right)} - n,$$

and then obtain

$$\lim_{x\to\infty} \frac{1-x\ln\left(1+\frac{1}{x}\right)}{\ln\left(1+\frac{1}{x}\right)} = \frac{1}{2}.$$

If one were to solve the problem in a different way, then a natural question related to convergence would be whether the sequence is bounded or not. The answer to this is given by the double inequality

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}, \tag{1}$$

which proves that the sequence is bounded and $x_n \in (0, 1)$. In view of this, one might ask if (1) can be refined to a similar pair of inequalities that incorporate 0.5 in the exponents. In other words, is sit true that, for a given $\varepsilon > 0$ and n sufficiently large, the following inequalities hold:

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}-\varepsilon} < e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}+\varepsilon}?$$

In order to answer this question, we will generalize (1) and show how the *generalized α-inequality* can be applied to various problems, namely: find a shorter proof of Stirling's formula

than the one given by D. S. Mitrinovic (see [3], pp. 181-84), solve the Olympiad problem mentioned before, and study the convergence of a general Euler-type series.

2. THE α-INEQUALITY

We prove the following

Proposition:

(a) If $0 \le \alpha < 0.5$, then there exists an $x(\alpha) \ge 0$ such that

$$\left(1+\frac{1}{x}\right)^{x+\alpha} < e, \quad x > x(\alpha);$$

(b) If $\alpha \ge 0.5$, then

$$\left(1+\frac{1}{x}\right)^{x+\alpha} > e, \quad x > 0.$$

Proof: For $\alpha \ge 0$, let $f_\alpha : (0, \infty) \to (0, \infty)$, $f_\alpha(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}$. Logarithmic differentiation of this function yields

$$f_\alpha'(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}\left[\ln\left(1+\frac{1}{x}\right) - \frac{\alpha+x}{x(x+1)}\right].$$

If we consider now the mapping $g_\alpha : (0, \infty) \to \mathbf{R}$, $g_\alpha(x) = \ln\left(1+\frac{1}{x}\right) - \frac{\alpha+x}{x(x+1)}$, then

$$g_\alpha'(x) = \frac{-x(1-2\alpha)+\alpha}{x^2(x+1)^2}.$$

We notice a couple of cases:

(i) If $\alpha \in [0, 0.5)$, then $g_\alpha'(x) < 0$ for all $x > x(\alpha) = \alpha/(1-2\alpha)$. Thus, g_α is nonincreasing on $(x(\alpha), \infty)$ and $g_\alpha(x) > \lim_{x \to \infty} g_\alpha(x) = 0$ for all $x > x(\alpha)$. This implies that $f_\alpha'(x) > 0$ for all $x > x(\alpha)$. Hence, f_α is strictly increasing on $(x(\alpha), \infty)$. Finally, using the fact that $\lim_{x \to \infty} f_\alpha(x) = e$, we infer that $f_\alpha(x) < e$ for all $x > x(\alpha)$.

(ii) If $\alpha \in [0.5, \infty)$, then $g_\alpha'(x) > 0$ for all $x > 0$. From this point, an argument similar to the one used before leads to the conclusion that $f_\alpha(x) > e$ for all $x > 0$.

Before we continue with our applications, let us note that the case $\alpha = 0.5$ is treated, among other inequalities involving exponentials, in [3, §3.6].

3. APPLICATIONS

A. If we let $\varepsilon \in (0, 0.5)$ and $\alpha = 0.5 - \varepsilon$ in (a), we see that $x_n > \frac{1}{2} - \varepsilon$ for all $n \ge [(1-2\varepsilon)/4\varepsilon]+1$. By (b), it is true that $x_n < \frac{1}{2} + \varepsilon$ for any $n \ge 1$; hence,

$$\left|x_n - \frac{1}{2}\right| < \varepsilon, \quad n \ge n(\varepsilon) = \left[\frac{1-2\varepsilon}{4\varepsilon}\right]+1,$$

which proves that the sequence converges, indeed, to 0.5.

B. It is well known that Euler's sequence

$$\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln n, \ n \geq 2,$$

is nondecreasing and converges to Euler's constant, $C = .57721566\ldots$. We show below that this fact is just a complex consequence of the α-inequality with $\alpha = 0, 1$ in the previous section. More generally, we use our proposition to study the convergence of the family of sequences:

$$\gamma_n(a) = \frac{1}{1+a} + \frac{1}{2+a} + \cdots + \frac{1}{n-1+a} - \ln n, \ n \geq 2, a \geq 0.$$

We will prove that $(\gamma_n(a))$ is convergent for any $a \geq 0$. Since

$$\gamma_{n+1}(a) - \gamma_n(a) = \frac{1}{n+a} - \ln\left(1+\frac{1}{n}\right),$$

if $a < 0.5$, then $\gamma_{n+1}(a) - \gamma_n(a) > 0$ for all $n \geq n(a) = \left[\frac{a}{1-2a}\right] + 2$, and if $a \geq 0.5$, then $\gamma_{n+1}(a) - \gamma_n(a) < 0$ for all $n \geq 2$. If we could prove that our sequence is also bounded, then convergence would follow automatically. Let us consider first the case when $a \in [0, 0.5)$. Since $a + 1 \geq 1$, we can write $\gamma_{k+1}(a+1) - \gamma_k(a+1) < 0$, $k \geq 2$. But

$$\gamma_k(a+1) = \gamma_{k+1}(a) + \ln\left(1+\frac{1}{k}\right) - \frac{1}{a+1}$$

implies that

$$\ln\left(1+\frac{1}{k}\right) - \ln\left(1+\frac{1}{k+1}\right) > \gamma_{k+2}(a) - \gamma_{k+1}(a), \ k \geq 2.$$

Now, if we let $k = 2, 3, \ldots, n-2$ and add these inequalities, we find that

$$\gamma_n(a) < \ln\frac{3}{2} + \gamma_3(a) - \ln\left(1+\frac{1}{n-1}\right) < \frac{1}{1+a} + \frac{1}{2+a} - \ln 2, \ n \geq 4,$$

which proves that our sequence is bounded and, hence, convergent. Denote its limit by $\gamma(a)$. Note that $\gamma(a) \in [m(a), M(a)]$, where $m(a) = \min\{\gamma_2(a), \ldots, \gamma_{n(a)}(a)\}$ and

$$M(a) = \max\left\{\gamma_2(a), \gamma_3(a), \frac{1}{1+a} + \frac{1}{2+a} - \ln 2\right\}.$$

For $a = 0$, $\gamma(0) = C$ and $n(0) = 2$; hence, $C \in [1 - \ln 2, 1.5 - \ln 2]$. Suppose now that $a = 0.5$. An easy computation gives

$$\gamma_n\left(\frac{1}{2}\right) = \gamma_{2n} - \gamma_n + 2\ln 2 - 2 \to C + 2\ln 2 - 2.$$

When $a \in (0.5, 1]$, we have

$$\gamma_n(a) \geq \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = \gamma_{n+1} + \ln\left(1+\frac{1}{n}\right) - 1$$

which implies $\gamma_n(a) \to \gamma(a) \in \left[C - 1, \frac{1}{1+a} - \ln 2\right]$. Finally, if $a \in (1, \infty)$, then

$$\gamma_n - \gamma_n(a) = a\left(\frac{1}{1+a} + \frac{1}{2(2+a)} + \cdots + \frac{1}{(n-1)(n-1+a)}\right)$$

$$< a\left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right) = \frac{a}{4} - \frac{a}{n};$$

hence, $\gamma_n(a) \to \gamma(a) \in \left[C - \frac{a}{4}, \frac{1}{1+a} - \ln 2 \right]$.

C. Stirling's formula asserts that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n.$$

It is well known that the result is closely related to the behavior of the gamma function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ for large values of x. This classical way of deriving Stirling's formula can be found, for example, in [1, pp. 20-24]. For different approaches, see also [2] and [4]. We use our proposition to give a proof which is different from the ones mentioned before. This proof uses an argument similar to, but shorter than, the one given by D. S. Mitrinovic. We will assume as known the following result due to Wallis:

$$\lim_{n \to \infty} \frac{(2n)!!}{(2n-1)!! \sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

For $\alpha > 0$, let $u_n(\alpha) = \frac{n!}{n^{n+\alpha} e^{-n}}$, $n \geq 2$. Then

$$\frac{u_n\left(\frac{1}{2}\right)}{u_{n+1}\left(\frac{1}{2}\right)} = \frac{1}{e} \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} > 1;$$

thus, $\left(u_n\left(\frac{1}{2}\right) \right)$ is nonincreasing and bounded below by 1. Therefore, $\lim_{n \to \infty} u_n\left(\frac{1}{2}\right) = u$ exists and is strictly positive. Note also that

$$\frac{u_n^2\left(\frac{1}{2}\right)}{u_{2n}\left(\frac{1}{2}\right)} = \frac{(2n)!! \sqrt{2}}{(2n-1)!! \sqrt{n}}.$$

If we let $n \to \infty$, we obtain $u = \sqrt{2\pi}$, which proves Stirling's formula. Note that in this formula the value $\alpha = 0.5$ is the best one, for

$$\lim_{n \to \infty} u_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < 0.5, \\ 0 & \text{if } \alpha > 0.5. \end{cases}$$

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❖❖❖

ON POLYNOMIALS RELATED TO DERIVATIVES OF THE GENERATING FUNCTION OF CATALAN NUMBERS

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1. INTRODUCTION AND SUMMARY

In [3] it has been shown that powers of the generating function $c(x)$ of Catalan numbers $\{C_n\}_{n \in \mathbb{N}_0} = \{1, 1, 2, 5, 14, 42, ...\}$, where $\mathbb{N}_0 := \{0, 1, 2, ...\}$ (nr. 1459 and A000108 of [8] and references of [3]) can be expressed in terms of a linear combination of 1 and $c(x)$ with coefficients replaced by certain scaled Chebyshev polynomials of the second kind. In this paper, derivatives of $c(x)$ are studied in a similar manner. The starting point is the following expression for the first derivative:

$$\frac{d\,c(x)}{dx} \equiv c'(x) = \frac{1}{x(1-4x)}(1 + (-1+2x)c(x)). \tag{1}$$

This equation is equivalent to the simple recurrence relation valid for C_n:

$$(n+2)C_{n+1} - 2(2n+1)C_n = 0, \quad n = -1, 0, 1, ..., \text{ with } C_{-1} = -1/2. \tag{2}$$

Equation (1) can, of course, also be found from the explicit form $c(x) = (1 - \sqrt{1-4x})/(2x)$. The result for the nth derivative is of the form

$$\frac{1}{n!}\frac{d^n c(x)}{dx^n} = \frac{1}{(x(1-4x))^n}(a_{n-1}(x) + b_n(x)c(x)), \tag{3}$$

with certain polynomials $a_{n-1}(x)$ of degree $n-1$ and $b_n(x)$ of degree n. These polynomials are found to be

$$b_n(x) = \sum_{m=0}^{n}(-1)^m B(n, m)x^{n-m}$$

with

$$B(n, m) := \binom{2n}{n}\binom{n}{m}\bigg/\binom{2m}{m}, \tag{4}$$

which defines a triangle of numbers for $n, m \in \mathbb{N}$, $n \geq m \geq 0$, where $\mathbb{N} := \{1, 2, 3, ...\}$. The first terms are depicted in Table 1 with $B(n, m) = 0$ for $n < m$. Another representation for the polynomials $b_n(x)$ is also found, i.e.,

$$b_n(x) = -2\sum_{k=0}^{n}C_{k-1}x^k(4x-1)^{n-k}. \tag{5}$$

Equating both forms of $b_n(x)$ leads to a formula involving convolutions of Catalan numbers with powers of an arbitrary constant $\lambda := (4x-1)/x$. This formula is given in (31). Equation (5) reveals the generating function of the polynomials $b_n(x)$ because it is a convolution of two functional sequences. The result is

$$g_b(x; z) := \sum_{n=0}^{\infty} b_n(x) z^n = \frac{\sqrt{1-4xz}}{1+(1-4x)z}.$$ (6)

TABLE 1. $B(n, m)$ **Central Binomial Triangle**

m n	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0	0
2	6	6	1	0	0	0	0	0	0	0	0
3	20	30	10	1	0	0	0	0	0	0	0
4	70	140	70	14	1	0	0	0	0	0	0
5	252	630	420	126	18	1	0	0	0	0	0
6	924	2772	2310	924	198	22	1	0	0	0	0
7	3432	12012	12012	6006	1716	286	26	1	0	0	0
8	12870	51480	60060	36036	12870	2860	390	30	1	0	0
9	48620	218790	291720	204204	87516	24310	4420	510	34	1	0
10	184756	923780	1385670	1108536	554268	184756	41990	6460	646	38	1

The other family of polynomials is

$$a_n(x) = \sum_{k=0}^{n} (-1)^k A(n+1, k+1) x^{n-k}$$

with the triangular array $A(n, m)$ defined for $m = 0$ by $A(n, 0) = C_n$, and for $n, m \in \mathbf{N}$ with $n \geq m > 0$ by the numbers

$$A(n, m) = \frac{1}{2} \binom{n}{m-1} \left[4^{n-m+1} - \binom{2n}{n} \Big/ \binom{2(m-1)}{m-1} \right].$$ (7)

The first terms of this triangular array of numbers are shown in Table 2 with $A(n, m) = 0$ for $n < m$. Both results (4) and (7) are solutions to recurrence relations which hold for $b_n(x)$ and $a_n(x)$ and their respective coefficients $B(n, m)$ and $A(n, m)$.

Another representation for the polynomials $a_n(x)$ is found to be

$$a_n(x) = \sum_{k=0}^{n} C_k x^k (4x - 1)^{n-k},$$ (8)

which shows that the generating function of these polynomials is

$$g_a(x; z) := \sum_{n=0}^{\infty} a_n(x) z^n = \frac{c(xz)}{1+(1-4x)z}.$$ (9)

Comparing (5) with (8) yields the following relation between these two types of polynomials

$$b_n(x) = (4x - 1)^n - 2x a_{n-1}(x), \quad n \in \mathbf{N}_0, \text{ with } a_{-1}(x) \equiv 0,$$ (10)

and between the coefficients

$$B(n, m) = \binom{n}{m} 4^{n-m} - 2A(n, m+1).$$ (11)

TABLE 2. $A(n, m)$ **Catalan Triangle**

n \ m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	2	5	1	0	0	0	0	0	0	0	0
3	5	22	9	1	0	0	0	0	0	0	0
4	14	93	58	13	1	0	0	0	0	0	0
5	42	386	325	110	17	1	0	0	0	0	0
6	132	1586	1686	765	178	21	1	0	0	0	0
7	429	6476	8330	4746	1477	262	25	1	0	0	0
8	1430	26333	39796	27314	10654	2525	362	29	1	0	0
9	4862	106762	185517	149052	69930	20754	3973	478	33	1	0
10	16796	431910	848830	781725	428772	152946	36646	5885	610	37	1

The triangle of numbers $A(n, m)$ is related to a rectangular array of integers $\hat{A}(n, m)$ with $\hat{A}(0, m) \equiv 1$, $\hat{A}(n, 0) = -C_n$ for $n \in \mathbf{N}$, and for $n \geq m \geq 1$ by

$$A(n, m) = -\hat{A}(n - m, m) + 2^{2(n-m)+1} \binom{n-1}{m-1}, \tag{12}$$

or with (7) for $m \in \mathbf{N}, n \in \mathbf{N}_0$, by

$$\hat{A}(n, m) = \frac{1}{2} \binom{n+m}{n+1} \left[\binom{2(n+m)}{n+m} \middle/ \binom{2(m-1)}{m-1} - 4^{n+1} \frac{m-1}{n+m} \right]. \tag{13}$$

Part of the array $\hat{A}(n, m)$ is shown in Table 3, where it is called $C4(n, m)$.

TABLE 3. $C4(n, m)$ **Catalan Array**

n \ m	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	-1	3	7	11	15	19	23
2	-2	10	38	82	142	218	310
3	-5	35	187	515	1083	1955	3195
4	-14	126	874	2934	7266	15086	27866
5	-42	462	3958	15694	44758	105102	216566
6	-132	1716	17548	80324	259356	679764	1546028
7	-429	6435	76627	397923	1435347	4154403	10338515
8	-1430	24310	330818	1922510	7663898	24281510	65635570
9	-4862	92378	1415650	9105690	39761282	136887322	399429602
10	-16796	352716	6015316	42438076	201483204	749032492	2346750900

It turns out that the mth column of the triangle of numbers $A(n, m)$ for $m = 0, 1, \ldots$ is determined by the generating function

$$c(x)\left(\frac{x}{1-4x}\right)^m.$$

The mth column of the triangle of numbers $B(n, m)$ for $m = 0, 1, \ldots$ is generated by

$$\frac{1}{\sqrt{1-4x}}\left(\frac{x}{1-4x}\right)^m.$$

This fact identifies the infinite dimensional matrices **A** and **B** as examples of Riordan matrices in the terminology of [7]. The matrix $\hat{\mathbf{A}}$ associated with $\hat{A}(n, m)$ is an example of a Riordan array.

Because differentiation of $c(x) = \sum_{k=0}^{\infty} C_k x^k$ leads to

$$\frac{1}{n!}\frac{d^m c(x)}{dx^n} = \sum_{k=0}^{\infty} C(n, k)x^k, \text{ with } C(n, k) := \frac{1}{n!}\prod_{j=1}^{n}(k+j)C_{n+k} = \frac{(2(n+k))!}{n!\,k!\,(n+k+1)!}, \qquad (14)$$

where $C(0, k) = C_k$, one finds, together with (3), the following identities for $n \in \mathbf{N}$, $p \in \{0, 1, \ldots, n-1\}$,

$$(D1): \sum_{k=0}^{p}(-1)^k C_k \binom{n}{p-k}\Bigg/\binom{2(n-p+k)}{n-p+k} = \frac{1}{2}\binom{n}{p+1}\left\{2^{2(p+1)}\Bigg/\binom{2n}{n} - 1\Bigg/\binom{2(n-p-1)}{n-p-1}\right\}$$

$$= A(n, n-p)\Bigg/\binom{2n}{n}, \qquad (15)$$

and for $n \in \mathbf{N}$, $k \in \mathbf{N}_0$,

$$(D2): \sum_{j=0}^{n}(-1)^j\left(\binom{n}{j}\Bigg/\binom{2j}{j}\right)\sum_{l=0}^{k}4^l\binom{n+l-1}{n-1}C_{k+j-l} = C(n, k)\Bigg/\binom{2n}{n}. \qquad (16)$$

The remainder of this paper provides proofs for the above statements.

2. DERIVATIVES

The starting point is equation (1) which can either be verified from the explicit form of the generating function $c(x)$ or by converting the recursion relation (2) for Catalan numbers into an equation for their generating function. A computation of

$$\frac{1}{(n+1)!}\frac{d^{n+1}c(x)}{dx^{n+1}} = \frac{1}{n+1}\frac{d}{dx}\left(\frac{1}{n!}\frac{d^n c(x)}{dx^n}\right),$$

with (3) taken as granted and equation (1), produces the following mixed relations between the quantities $a_n(x)$ and $b_n(x)$ and their first derivatives, valid for $n \in \mathbf{N}_0$,

$$(n+1)a_n(x) = x(1-4x)a'_{n-1}(x) + b_n(x) + n(8x-1)a_{n-1}(x), \qquad (17)$$

$$(n+1)b_{n+1}(x) = x(1-4x)b'_n(x) + (-(n+1) + 2(1+4n)x)b_n(x), \qquad (18)$$

with inputs $a_{-1}(x) \equiv 0$ and $b_0(x) \equiv 1$.

From (18), it is clear by induction that $b_n(x)$ is a polynomial of degree n. Again by induction, the same statement holds for $a_n(x)$ in (17). Therefore, we write, for $n \in \mathbf{N}_0$,

$$a_n(x) = \sum_{k=0}^{n} (-1)^k a(n, k) x^{n-k}, \tag{19}$$

$$b_n(x) = \sum_{k=0}^{n} (-1)^k B(n, k) x^{n-k}, \tag{20}$$

with the triangular arrays of numbers $a(n, k)$ and $B(n, k)$ with row number n and column number $k \le n$. The triangular array $a(n, k)$ will later be enlarged to another one which will then be called $A(n, k)$.

We first solve $b_n(x)$ in (18) by inserting (20) and deriving the recursion relation for the coefficients $B(n, m)$ after comparing coefficients of x^{n+1}, x^0, and x^{n-k} for $k = 0, 1, ..., n-1$.

$$x^{n+1}: \quad (n+1)B(n+1, 0) = 2(2n+1)B(n, 0), \tag{21}$$

$$x^0: \quad B(n+1, n+1) = B(n, n), \tag{22}$$

$$x^{n-k}: \quad (n+1)B(n+1, k+1) = (k+1)B(n, k) + 2(2(n+k)+3)B(n, k+1). \tag{23}$$

With the input $B(0, 0) = 1$, one deduces from (21) for the leading coefficient of $b_n(x)$

$$B(n, 0) = 2^n \frac{(2n-1)!!}{n!} = \frac{(2n)!}{n!n!} = \binom{2n}{n}, \tag{24}$$

and from (22)

$$B(n, n) \equiv 1, \quad \text{i.e.,} \quad b_n(0) = (-1)^n. \tag{25}$$

The double factorial $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$ appeared in (24).

In order to solve (23), we conjecture from Table 1 that, for $n, m \in \mathbf{N}$,

$$B(n, m) = 4B(n-1, m) + B(n-1, m-1), \tag{26}$$

with input $B(n, 0) = \binom{2n}{n}$ from (24).

If we use this conjecture in (23), written with $n \to n-1$, $k \to m-1$, we are led to consider the simple recursion

$$B(n, m) = \frac{n+1-m}{2(2m-1)} B(n, m-1). \tag{27}$$

The solution of this recursion is, for $n, m \in \mathbf{N}_0$,

$$B(n, m) = \frac{1}{2^m(2m-1)!!} \frac{n!}{(n-m)!} \binom{2n}{n} = \frac{m!n!}{(2m)!(n-m)!} \binom{2n}{n} = \binom{2n}{n}\binom{n}{m} \bigg/ \binom{2m}{m}. \tag{28}$$

With the Pochhammer symbol $(a)_n := \Gamma(n+a)/\Gamma(a)$, this result can also be written as

$$B(n, m) = ((2m+1)/2)_{n-m} 4^{m-n} / (n-m)!.$$

This result satisfies (21), i.e., (24), as well as (22), i.e., (25). It is also the solution to (23) provided we prove the conjecture (26) using $B(n, m)$ in (28). This can be done by inserting

$$B(n, m) = \frac{(2n)!m!}{(2m)!n!(n-m)!}$$

in (26). Thus, we have proved the following proposition.

Proposition 1: We have

$$b_n(x) = \sum_{k=0}^{n} (-1)^k B(n, k) x^{n-k}, \quad \text{where } B(n, k) = \binom{2n}{n}\binom{n}{k}\Big/\binom{2k}{k}.$$

This triangle of numbers as shown in Table 1 appears as A046521 in the database [8].

One can derive another explicit representation for the polynomials $b_n(x)$ by using (27) in (20):

$$(1 - 4x)b_n'(x) + 2(2n - 1)b_n(x) + 2\binom{2n}{n}x^n = 0. \tag{29}$$

This leads, together with (18), to the following inhomogeneous recursion relation for $b_n(x)$:

$$b_{n+1}(x) = (4x - 1)b_n(x) - 2C_n x^{n+1}, \quad b_0(x) \equiv 1. \tag{30}$$

Equation (29) can also be solved as first-order linear and inhomogeneous differential equation for $b_n(x)$.

Proposition 2: We have

$$b_n(x) = -2\sum_{k=0}^{n} C_{k-1}x^k(4x - 1)^{n-k},$$

where the C_k's are the Catalan numbers for $k \in N_0$ and $C_{-1} = -1/2$.

Proof: Iteration of (30). □

Proposition 3: The generating function $g_b(x; z) := \sum_{n=0}^{\infty} b_n(x)x^n$ for $\{b_n(x)\}$ is given by (6).

Proof: The alternative form of $b_n(x)$ given by equation (5) is a convolution of the functional sequences $\{-2C_{k-1}x^k\}_{n \in N_0}$ and $\{(4x - 1)^n\}_{n \in N_0}$, with generating functions $1 - 2xzc(xz) = \sqrt{1 - 4xz}$ and $1/(1 + (1 - 4x)z)$, respectively. Therefore, $g_b(x; z)$ is the product of these two generating functions. □

Comparing this alternative form (5) for $b_n(x)$ with the one given by (20), together with (28), proves the following identity in n and $\lambda := (4x - 1)/x$. The term $k = 0$ in the sum (5) has been written separately.

Corollary 1 (convolution of Catalan sequence and the sequence of powers of λ): For $n \in N$ and $\lambda \neq \infty$,

$$s_{n-1}(\lambda) := \lambda^{n-1}\sum_{k=0}^{n-1}\frac{C_k}{\lambda^k} = \frac{1}{2}\left(\lambda^n - \binom{2n}{n}\sum_{k=0}^{n}(-1)^k(4 - \lambda)^k\binom{n}{k}\Big/\binom{2k}{k}\right). \tag{31}$$

Therefore, the generating function for the sequence $s_n(\lambda)$ is

$$g(\lambda; x) := \sum_{n=0}^{\infty} s_n(\lambda)x^n = c(x)/(1 - \lambda x).$$

From the generating function, the recurrence relation is found to be $s_n(\lambda) = \lambda s_{n-1}(\lambda) + C_n$, $s_{-1}(\lambda) \equiv 0$. The connection with the polynomial $b_n(x)$ is

$$s_n(\lambda) = \frac{1}{2}(\lambda^{n+1} - (4 - \lambda)^{n+1}b_{n+1}(1/(4 - \lambda))).$$

The case $\lambda = 0$ $(x = 1/4)$ is also covered by this formula. It produces from $s_n(0) = C_n$ the following identity.

Example 1: Case $\lambda = 0$ $(x = 1/4)$,

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 4^k \left/ \binom{2k}{k} \right. = \frac{1}{2n-1}. \tag{32}$$

This identity occurs in one of the exercises 2.7, 2, page 32 of [4].

We note that from (5) one has $-2b_{n+1}(1/4) = C_n / 4^n$. The large n behavior of this sequence is known to be (see [2], Exercise 9.60):

$$C_n / 4^n \sim \frac{1}{\sqrt{\pi}} \frac{1}{n^{3/2}}.$$

If one puts $4x - 1 = x$, i.e., $x = 1/3$, in (5), one can identify the partial sum $s_n(1)$ of Catalan numbers:

$$s_n(1) := \sum_{k=0}^{n} C_k = \frac{1}{2}(1 - 3^{n+1} b_{n+1}(1/3)). \tag{33}$$

This sequence $\{1, 2, 4, 9, 23, 65, 197, 626, 2056, ...\}$ appears as A014137 in the web encyclopedia [8]. If one puts $\lambda - 1$ in Corollary 1, one also finds the following example.

Example 2:

$$2s_{n-1}(1) = 1 + \binom{2n}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 3^k \left/ \binom{2k}{k} \right. . \tag{34}$$

Another interesting example is the case $\lambda = 4$ $(x = \infty)$. Here one finds a simple result for the convolution of Catalan's sequence with powers of 4.

Example 3: $\lambda = 4$ $(x = \infty)$,

$$2s_{n-1}(4) = 4^n - \binom{2n}{n}. \tag{35}$$

This sequence $\{1, 5, 22, 93, 386, 1586, 6476, ...\}$ appears in the book [8] as Nr. 3920 and as A000346 in the web encyclopedia [8]. It will show up again in this work as $A(n+1, 1)$, the second column in the $A(n, m)$ triangle (see Table 2).

The sequence for $\lambda = -1$ $(x = 1/5)$ is also nonnegative, as can be seen by writing

$$s_{2k}(-1) = C_2 + \sum_{l=2}^{k} (C_{2l} - C_{2l-1}) \text{ for } k \in \mathbb{N}$$

and

$$s_{2k+1}(-1) = \sum_{l=1}^{k} (C_{2l+1} - C_{2l}),$$

and using

$$\Delta C_n := C_n - C_{n-1} = 3\frac{n-1}{n+1} C_{n-1} \geq 0.$$

This is the sequence $\{1, 0, 2, 3, 11, 31, 101, 328, 1102, 3760, ...\}$ which appears now as A032357 in the web encyclopedia [8].

Recursion (26) for $B(n, m)$ can be transformed into an equation for the generating function for the sequence appearing in the mth column of the $B(n, m)$ triangle

$$G_B(m; x) := \sum_{n=m}^{\infty} B(n, m)x^n, \tag{36}$$

with input

$$G_B(0; x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

the generating function for the central binomial numbers. So (26) implies, for $m \in \mathbf{N}_0$,

$$G_B(m; x) = \left(\frac{x}{1-4x}\right)^m \frac{1}{\sqrt{1-4x}}. \tag{37}$$

For $x\frac{d}{dx}G_B(m; x)$, see (53). Therefore, we have proved the following proposition.

Proposition 4 (column sequences of the $B(n, m)$ triangle): The sequence $\{B(n, m)\}_{n=m}^{\infty}$, defined for fixed $m \in \mathbf{N}_0$ and $n \in \mathbf{N}_0$ by (28), is the convolution of the central binomial sequence

$$\left\{\binom{2k}{k}\right\}_{k \in \mathbf{N}_0}$$

and the mth convolution of the (shifted) power sequence $\{0, 1, 4^1, 4^2, \ldots\}$.

Note 1: The infinite dimensional matrix **B** with elements $B(n, m)$ given for $n \geq m \geq 0$ by (28) and $B(n, m) \equiv 0$ for $n < m$ is an example of a Riordan matrix [7]. With the notation of this reference,

$$\mathbf{B} = \left(\frac{1}{\sqrt{1-4x}}, \frac{x}{1-4x}\right).$$

Note 2:(Sheffer-type identities from Riordan matrices): Triangular Riordan matrices

$$\mathbf{M} = (M_{i,j})_{i \geq j \geq 0} = (g(x), f(x)),$$

$M_{i,j} = 0$ for $j > i$, in the notation of [7], lead to polynomials that satisfy Sheffer-type identities (see [5] and its references, and also [1]),

$$S_n(x+y) = \sum_{k=0}^{n} S_k(y)P_{n-k}(x) = \sum_{k=0}^{n} P_k(y)S_{n-k}(x), \tag{38}$$

$$P_n(x+y) = \sum_{k=0}^{n} P_k(y)P_{n-k}(x) = \sum_{k=0}^{n} P_k(x)P_{n-k}(y), \tag{39}$$

where the polynomials $S_n(x)$ and $P_n(x)$ are defined by

$$S_n(x) = \sum_{m=0}^{n} M_{n,m} \frac{x^m}{m!}, \ n \in \mathbf{N}_0, \quad P_n(x) = \sum_{m=1}^{n} P_{n,m} \frac{x^m}{m!}, \ n \in \mathbf{N}, \ P_0(x) \equiv 1, \tag{40}$$

with $P_{n,m} := [z^n](f^m(z))$, $n \geq m \geq 1$. Here $g(x)$ defines the first column of **M**: $M_{n,0} = [x^n]g(x)$.

If one uses $s_n(x) := n! S_n(x)$ and $p_n(x) := n! P_n(x)$, one obtains the Sheffer identities (also called binomial identities) treated in [5]. Then $s_n(x)$ is Sheffer for $(1/g(\bar{f}(t)), \bar{f}(t))$, and $p_n(x)$ is

associated to $\bar{f}(t)$—or Sheffer for $(1, \bar{f}(t))$—in the terminology of [5]. Here $\bar{f}(t)$ stands for the inverse of $f(t)$ as a function.

Let us give the relation between $g_b(x; z)$ and $G_B(m; x)$.

Proposition 5: We have

$$g_b(x; z) = \sum_{m=0}^{\infty} (-1)^m G_B(m; xz) \left(\frac{1}{x}\right)^m. \tag{41}$$

Proof: One inserts the value of $b_n(x)$ given in (20) into the definition (6) of $g_b(x; z)$ and rewrites the Cauchy sum as two infinite sums which are then interchanged. Finally, the definition of $G_B(m; x)$ in (36) is used. $\square$

One can check (41) by using the explicit form of $G_B(m; xz)$ given in (36) and comparing with (6).

In a similar vein, we can solve $a_n(x)$ in (17) with $b_n(x)$ given by (20) and (28). The coefficients $a(n, k)$, defined by (19), have to satisfy, after comparing coefficients of x^n, x^0, and x^{n-k} for $k = 1, 2, ..., n-1$ and $n \in \mathbb{N}_0$:

$$x^n: \quad a(n, 0) = 4a(n-1, 0) + C_n, \tag{42}$$

$$x^0: \quad (n+1)a(n, n) = 1 + na(n-1, n-1), \tag{43}$$

$$x^{n-k}: \quad (n+1)a(n, k) = ka(n-1, k-1) + 4(n+1+k)a(n-1, k) + B(n, k). \tag{44}$$

In (42) we have used (24), i.e., $B(n, 0) = (n+1)C_n$; in (43) we have used (25), i.e., $B(n, n) \equiv 1$. From (42) one finds, with input $a(0, 0) = 1$,

$$a(n, 0) = \sum_{k=0}^{n} C_k 4^{n-k}, \tag{45}$$

and from (43),

$$a(n, n) \equiv 1 \quad \text{or} \quad a_n(0) = (-1)^n. \tag{46}$$

Note that $a(n, 0) = s_n(4)$ of (31) with solution (35). It is convenient to define $a(n-1, -1) := C_n$, $n \in \mathbb{N}_0$. Then the sequence $\{a(n, 0)\}_{-1}^{\infty}$ is, with $a(-1, 0) := 0$, the convolution of the sequence $\{a(k, -1)\}_{-1}^{\infty}$ and the shifted power sequence $\{0, 1, 4^1, 4^2, ...\}$. Before solving (44), with $B(n, k)$ from (28) inserted, we add to the triangular array of numbers $a(n, m)$ the $m = -1$ column and an extra row for $n = -1$, and define a new enlarged triangular array for $n, m \in \mathbb{N}_0$ as

$$A(n, m) := a(n-1, m-1) \tag{47}$$

with $A(n, 0) = a(n-1, -1) = C_n$ and $A(0, m) = a(-1, m-1) = \delta_{0, m}$. An inspection of the $A(n, m)$ triangular array, partly depicted in Table 2, leads to the conjecture

$$A(n, m) = 4A(n-1, m) + A(n-1, m-1), \tag{48}$$

with $A(n, 0) = C_n$ and $A(n, m) \equiv 0$ for $n < m$. This recursion relation can be used to extend the array $A(n, m)$ to negative integer values of m. This conjecture is correct for $A(n+1, 1) = a(n, 0)$ found in (45), as well as for $A(n+1, n+1) = a(n, n) \equiv 1$ known from (46). The generating function for the sequence appearing in the mth column,

$$G_A(m; x) := \sum_{n=m}^{\infty} A(n, m)x^n, \tag{49}$$

satisfies, due to (48), $G_A(m; x) = \frac{x}{1-4x} G_A(m-1; x)$, remembering that $A(m-1, m) \equiv 0$ and that $G_A(0; x) = c(x)$. Therefore,

$$G_A(m; x) = \left(\frac{x}{1-4x} \right)^m c(x). \tag{50}$$

Note 3: The infinite dimensional matrix **A** with elements $A(n, m)$ given for $n \geq m \geq 0$ by (48) and $A(n, m) \equiv 0$ for $n < m$ is another example of a Riordan matrix, written in the notation of [7] as $(c(x), x/(1-4x))$.

Because of (37) and $\sqrt{1-4x}\, c(x) = 2 - c(x)$, these generating functions of the conjectured $A(n, m)$ column sequences obey

$$G_A(m; x) = (2 - c(x))G_B(m; x). \tag{51}$$

If we use the conjecture (48) in (44), which is written with (47) in the form

$$(n+1) A(n+1, m+1) = m A(n, m) + 4(n+m+1) A(n, m+1) + B(n, m)$$

for $n \in \mathbf{N}_0$, $m \in \{1, 2, ..., n-1\}$, we have

$$m A(n+1, m+1) - (n+1) A(n, m) + B(n, m) = 0. \tag{52}$$

This recursion relation can be written with the help of the generating functions (36) and (49) as

$$\left(x\frac{d}{dx} + 1 \right) G_A(m; x) - \frac{m}{x} G_A(m+1; x) = G_B(m; x), \tag{53}$$

or with (50) (i.e., the conjecture) as

$$\left(x\frac{d}{dx} + 1 - \frac{m}{1-4x} \right) G_A(m; x) = G_B(m; x). \tag{54}$$

Together with (51), this means

$$x\frac{d}{dx}((2 - c(x)) G_B(m; x)) = \left[\left(\frac{m}{1-4x} - 1 \right)(2 - c(x)) + 1 \right] G_B(m; x). \tag{55}$$

If we can prove this equation with $G_B(x)$ given by (37), we have shown that (44) is equivalent to the conjecture (48). In order to prove (55), we first compute from (37) for $m \in \mathbf{N}_0$,

$$x\frac{d}{dx} G_B(m; x) = \left(2 + \frac{m}{x} \right) G_B(m+1; x) = \frac{2x+m}{1-4x} G_B(m; x). \tag{56}$$

With this result, (55) reduces to

$$\left(-xc'(x) + (2 - c(x))\frac{1-2x}{1-4x} - 1 \right) G_B(m; x) = 0, \tag{57}$$

and with (1), the factor in front of $G_B(m; x)$ vanishes identically for $x \neq 1/4$. Therefore, we have proved the following two propositions concerning the column sequences of the $A(n, m)$ triangular array and the triangular $A(n, m)$ array, respectively.

Proposition 6: The triangular array of numbers $A(n, m)$, defined for $n, m \in \mathbf{N}_0$ by equation (48), $A(n, 0) = C_n$, $A(n, m) \equiv 0$ for $n < m$ has as its mth column sequence $\{A(n, m)\}_{n=m}^{\infty}$ the convolution of the Catalan sequence and the mth convolution of the shifted power sequence $\{0, 1, 4^1, 4^2, \ldots\}$.

Proof: Use (50) with (49). $\square$

Proposition 7: The triangular array $A(n, m)$ of Proposition 6 coincides with the one defined by (47) and (42), (43) and (44) with $B(n, m)$ given by (28).

Proof: On one hand, $a(n, 0) = A(n+1, 1)$ and $a(n, n) = A(n+1, n+1) \equiv 1$ of (42) and (43), i.e., (45) and (46), respectively, satisfy (48). On the other hand, (44) is rewritten with the aid of (47) as (52), and (52) has been proved by (53)-(57). $\square$

Alternatively, one can use the now proven conjecture (48), together with (47), in (44) and derive for $n \in \mathbf{N}_0$, $m \in \mathbf{N}_0$,

$$4ma(n-1, m) = (n+1-m)a(n-1, m-1) - B(n, m). \tag{58}$$

This is written in terms of the polynomials $a_{n-1}(x)$ of (19) and $b_n(x)$ of (20) as

$$x(1-4x)a'_{n-1}(x) + (1-4x+4nx)a_{n-1}(x) - \binom{2n}{n}x^n + b_n(x) = 0. \tag{59}$$

With this result, (17) becomes an inhomogeneous recursion relation for $a_n(x)$:

$$a_n(x) = (4x-1)a_{n-1}(x) + C_n x^n, \quad a_0(x) \equiv 1. \tag{60}$$

Moreover, (59) can also be considered as an inhomogeneous linear differential equation for $a_{n-1}(x)$ with given $b_n(x)$. To find the solution this way is, however, a bit tedious. Let us give an alternative form for $a_n(x)$ in the following proposition.

Proposition 8: The solution of the recursion relation (60) is given by (8).

Proof: Iteration of (60). $\square$

Next, we give a corollary.

Corollary 2: The generating function $g_a(x; z) := \sum_{n=0}^{\infty} a_n(x) z^n$ is given by (9).

Proof: Equation (8) above shows that $a_n(x)$ is a convolution of the functional sequences $\{C_k x^k\}_{n \in \mathbf{N}_0}$ and $\{(4x-1)^k\}_{k \in \mathbf{N}_0}$ with generating functions $c(xz)$ and $1/(1+(1-4x)z)$. Therefore, $g_a(x; z)$ is the product of these generating functions. $\square$

We now have a relation between $g_a(x; z)$ and $G_A(m; x)$.

Proposition 9:

$$g_a(x; z) = \frac{1}{1-4xz} \sum_{m=0}^{\infty} (-1)^m G_A(m; xz) \left(\frac{1}{x}\right)^m. \tag{61}$$

Proof: Analogous to the proof of Proposition 5. $\square$

One can check (61) by putting in the explicit form (50) of $G_A(m; x)$ and compare with (9). Let us state the relation between $b_n(x)$ and $a_{n-1}(x)$ as Proposition 10.

Proposition 10: For $n \in \mathbf{N}_0$ and $a_{-1}(x) \equiv 0$, the relation between $b_n(x)$ and $a_{n-1}(x)$ is given by (10).

Proof: The alternative expressions (5) and (8) for these two families of polynomials are used. One splits off the $k = 0$ term in (5) with $C_{-1} = -1/2$ from the sum and shifts the summation variable. $\square$

Corollary 3: The coefficients of the triangular arrays $A(n, m)$ and $B(n, m)$ are related as given by (11).

Proof: The relation (10) between the polynomials is, with the help of (19) and (20), written for the coefficients $a(n-1, m)$, or by (47) for $A(n, m+1)$ and $B(n, m)$. $\square$

It remains to compute the explicit expression for the coefficients $a(n, k)$ of $a_n(x)$ defined by (19). Because of (47), it suffices to determine $A(n, m)$.

Corollary 4: The triangular array numbers $A(n, m)$ are given explicitly by formula (7).

Proof: The formula (4) written for $B(n, m-1)$ is used in relation (11). $\square$

Note 4: This formula for $A(n, m)$ satisfies indeed the recursion relation (48) with the given input. The first term,

$$\frac{1}{2} 4^{n-m+1} \binom{n}{m-1}$$

satisfies it because of the binomial identity

$$\binom{n}{m-1} = \binom{n-1}{m-1} + \binom{n-1}{m-2}.$$

For the second term of $A(n, m)$ in (7) one has to prove

$$\binom{n}{m-1}\binom{2n}{n} = 4\binom{n-1}{m-1}\binom{2(n-1)}{n-1} + \binom{n-1}{m-2}\binom{2(n-1)}{n-1}\frac{2(2m-3)}{m-1},$$

or after division by $\binom{2(n-1)}{n-1}$,

$$\frac{2n-1}{n}\binom{n}{m-1} = 2\binom{n-1}{m-1} + \binom{n-1}{m-2}\frac{2m-3}{m-1},$$

which reduces to the trivial identity $2n - 1 = 2(n - m + 1) + 2m - 3$. Both terms together, i.e., (7), satisfy the input $A(n, n) \equiv 1$.

Note 5: $A(n, m)$ was found originally after iteration in the form (with $n \geq m > 0$ and $(-1)!! := 1$)

$$A(n, m) = 2 \cdot 4^{n-m} \binom{n}{m-1} - \frac{\prod_{k=1}^{m}(2(n-m)+2k-1)}{(2m-3)!!} C_{n-m}. \tag{62}$$

$A(n, 0) = C_n$. It is easy to establish the equivalence with (7).

In the original derivation of the formula (7) for $A(n, m)$, it turned out to be convenient to introduce a rectangular array of integers $\hat{A}(n, m)$ for $n, m \in \mathbf{N}_0$ as follows: $\hat{A}(0, m) \equiv 1$, $\hat{A}(n, 0) :=$ $-C_n$ for $n \in \mathbf{N}$, and for $m \in \mathbf{N}$ and $n \in \mathbf{N}_0$, $\hat{A}(n, m)$ is defined by (12) or, equivalently, by (13). The $A(n, m)$ recursion (48) translates (with the help of the Pascal-triangle identity) into

$$\hat{A}(n,m) = 4\,\hat{A}(n-1,m) + \hat{A}(n,m-1). \tag{63}$$

This leads, after iteration and use of $\hat{A}(0,m) \equiv 1$ from (12) with $A(n,n) \equiv 1$, to

$$\hat{A}(n,m) = 4^n \sum_{k=0}^{n} \hat{A}(k,m-1)/4^k. \tag{64}$$

Thus, the following proposition describes column sequences of the $\hat{A}(n,m) \equiv C4(n,m)$ array.

Proposition 11: The mth column sequence of the $\hat{A}(n,m)$ array, $\{\hat{A}(n,m)\}_{n \in N_0}$, is the convolution of the sequence $\{\hat{A}(n,0)\}_{n \in N_0} = \{1,-1,-2,-5,...\}$, generated by $2 - c(x)$, and the mth convolution of the power sequence $\{4^k\}_{k \in N_0}$.

Proof: Iteration of (64) with the $\hat{A}(n,0)$ input. $\square$

Corollary 5: The ordinary generating function of the mth column sequence of the $\hat{A}(n,m)$ array (13) is given by

$$G_{\hat{A}}(m;x) := \sum_{n=0}^{\infty} \hat{A}(n,m)x^n = (2 - c(x))\left(\frac{1}{1-4x}\right)^m \tag{65}$$

for $m \in N_0$.

Proof: Use Proposition 11 written for generating functions. $\square$

Because of the convolution of the (negative) Catalan sequence with powers of 4, we shall call this $\hat{A}(n,m)$ array also $C4(n,m)$. A part of it is shown in Table 3 above. The second column sequence is given by

$$\hat{A}(n,1) \equiv C4(n,1) = \binom{2n+1}{n}$$

and appears as nr. 2848 in the book [8], or as A001700 in the web encyclopedia [8]. The sequence of the third column $\{\hat{A}(n,2) \equiv C4(n,2)\}_{n \in N_0} = \{1,7,38,187,...\}$ is, from (64) and (62) with (12), determined by

$$4^n \sum_{k=0}^{n} \binom{2k+1}{k} \Big/ 4^k = (2n+3)(2n+1)C_n - 2^{2n+1},$$

and is listed as A000531 in the web encyclopedia [8]. There the fourth column sequence is now listed as A029887.

Note 6: The infinite dimensional lower triangular matrix $\tilde{A}$ related to the array $\hat{A}(n,m) \equiv C4(n,m)$ by $\tilde{A}(n,m) := \hat{A}(n-m,m+1)$ for $n \geq m \geq 0$ and $\tilde{A}(n,m) := 0$ for $n < m$ is again an example of a Riordan matrix [7]. In the notation of [7], $\tilde{A} = (c(x)/\sqrt{1-4x}, x/\sqrt{1-4x})$.

Finally, we derive identities by using, for $n \in N_0$, equation (14) for the left-hand side of (3) and the results for $a_{n-1}(x)$ and $b_n(x)$ for the right-hand side. Because there are no negative powers of x on the left-hand side of (3), such powers have to vanish on the right-hand side. This leads to the first family of identities. Because

$$(1-4x)^{-n} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} 4^k x^k,$$

with Pochhammer's symbol defined after (28), this means that $x^p](a_{n-1}(x) + b_n(x)c(x))$, the coefficient proportional to x^p, has to vanish for $p = 0, 1, ..., n-1$, $n \in \mathbb{N}$. This requirement reads

$$(-1)^{n-1-p} a(n-1, n-1-p) + \sum_{k=0}^{p} (-1)^{n-k} B(n, n-k) C_{p-k} \equiv 0. \tag{66}$$

The sum is restricted to $k \le p$ $(< n)$ because no number C_l with negative index is found in $c(x)$. Inserting the known coefficients produces (15).

Proposition 12: For $n \in \mathbb{N}$ and $p \in \{0, 1, ..., n-1\}$ identity $(D1)$, given by (15), holds.

    Proof: With (47), (66) becomes

$$\sum_{k=0}^{p} (-1)^{p-k} C_{p-k} B(n, n-k) = A(n, n-p), \tag{67}$$

which is $(D1)$ of (15) if the summation index k is changed into $p-k$, and the symmetry of the binomial coefficients is used. $\square$

Example 4: Take $p = n-1 \in \mathbb{N}_0$:

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1} = 4^n \Big/ \binom{2n}{n} - 1 = 2A(n, 1) \Big/ \binom{2n}{n}. \tag{68}$$

With this identity we have found a sum representation for the convolution of the Catalan sequence and powers of 4:

$$s_{n-1}(4) := 4^{n-1} \sum_{k=0}^{n-1} C_k / 4^k = \frac{1}{2} \binom{2n}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1}$$

[cf. (35) with (31)].

    The second family of identities, $(D2)$ of (16), results from comparing powers x^k with $k \in \mathbb{N}_0$ on both sides of (3) after expansion of $(1-4x)^{-n}$ as given above in the text before (66). Only the second term $b_n(x)c(x)$ contributes because $a_{n-1}(x)/x^n$ has only negative powers of x. Thus, with definition (14), one finds, for $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$C(n, k) = \sum_{l=0}^{k} \frac{(n)_l 4^l}{l!} \sum_{j=0}^{n} (-1)^{n-j} B(n, n-j) C_{n-j+k-l}, \tag{69}$$

which is, after interchange of the summations and insertion of $B(n, n-j)$ from (4), the desired identity $(D2)$ if also the summation index j is changed to $n-q$.

    Thus, we have shown

Proposition 13: For $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, identity $(D2)$ of (16) with $C(n, k)$ defined by (14) holds true.

Example 5: Take $k = 0$, $n \in \mathbb{N}$. Then we have

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \equiv 1, \tag{70}$$

which is elementary.

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❖❖❖

SOME IDENTITIES INVOLVING THE FIBONACCI POLYNOMIALS*

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1. INTRODUCTION AND RESULTS

As usual, the Fibonacci polynomials $F(x) = \{F_n(x)\}$, $n = 0, 1, 2, \ldots$, are defined by the second-order linear recurrence sequence

$$F_{n+2}(x) = x F_{n+1}(x) + F_n(x) \tag{1}$$

for $n \geq 0$ and $F_0(x) = 0$, $F_1(x) = 1$. Let

$$\alpha = \frac{x + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{x - \sqrt{x^2 + 4}}{2}$$

denote the roots of the characteristic polynomial $\lambda^2 - x\lambda - 1$ of the sequence $F(x)$, then the terms of the sequence $F(x)$ (see [2]) can be expressed as

$$F_n(x) = \frac{1}{\alpha - \beta} \{\alpha^n - \beta^n\}$$

for $n = 0, 1, 2, \ldots$.

If $x = 1$, then the sequence $F(1)$ is called the Fibonacci sequence, and we shall denote it by $F = \{F_n\}$.

The various properties of $\{F_n\}$ were investigated by many authors. For example, Duncan [1] and Kuipers [3] proved that $(\log F_n)$ is uniformly distributed mod 1. Robbins [4] studied the Fibonacci numbers of the forms $px^2 \pm 1$ and $px^3 \pm 1$, where p is a prime. The second author [5] obtained some identities involving the Fibonacci numbers. The main purpose of this paper is to study how to calculate the summation involving the Fibonacci polynomials:

$$\sum_{a_1 + a_2 + \cdots + a_k = n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot \cdots \cdot F_{a_k+1}(x), \tag{2}$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, and k is any positive integer.

Regarding (2), it seems that it has not been studied yet, at least I have not seen expressions like (2) before. The problem is interesting because it is a generalization of [5], and it can also help us to find some new convolution properties for $F(x)$. In this paper we use the generating function of the sequence $F(x)$ and its partial derivative to study the evaluation of (2), and give an interesting identity for any fixed positive integers k and n. That is, we shall prove the following proposition.

Proposition: Let $F(x) = \{F_n(x)\}$ be defined by (1). Then, for any positive integers k and n, we have the calculating formula

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdots \cdot F_{a_k+1}(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1} \cdot x^{n-2m},$$

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$, and $[z]$ denotes the greatest integer not exceeding z.

From this proposition, we may immediately deduce the following several corollaries.

Corollary 1: For any positive integers k and n, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{a_1} \cdot F_{a_2} \cdots \cdot F_{a_k} = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}.$$

Corollary 2: For any positive integers k and n, we have

$$\sum_{a_1+\cdots+a_k=n+k} F_{2a_1} \cdot F_{2a_2} \cdots \cdot F_{2a_k} = 3^k \cdot 5^{\frac{n-k}{2}} \cdot \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}}{5^m}.$$

Corollary 3: The identity

$$\sum_{a_1+\cdots+a_k=n+k} F_{3a_1} \cdot F_{3a_2} \cdots \cdot F_{3a_k} = 2^{2n+k} \cdot \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}}{16^m}.$$

holds for all positive integers k and n.

Corollary 4: Let k and n be positive integers. Then

$$\sum_{a_1+\cdots+a_k=n+k} F_{4a_1} \cdot F_{4a_2} \cdots \cdot F_{4a_k} = 3^n \cdot 7^k \cdot 5^{\frac{n-k}{2}} \cdot \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}}{45^m}.$$

Corollary 5: Let k and n be positive integers. Then

$$\sum_{a_1+\cdots+a_k=n+k} F_{5a_1} \cdot F_{5a_2} \cdots \cdot F_{5a_k} = 5^k \cdot 11^n \cdot \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}}{121^m}.$$

In fact, for any positive integer m, using the proposition, we can give an exact calculating formula for

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{ma_1} \cdot F_{ma_2} \cdots \cdot F_{ma_k}.$$

2. PROOF OF THE PROPOSITION

In this section we shall complete the proof of the proposition. First, note that

$$F_n(x) = \frac{1}{\sqrt{x^2+4}} \left[\left(\frac{x+\sqrt{x^2+4}}{2} \right)^n - \left(\frac{x-\sqrt{x^2+4}}{2} \right)^n \right],$$

so we can easily deduce that the generating function of $F(x)$ is

$$G(t, x) = \frac{1}{1 - xt - t^2} = \frac{1}{(\alpha - t)(\beta - t)}$$

$$= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^{n+1} - \beta^{n+1}) \cdot t^n = \sum_{n=0}^{\infty} F_{n+1}(x) \cdot t^n. \tag{3}$$

Let $\frac{\partial G^k(t, x)}{\partial x^k}$ denote the kth partial derivative of $G(t, x)$ for x, and $F_n^{(k)}(x)$ denote the kth derivative of $F_n(x)$. Then from (3) we have

$$\frac{\partial G(t, x)}{\partial x} = \frac{t}{(1 - xt - t^2)^2} = \sum_{n=0}^{\infty} F_{n+1}^{(1)}(x) \cdot t^n,$$

$$\frac{\partial G^2(t, x)}{\partial x^2} = \frac{2! \cdot t^2}{(1 - xt - t^2)^3} = \sum_{n=0}^{\infty} F_{n+1}^{(2)}(x) \cdot t^n, \tag{5}$$

$$\cdots$$

$$\frac{\partial G^{k-1}(t, x)}{\partial x^{k-1}} = \frac{(k-1)! \cdot t^{k-1}}{(1 - xt - t^2)^k} = \sum_{n=0}^{\infty} F_{n+1}^{(k-1)}(x) \cdot t^n = \sum_{n=0}^{\infty} F_{n+1}^{(k-1)}(x) \cdot t^{n+k-1},$$

where we have used the fact that $F_{n+1}(x)$ is a polynomial of degree n.

For any two absolutely convergent power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, note that

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{u+v=n} a_u b_v \right) x^n.$$

So from (5) we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{a_1 + \cdots + a_k = n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdots F_{a_k+1}(x) \right) \cdot t^n = \left(\sum_{n=0}^{\infty} F_{n+1}(x) \cdot t^n \right)^k$$

$$= \frac{1}{(1 - xt - t^2)^k} = \frac{1}{(k-1)! \cdot t^{k-1}} \frac{\partial G^{k-1}(t, x)}{\partial x^{k-1}} = \frac{1}{(k-1)!} \sum_{n=0}^{\infty} F_{n+k}^{(k-1)}(x) \cdot t^n. \tag{6}$$

Equating the coefficients of t^n on both sides of equation (6), we obtain the identity

$$\sum_{a_1 + a_2 + \cdots + a_k = n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdots F_{a_k+1}(x) = \frac{1}{(k-1)!} \cdot F_{n+k}^{(k-1)}(x). \tag{7}$$

On the other hand, note that from the combinatorial identity

$$\binom{n-m+1}{m} = \binom{n-m}{m} + \binom{n-m}{m-1}, \tag{8}$$

the recurrence formula $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, and by mathematical induction, we can easily deduce

$$F_{n+1}(x) = \sum_{m=0}^{[\frac{n}{2}]} \binom{n-m}{m} \cdot x^{n-2m}. \tag{9}$$

In fact, from the definition of $F_n(x)$, we know that (9) is true for $n = 0$ and $n = 1$. Assume (9) is true for all integers $0 \le n \le k$. Then, for $n = k + 1$, applying (8) and the inductive hypothesis we immediately obtain

$$\sum_{m=0}^{[\frac{k+1}{2}]} \binom{k+1-m}{m} \cdot x^{k+1-2m} = 1 + \sum_{m=0}^{[\frac{k-1}{2}]} \binom{k-m}{m+1} \cdot x^{k-1-2m}$$

$$= 1 + \sum_{m=0}^{[\frac{k-1}{2}]} \binom{k-1-m}{m+1} \cdot x^{k-1-2m} + \sum_{m=0}^{[\frac{k-1}{2}]} \binom{k-1-m}{m} \cdot x^{k-1-2m}$$

$$= \sum_{m=0}^{[\frac{k+1}{2}]} \binom{k-m}{m} \cdot x^{k+1-2m} + \sum_{m=0}^{[\frac{k-1}{2}]} \binom{k-1-m}{m} \cdot x^{k-1-2m}$$

$$= x \sum_{m=0}^{[\frac{k}{2}]} \binom{k-m}{m} \cdot x^{k-2m} + \sum_{m=0}^{[\frac{k-1}{2}]} \binom{k-1-m}{m} \cdot x^{k-1-2m}$$

$$= xF_{k+1}(x) + F_k(x) = F_{k+2}(x),$$

where we have used $\binom{k-m}{m} = 0$ if $m > \frac{k}{2}$. So by induction we know that (9) is true for all non-negative integer n.

From (9) we can deduce that the $(k-1)^{\text{th}}$ derivative of $F_{n+k}(x)$ is

$$F_{n+k}^{(k-1)}(x) = \left(\sum_{m=0}^{[\frac{n+k-1}{2}]} \binom{n+k-1-m}{m} \cdot x^{n+k-1-2m} \right)^{(k-1)} = \sum_{m=0}^{[\frac{n}{2}]} \frac{(n+k-1-m)!}{m! \cdot (n-2m)!} x^{n-2m}. \qquad (10)$$

Combining (7) and (10), we obtain the identity

$$\sum_{a_1+\cdots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdots \cdots F_{a_k+1}(x) = \sum_{m=0}^{[\frac{n}{2}]} \binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1} \cdot x^{n-2m}.$$

This completes the proof of the Proposition.

Proof of the Corollaries: Taking $x = 1$ in the Proposition and noting that $F_0 = 0$, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(1) \cdot F_{a_2+1}(1) \cdots \cdots F_{a_k+1}(1) = \sum_{a_1+1+a_2+1+\cdots+a_k+1=n+k} F_{a_1+1} \cdot F_{a_2+1} \cdots \cdots F_{a_k+1}$$

$$= \sum_{a_1+a_2+\cdots+a_k=n+k} F_{a_1} \cdot F_{a_2} \cdots \cdots F_{a_k} = \sum_{m=0}^{[\frac{n}{2}]} \binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}.$$

This proves Corollary 1.

Taking $x = -\sqrt{5}, 4, -3\sqrt{5}$, and 11, respectively, in the Proposition, and noting that

$$F_n(-\sqrt{5}) = \frac{(-1)^{n+1}}{3} \left[\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right] = \frac{(-1)^{n+1}\sqrt{5}}{3} \cdot F_{2n},$$

$$F_n(4) = \frac{1}{2\sqrt{5}} \left[(2+\sqrt{5})^n - (2-\sqrt{5})^n \right] = \frac{1}{2\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{3n} - \left(\frac{1-\sqrt{5}}{2} \right)^{3n} \right] = \frac{1}{2} \cdot F_{3n},$$

$$F_n(-3\sqrt{5}) = \frac{(-1)^{n+1}}{7} \left[\left(\frac{7+3\sqrt{5}}{2} \right)^n - \left(\frac{7-3\sqrt{5}}{2} \right)^n \right] = \frac{(-1)^{n+1}\sqrt{5}}{7} \cdot F_{4n},$$

and

$$F_n(11) = \frac{1}{5\sqrt{5}}\left[\left(\frac{11+5\sqrt{5}}{2}\right)^n - \left(\frac{11-5\sqrt{5}}{2}\right)^n\right] = \frac{1}{5\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{5n} - \left(\frac{1-\sqrt{5}}{2}\right)^{5n}\right] = \frac{1}{5}\cdot F_{5n},$$

we may immediately deduce Corollary 2, Corollary 3, Corollary 4, and Corollary 5.

This completes the proof of the Corollaries.

ACKNOWLEDGMENT

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HEPTAGONAL NUMBERS IN THE LUCAS SEQUENCE AND DIOPHANTINE EQUATIONS $x^2(5x - 3)^2 = 20y^2 \pm 16$

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1. INTRODUCTION

The numbers of the form $\frac{m(5m-3)}{2}$, where m is any positive integer, are called *heptagonal numbers*. That is, 1, 7, 18, 34, 55, 81, ..., listed in [4] as sequence number 1826. In this paper, it is established that 1, 4, 7, and 18 are the only generalized heptagonal numbers (where m is any integer) in the *Lucas sequence* $\{L_n\}$. As a result, the Diophantine equations of the title are solved. Earlier, Cohn [1] identified the squares (listed in [4] as sequence number 1340) and Luo (see [2] and [3]) identified the triangular and pentagonal numbers (listed in [4] as sequence numbers 1002 and 1562, respectively) in $\{L_n\}$.

2. IDENTITIES AND PRELIMINARY LEMMAS

We have the following well-known properties of $\{L_n\}$ and $\{F_n\}$:

$$L_{-n} = (-1)^n L_n \quad \text{and} \quad F_{-n} = (-1)^{n+1} F_n; \tag{1}$$

$$2 \mid L_n \text{ iff } 3 \mid n \quad \text{and} \quad 3 \mid L_n \text{ iff } n \equiv 2 \ (\text{mod } 4); \tag{2}$$

$$L_n^2 = 5F_n^2 + 4(-1)^n. \tag{3}$$

If $m \equiv \pm 2 \ (\text{mod } 6)$, then the congruence

$$L_{n+2km} \equiv (-1)^k L_n \ (\text{mod } L_m) \tag{4}$$

holds, where k is an integer.

Since N is generalized heptagonal if and only if $40N + 9$ is the square of an integer congruent to 7 (mod 10), we identify those n for which $40L_n + 9$ is a perfect square. We begin with

Lemma 1: Suppose $n \equiv 1, 3, \pm 4$, or $\pm 6 \ (\text{mod } 18200)$. Then $40L_n + 9$ is a perfect square if and only if $n \equiv 1, 3, \pm 4$, or ± 6.

Proof: To prove this, we adopt the following procedure: Suppose $n \equiv \varepsilon \ (\text{mod } N)$ and $n \neq \varepsilon$. Then n can be written as $n = 2 \cdot \delta \cdot 2^\theta \cdot g + \varepsilon$, where $\theta \geq \gamma$ and $2 \nmid g$. And since, for $\theta \geq \gamma$, $2^{\theta+s} \equiv 2^\theta \ (\text{mod } p)$, taking

$$m = \begin{cases} \mu \cdot 2^\theta & \text{if } \theta \equiv \zeta \ (\text{mod } s), \\ 2^\theta & \text{otherwise,} \end{cases}$$

we get that

$$m \equiv c \ (\text{mod } p) \quad \text{and} \quad n = 2km + \varepsilon, \text{ where } k \text{ is odd.} \tag{5}$$

Now, by (4), (5), and the fact that $m \equiv \pm 2 \ (\text{mod } 6)$, we have

$$40L_n + 9 = 40L_{2km+\varepsilon} + 9 \equiv 40(-1)^k L_\varepsilon + 9 \ (\text{mod } L_m).$$

Since either m or n is not congruent to 2 modulo 4 we have, by (3), the Jacobi symbol

$$\left(\frac{40L_n+9}{L_m}\right) = \left(\frac{-40L_\varepsilon+9}{L_m}\right) = \left(\frac{L_m}{M}\right). \tag{6}$$

But, modulo M, $\{L_n\}$ is periodic with period P (i.e., $L_{n+Pt} \equiv L_n \pmod{M}$ for all integers $t \geq 0$). Thus, from (1) and (5), we have $\left(\frac{L_m}{M}\right) = -1$. Therefore, by (6), it follows that $\left(\frac{40L_n+9}{L_m}\right) = -1$ for $n \neq \varepsilon$, showing that $40L_n+9$ is not a perfect square. For each value of $n = \varepsilon$, the corresponding values are tabulated in Table A.

TABLE A

ε	N	δ	γ	s	p	μ	ζ (mod s)	c (mod p)	M	P
1	$2^2 \cdot 5$	5	1	4	30	5	2, 3	2, ±10, 16	31	30
3	$2^2 \cdot 5 \cdot 13$	$5 \cdot 13$	1	20	50	$5 \cdot 13$	3, ±5, 9, 13, 19.	±2, ±4, ±16, ±20, ±22, ±24.	151	50
						5	6, 8, 16, 18.			
±4	$2^2 \cdot 5^2$	5^2	1	36	270	5^2	7, 16, 34, 35.	2, 8, ±20, ±40, 46, 62, 64, ±80, 94, 98, ±110, 122, 124, 130, 136, 152, 166, 182, 212, 218, 226, 244, 256, 260.	271	270
						5	2, ±4, ±5, ±9, 10, 11, ±13, 14, 28, 30.			
±6	$2^3 \cdot 5^2 \cdot 7$	$5^2 \cdot 7$	2	12	156	5^2	0, 10.	4, 8, 16, 64, 80, 100.	79	78
						5	±5, 9, 11.			

Since the L.C.M. of $(2^5 \cdot 5, 2^2 \cdot 5 \cdot 13, 2^2 \cdot 5^2, 2^3 \cdot 5^2 \cdot 7) = 18200$, Lemma 1 follows from Table A. $\square$

Lemma 2: $40L_n+9$ is not a perfect square if $n \neq 1, 3, \pm 4$, or ± 6 (mod 18200).

Proof: We prove the lemma in different steps, eliminating at each stage certain integers n congruent modulo 18200 for which $40L_n+9$ is not a square. In each step, we choose an integer M such that the period P (of the sequence $\{L_n\}$ mod M) is a divisor of 18200 and thereby eliminate certain residue classes modulo P. We tabulate these in the following way (Table B).

TABLE B

Period P	Modulus M	Required values of n where $\left(\dfrac{40L_n+9}{m}\right) = -1$	Left out values of n (mod k) where k is a positive integer
10	11	± 2, 9.	0, 1, ± 3, 4, 5 or 6 (mod 10)
50	101	0, 11, ± 15, ± 16, 17, ± 20, ± 24, 27, 43, 45, 47.	1, 3, ± 4, ± 6, ± 10, 13, 21, 23, 25 or 31 (mod 50)
	151	5, 7, ± 14, 33, 37, 41.	
100	3001	± 10, 13, 21, 23, ± 44, 53, 71, 75.	1, 3, ± 4, ± 6, 25, 31, ± 40, ± 46, 51, 63, 73 or 81 (mod 100)
14	29	0, 5, 13.	1, 3, ± 4, ± 6, ± 104, ± 246, 281, ± 340 (mod 700)
28	13	9, ± 10, ± 12, 15, 17, 21, 23, 25.	
70	71	11, 15, 31, 53, 63.	
	911	± 16, ± 20.	
700	701	± 60, ± 106, ± 146, ± 204, 231, ± 254, ± 304, ± 306, 563, 651.	
350	54601	323	
26	521	0, ± 8, ± 9, ± 10, ± 11, ± 12, 19.	1, 3, ± 4, ± 6, ± 2346 or 7281 (mod 9100)
52	233	± 5, ± 20, ± 21, ± 24, 29, 39, 49.	
130	131	23, ± 30, 33, 51, ± 54, ± 56, 91, 103, 111.	
	24571	53.	
650	3251	± 46, ± 106, ± 154, ± 256, ± 306.	
910	50051	± 386.	
8	3	0, 5, 7.	1, 3, ± 4, ± 6 (mod 18200)
40	41	± 14.	
728	232961	± 202.	
1400	28001	281.	

3. MAIN THEOREM

Theorem:

(a) L_n is a generalized heptagonal number only for $n = 1, 3, \pm 4$, or ± 6.

(b) L_n is a heptagonal number only for $n = 1, \pm 4$, or ± 6.

Proof:

(a) The first part of the theorem follows from Lemmas 1 and 2.

(b) Since an integer N is heptagonal if and only if $40N + 9 = (10m - 3)^2$, where m is a positive integer, we have the following table. $\square$

TABLE C

n	1	3	± 4	± 6
L_n	1	4	7	18
$40L_n+9$	7^2	13^2	17^2	27^2
m	1	-1	2	3
F_n	1	2	± 3	± 8

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

It is well known that if $x_1 + y_1\sqrt{D}$ (where D is not a perfect square, x_1, y_1 are least positive integers) is the fundamental solution of Pell's equation $x^2 - Dy^2 = \pm 1$, then the general solution is given by $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$. Therefore, by (3), it follows that

$$L_{2n} + \sqrt{5}F_{2n} \text{ is a solution of } x^2 - 5y^2 = 4, \tag{7}$$

while

$$L_{2n+1} + \sqrt{5}F_{2n+1} \text{ is a solution of } x^2 - 5y^2 = -4. \tag{8}$$

We have the following two corollaries.

Corollary 1: The solution set of the Diophantine equation

$$x^2(5x-3)^2 = 20y^2 - 16 \tag{9}$$

is $\{(1, \pm 1), (-1, \pm 2)\}$.

Proof: Writing $X = x(5x-3)/2$, equation (9) reduces to the form

$$X^2 = 5y^2 - 4 \tag{10}$$

whose solutions are, by (8), $L_{2n+1} + \sqrt{5}F_{2n+1}$ for any integer n.

Now $x = m$, $y = b$ is a solution of (9) $\Leftrightarrow \frac{m(5m-3)}{2} + \sqrt{5}b$ is a solution of (10) and the corollary follows from Theorem 1(a) and Table C. $\square$

Similarly, we can prove the following.

Corollary 2: The solution set of the Diophantine equation

$$x^2(5x-3)^2 = 20y^2 + 16$$

is $\{(2, \pm 3), (3, \pm 8)\}$.

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NULLSPACE-PRIMES AND FIBONACCI POLYNOMIALS

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1. INTRODUCTION

A nonzero $m \times n$ (0, 1)-matrix A is called a *nullspace matrix* if each entry (i, j) of A has an even number of 1's in the set of entries consisting of (i, j) and its rectilinear neighbors. It is called a nullspace matrix since the existence of an $m \times n$ nullspace matrix implies the closed neighborhood matrix of the $m \times n$ grid graph is singular over $GF(2)$. By *closed neighborhood matrix*, we mean the adjacency matrix of the graphs with 1's down the diagonal.

In Sections 2 and 3, we review the relationship of the Fibonacci polynomials to nullspace matrices. In Section 3, we define *composite* and *prime* nullspace matrices and present some number sequences related to the nullspace matrices and pose a question analogous to the famous question about whether or not there exist infinitely many prime Fibonacci numbers.

2. BACKGROUND

In this paper, all polynomials are over the binary field $GF(2)$. When no confusion results, we denote the all-zero n-vector simply by 0. See Table 1 for an example of a nullspace matrix.

TABLE 1. A 4×4 Nullspace Matrix

```
1 0 0 0
1 1 0 0
1 0 1 0
0 1 1 1
```

If we choose a nonzero vector $w \in F^n$, where F^n is the binary *n-tuple* space and let w be the first row of a matrix A, for each $i > 1$ there is a unique way to choose the ith row to make the number of 1's in the closed neighborhood of each entry in the $(i-1)^{\text{st}}$ row even. If r_i is the ith row, the unique way of doing this is given by

$$r_i = Br_{i-1} + r_{i-2}, \quad i \geq 2, \quad r_0 = 0, r_1 = w, \tag{1}$$

where $B = [b_{ij}]$ is the $n \times n$ tridiagonal (0, 1)-matrix with $b_{ij} = 1$ if and only if $|i - j| \leq 1$ (and the r_i's in (1) are written as column vectors). If $r_{m+1} = 0$ for some positive integer m, then $r_1, r_2, ..., r_m$ are the rows of an $m \times n$ nullspace matrix. We can also compute the entries of r_i one at a time by $r_i[j] = r_{i-1}[j] + r_{i-1}[j-1] + r_{i-1}[j+1] + r_{i-2}[j] \mod 2$.

It follows from the definitions that $r_i = f_i(B)w$ for $i = 0, 1, 2, ...$, where f_i is the ith Fibonacci polynomial over $GF(2)$:

$$f_i = xf_{i-1} + f_{i-2}, \quad i \geq 2, \quad f_0 = 0, f_1 = 1. \tag{2}$$

In this paper, we are interested in building large nullspace matrices from smaller ones. A fundamental property of nullspace matrices is given in the following simple proposition.

Proposition 1: Let n and k be positive integers with $k+1$ a multiple of $n+1$. If there exists an $n \times n$ nullspace matrix, then there also exists a $k \times k$ nullspace matrix.

To see this another way, if $k+1 = q(n+1)$ where q is a positive integer, and if A is an $n \times n$ nullspace matrix, then a $k \times k$ nullspace matrix can be constructed by letting row and column numbers $n+1$, $2(n+1), \ldots, (q-1)(n+1)$ have all entries equal to zero, creating a $q \times q$ array of $n \times n$ squares, putting A in one of the $n \times n$ squares and filling in the rest of them by "reflecting" across the lines of zeros. That is, one can take the 4×4 nullspace matrix from Table 1 and construct a 9×9 nullspace matrix; see Table 2.

TABLE 2. A 9×9 Nullspace Matrix

```
1 0 0 0 0 0 0 0 1
1 1 0 0 0 0 0 1 1
1 0 1 0 0 0 1 0 1
0 1 1 1 0 1 1 1 0
0 0 0 0 0 0 0 0 0
0 1 1 1 0 1 1 1 0
1 0 1 0 0 0 1 0 1
1 1 0 0 0 0 0 1 1
1 0 0 0 0 0 0 0 1
```

3. NULLSPACE-PRIMES

We call a nullspace matrix that has at least one row or column of zeros a *composite nullspace matrix*, otherwise we say it is a *prime nullspace matrix*. We say that an integer n is *nullspace-prime* if there exists an $(n-1) \times (n-1)$ nullspace matrix, but for no proper divisor m of n does there exist an $(m-1) \times (m-1)$ nullspace matrix. With the aid of a computer, we have determined that the first few nullspace-primes are 5, 6, 17, 31, 33, 63, 127, 129, 171, 257, 511, 683. This sequence does not match any in *Sloane's Encyclopedia of Integer Sequences*. Other nullspace-primes include 2047, 2731, 2979, 3277, 3641, and 8191. We prove below that 6 is, in fact, the only even nullspace-prime. It is easy to see that there exists an $n \times n$ nullspace matrix if and only if n is one less than a multiple of a nullspace-prime.

One could use a simple (albeit, rather slow) sieving algorithm to determine if an integer n is a nullspace-prime, assuming we know that there exists an $(n-1) \times (n-1)$ nullspace matrix (which can be determined in $O(n \log^2 n)$ time [1]). For example, 693 is not a nullspace-prime since 693 modulo $33 = 0$, though there does exist a 692×692 nullspace matrix.

We say two polynomials $p_1(x)$ and $p_2(x)$ are *conjugates* if $p_1(x+1) = p(x)$. If $p(x)$ is an irreducible polynomial, we say that the *Fibonacci index* of $p(x)$ is t if t is the smallest positive integer such that $p(x)$ divides $f_t(x)$. The following is from [1].

Theorem 2 [1]: There exists an $n \times m$ nullspace matrix if and only if $f_{n+1}(x)$ and $f_{m+1}(x+1)$ are not relatively prime.

Theorem 2 is a special case of the following result (letting $r = 0$ in Proposition 3 below yields Theorem 2), also from [1].

Proposition 3: Let X be the closed neighborhood matrix of the $m \times n$ grid graph. If r is the degree of the greatest common divisor of $f_{n+1}(x+1)$ and $f_{m+1}(x)$, then the fraction of $n \times 1$ 0-1 vectors z having solutions y to the equation $Xy - z$ is 2^{-r}.

Proposition 3 was proved using the Primary Decomposition Theorem for linear operators, also known as the Spectral Decomposition Theorem (cf. [4]).

To illustrate Theorem 2, there exists a 16×16 nullspace matrix because $f_{17}(x)$ has the self-conjugate irreducible factor and there exists a 32×32 nullspace matrix because $f_{33}(x)$ has the conjugate pair of irreducible factors $x^5 + x^4 + x^3 + x + 1$ and $x^5 + x^3 + x^2 + x + 1$.

Using Theorem 2, we can prove that there is only one even nullspace-prime.

Fact 4: The only even nullspace-prime is 6.

Proof: As there do not exist 1×1 or 3×3 nullspace matrices, 2 and 4 are not nullspace-primes. Let $n > 6$ be an even integer and suppose n were a nullspace-prime. Then there exists an $(n-1) \times (n-1)$ nullspace matrix. Hence, by Theorem 2, $f_n(x)$ and $f_n(x+1)$ have a common factor. It was shown in Lemma 4, part (3), of [1] (using induction), that $f_{2n} = xf_n^2$ for all $n \geq 0$. Lemma 4, part (5), of [1] states that $f_{mn}(x) = f_m(x)f_n(xf_m(x))$, for $m, n \geq 0$. It follows that either there exists an $(\frac{n}{2} - 1) \times (\frac{n}{2} - 1)$ nullspace matrix, in which case n is not a nullspace-prime, or that x and $x+1$ are a conjugate pair of factors of $f_n(x)$ and $f_n(x+1)$. Using Lemma 4 of [1] and induction, it is not hard to prove that $x+1$ is a factor of f_k if and only if $3|k$ and this property also happens to be a special case of Proposition 5(b) of [1]. Hence, we have that $6|n$, which implies that n is not a nullspace-prime. $\square$

For completeness, we note that Proposition 5(b) from [1] states that, if $p(x)$ is an irreducible polynomial other than 1 or x with Fibonacci index t, then $p(x)|f_r(x)$ if and only if $t|r$. The proof of this property is based on Lemma 4 of [1].

We state a theorem from [3] that follows from results in [1]. Recall that B is the $n \times n$ tri-diagonal matrix defined in Section 2.

Theorem 5 [3]: The set of all vectors w that can be the first row of an $m \times n$ nullspace matrix is equal to the nullspace N_{m+1} of $f_{m+1}(B)$. If $d_{m+1}(x)$ is the greatest common divisor of $f_{n+1}(x+1)$ and $f_{m+1}(x)$, then the nullspace of $d_{m+1}(B)$ is equal to N_{m+1} and has dimension equal to the degree of d_{m+1}.

As can be concluded from the results in [1] and [3], if an $m \times n$ nullspace matrix has a row of zeros and if the first such row is the $(j+1)^{\text{st}}$, then $j+1$ divides $m+1$ and row r is all zeros if and only if r is a multiple of $j+1$. The same is true of columns (with n in place of m), since a matrix is a nullspace matrix if and only if its transpose is.

As we noted above, 63 is a nullspace-prime, so there is no way to "piece together" square nullspace matrices to get a 62×62 nullspace matrix. But there does exist a 6×8 nullspace matrix. A 9×7 array of this nullspace matrix and its reflections, with rows and columns of zeros in between, can be used to construct a composite 62×62 nullspace matrix. Therefore, if n is a nullspace-prime, there may exist an $(n-1) \times (n-1)$ composite nullspace matrix. But it is not hard

to show that there must also exist an $(n-1) \times (n-1)$ prime nullspace matrix (the sum of the 62×62 composite nullspace matrix and its 90 degree rotation is a prime nullspace matrix). This situation, and more, is described in the following theorem; an example is given following the proof of the theorem.

Theorem 6: Let n be an even positive integer and let $d_{n+1}(x)$ have positive degree and be the greatest common divisor of $f_{n+1}(x)$ and $f_{n+1}(x+1)$. Then:

(1) Every $n \times n$ nullspace matrix is prime if and only if every irreducible factor of $d_{n+1}(x)$ has Fibonacci index equal to $n+1$.

(2) Every $n \times n$ nullspace matrix is composite if and only if $d_{n+1}(x)$ divides $f_{t+1}(x)$ for some $t+1 \neq 0$ less than $n+1$.

Proof: Let $p_1, p_2, ..., p_k$ be the irreducible factors of d_{n+1}, and let W_i be the nullspace of $p_i(B)$ for $i = 1, 2, ..., k$, where B is the tridiagonal matrix defined above. We note that the W_i intersection $W_j = \{0\}$ for $i \neq j$, and that each W_i is invariant under multiplication by B ($B\alpha_i \in W_i$ for each $\alpha_i \in W_i$). So the nullspace of $d_{n+1}(B)$ is equal to the direct sum $W_1 \oplus W_2 \oplus \cdots \oplus W_k$. By Theorem 5, this is equal to the set of vectors that can be the first row of an $n \times n$ nullspace matrix. Choose a nonzero vector $\alpha_i \in W_i$ for each i. Let f be any polynomial. Then $f(B)(\alpha_1 + \alpha_2 + \cdots \alpha_k) = 0$ if and only if $f(B)\alpha_i = 0$ for each i, and this happens if and only if f is divisible by p_i for each i. If some p_i has Fibonacci index $t+1$ where $t \leq n$, then every nonzero vector in W_i is the first row of an $n \times n$ nullspace matrix with $(t+1)^{\text{st}}$ row all zeros. If there is no such t, then each $n \times n$ nullspace matrix is prime, establishing (1).

Letting the polynomial f above be d_{n+1}, it is clear that if d_{n+1} divides f_{t+1} for some $t \neq 0$ less than n, then every $n \times n$ nullspace matrix has $(t+1)^{\text{st}}$ row all zeros. And if there is no such t, then the vector $\alpha_1 + \alpha_2 + \cdots + \alpha_k$, where α_i is a nonzero vector in W_i for each i, is the first row of a prime $n \times n$ nullspace matrix. $\square$

For example, every 32×32 nullspace matrix is prime because

$$d_{33}(x) = (x^5 + x^4 + x^3 + x + 1)^2 (x^5 + x^3 + x^2 + x + 1)^2$$

and each of these factors has Fibonacci index 33. But no 98×98 nullspace matrix is prime because $d_{99}(x) = d_{33}(x)$. Every 98×98 nullspace matrix has row and column numbers 33 and 66 with all entries zero.

Corollary 7: If $n+1$ is a prime number, then there exists no $n \times n$ composite nullspace matrix.

We now pose our main open question.

Question 1: Are there an infinite number of nullspace-primes?

One might also ask whether or not a polynomial time algorithm exists to determine if an integer is nullspace-prime or not.

What more can be said about the distribution of nullspace-primes? From the few listed above, we can see that many take the form $2^k \pm 1$, but there are many nullspace primes that are not of this form; being of this form does not guarantee being nullspace-prime, take for example 65. In general, Fibonacci polynomials of the form $f_{2^k+1}(x)$ and $f_{2^k-1}(x)$ have many distinct factors [3], as do those with indices that are of the form $(2^k \pm 1)/p$, where p is a "small" prime. For example,

f_{171} has eleven distinct nontrivial factors and f_{683} has 31 distinct nontrivial factors. Thus, it is not surprising, given the results from [3], that many of these indices turn out to be nullspace-primes. What is not fully understood is how to characterize more precisely when an integer if nullspace-prime, even if it is of the form $2^k \pm 1$.

4. SUPER NULLSPACE-PRIMES

Define n to be *super nullspace-prime* if there exists no $(n-1) \times (n-1)$ composite nullspace matrix and there exists an $(n-1) \times (n-1)$ nullspace matrix. As mentioned above, 63 is nullspace-prime, but not super nullspace-prime. But 33 is super nullspace-prime because there does not exist a 2×10 nullspace matrix. Or, using Theorem 6(1), we see that

$$d_{33}(x) = (x^5 + x^4 + x^3 + x + 1)^2 (x^5 + x^3 + x^2 + x + 1)^2$$

and each of these two factors has Fibonacci index 33. The integers 5, 6, 17, 31, 33, 127, 129, 171, 257, 511, 683 are super nullspace-prime. Of course, although 29 is prime, 29 is not nullspace-prime or super nullspace-prime since there does not exist a 28×28 nullspace matrix.

We know from [3] that, if $n = 2^k$ where $k > 3$, or $n = 2^k - 2$ where $k > 3$, that there exists an $n \times n$ nullspace matrix. Thus, if n is prime and either $n-1 = 2^k$ or $n-1 = 2^k - 2$, then $n+1$ is super nullspace-prime, such as $n = 257$. But it seems likely that in order to determine whether an integer is super nullspace-prime requires factoring that integer or computing the Fibonacci indices of a number of polynomials, if we use the criteria described in Theorem 6(1), neither of which we know how to do efficiently (i.e., in polynomial time).

Conjecture 2: There are an infinite number of super nullspace-primes.

Note that, if the conjecture is false, then there are only finitely many Mersenne primes. We leave as an open problem determining how many *super nullspace composites* there are: integers, such as 99, which are such that there exists an $(n-1) \times (n-1)$ nullspace matrix and every $(n-1) \times (n-1)$ nullspace matrix is composite. Likewise, how many integers, such as 63, are nullspace-prime but not super nullspace-prime?

ACKNOWLEDGMENT

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PATH-COUNTING AND FIBONACCI NUMBERS

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1. INTRODUCTION

Consider the set of points (i, j) given by nonnegative integers i and j. This lattice may be viewed as an unbounded rectangle with boundary consisting of points $(i, 0)$ on a horizontal x-axis and points $(0, j)$ on a vertical y-axis. There are many systematic ways to draw paths from this boundary into the interior of the lattice. Enumerations of such paths yield arrays associated with Fibonacci numbers and other recurrence sequences. Such enumerations also apply to various classes of compositions of nonnegative integers. In order to investigate such enumerations, we begin with some notation:

$R = \{(i, j) : i \text{ and } j \text{ are nonnegative integers}\}$,
$R^+ = \{(i, j) : i \text{ and } j \text{ are positive integers}\}$,
$R^0 = R - R^+$.

Suppose G is a circuit-free graph on R, directed so that for each (i, j) in R^+ every path to (i, j) is rooted in a vertex in R^0. Each edge entering (i, j) has a tail (x, y); let

$$E(i, j) = \{(x_{i,j}(k), y_{i,j}(k)) : k = 1, 2, \ldots, n(i, j)\}$$

be the set of tails. Suppose now that a number $R(i, j)$ is assigned to each (i, j) in R^0, and for each (i, j) in R^+ define inductively

$$R(i, j) = \sum_{k=1}^{n(i, j)} R(p_k), \tag{1}$$

where the points p_k are the vertices in $E(i, j)$. The numbers $R(i, j)$ comprise a rectangular array:

$$
\begin{array}{ccccc}
\vdots \\
R(0, 2) & R(1, 2) & R(2, 2) & R(3, 2) & \cdots \\
R(0, 1) & R(1, 1) & R(2, 1) & R(3, 1) & \cdots \\
R(0, 0) & R(1, 0) & R(2, 0) & R(3, 0) & \cdots
\end{array}
$$

which can be expressed in triangular form:

$$
\begin{array}{ccccc}
 & & R(0, 0) & & \\
 & R(1, 0) & & R(0, 1) & \\
R(2, 0) & & R(1, 1) & & R(0, 2) \\
\cdots & & & &
\end{array}
$$

or

$$
\begin{array}{ccccc}
 & & T(0, 0) & & \\
 & T(1, 0) & & T(1, 1) & \\
T(2, 0) & & T(2, 1) & & T(2, 2) \\
\cdots & & & &
\end{array}
$$

Explicitly,

$$T(i, j) := R(i - j, j) \text{ for } 0 \le j \le i. \tag{2}$$

Henceforth, except for Examples 3C and 3D, we posit that for all (i, j) in R^0, the number $R(i, j)$ is the out-degree of (i, j), satisfying

$$R(i, 0) = 1 \quad \text{and} \quad R(0, j) \in \{0, 1\}. \tag{3}$$

Then, for (i, j) in R^+, the number of paths to (i, j) is $R(i, j)$, hence $T(i + j, j)$. The initial values $R(i, j)$ for (i, j) in R^0 imply initial values $T(i, 0)$ and $T(i, i)$ for $i \geq 0$; these values occupy the outermost wedge of the triangular array T. We call $\{R(i, j)\}$ the *path-counting rectangle of G*, and $\{T(i, j)\}$ the *path-counting triangle of G* for the given initial values. For reasons of notational convenience hereafter, define

$$R(i, j) = 0 \text{ if } i < 0 \text{ or } j < 0; \tag{4}$$

$$T(i, j) = 0 \text{ if } i < 0 \text{ or } j < 0 \text{ or } j > i. \tag{5}$$

2. INTEGER STRINGS AND COMPOSITIONS

In this section we restrict attention to path-counting under these conditions:

(i) $T(i, 0) = 1$ for $i \geq 0$;

(ii) for (i, j) in R^+, each (x, y) in $E(i, j)$ has the form $(i - 1, j + q)$, where q is an element of a prescribed set Q of nonnegative integers.

By (1) and (2),

$$R(i, j) = \sum_{k=1}^{n} R(i - 1, j + q_k), \tag{6}$$

$$T(i, j) = \sum_{k=1}^{n} T(i - q_k - 1, j + q_k). \tag{7}$$

Theorem 1: Let Q be a nonempty set of nonnegative integers, and let i and j be positive integers. If $0 \in Q$, then the number of strings $(s_1, s_2, ..., s_m)$ of nonnegative integers s_k satisfying the three conditions,

(a) $s_k - s_{k-1} \in Q$ for $k = 2, 3, ..., m$,

(b) $s_m = j$,

(c) $m = i + 1$,

is given as in (6) by $R(i, j)$ or, equivalently, by $T(i + j, j)$. If $0 \notin Q$, then the number of strings $(s_1, s_2, ..., s_m)$ of nonnegative integers s_k satisfying *(a)*, *(b)*, and

(c)' $m \leq i + 1$

is given as in (6) by $R(i, j)$ or, equivalently, by $T(i + j, j)$.

   Proof:
   Case 1: $0 \in Q$. The paths counted by $R(i, j)$ consist of edges $(k - 1, j_k)$-to-(k, j_{k+1}), where $j_{k+1} - j_k \in Q$ for $k = 1, 2, ..., i$, and $j_{i+1} = j$. Let $s_k = j_{k+1} - j_k$ for $k = 1, 2, ..., i + 1$. Then $(s_1, s_2, ..., s_m)$ is a string of the sort described. Conversely, for $m = i + 1$, each such string yields a path with initial point $(0, j_1)$ for some $j_1 \geq 0$ and terminal point (i, j), where $j = s_{i+1}$. This one-to-one correspondence between the paths and strings establishes that the number of strings is $R(i, j)$.

Case 2: $0 \notin Q$. Here, the initial point of a path can be of the form $(i_0, 0)$, where $0 \le i_0 \le i - 1$. The one-to-one correspondence holds, but the length of a string can be $\le i + 1$. $\square$

Corollary 1A: Suppose $T(i, i) = 0$ for $i \ge 1$. If $0 \in Q$, then $R(i, j)$, hence also $T(i + j, j)$, is the number of compositions of j consisting of i parts in the set Q. If $0 \notin Q$, then $R(i, j)$, hence also $T(i + j, j)$, is the number of compositions of j consisting of at most i parts, all in the set Q.

Proof: If $0 \in Q$, the i differences $j_{k+1} - j_k$ in the proof of Theorem 1 lie in Q and have sum j. Thus, there is a one-to-one correspondence between the paths counted by $R(i, j)$ and the compositions. If $0 \notin Q$, the same argument applies, except that the root of a path to (i, j) may be a point $(h, 0)$ for $0 \le h \le i - 1$, and the corresponding number of parts is $i - h$. $\square$

The following two corollaries have similar, omitted, proofs.

Corollary 1B: Suppose $h \ge 1$, $T(i, i) = 1$ for $i \le h$, and $T(i, i) = 0$ for $i > h$. If $0 \in Q$, then $R(i, j)$, hence also $T(i + j, j)$, is the number of compositions of the numbers $j, j - 1, j - 2, \ldots, j - h$ consisting of i parts in the set Q. If $0 \notin Q$, then $R(i, j)$, hence also $T(i + j, j)$, is the number of compositions of the numbers $j, j - 1, j - 2, \ldots, j - h$ consisting of at most i parts, all in the set Q.

Corollary 1C: Suppose $T(i, i) = 1$ for all $i \ge 0$. If $0 \in Q$, then $R(i, j)$, hence also $T(i + j, j)$, is the number of compositions of the numbers $0, 1, 2, \ldots, j$ consisting of i parts in the set Q. If $0 \notin Q$, then $R(i, j)$, hence also $T(i + j, j)$, is the number of compositions of the numbers $0, 1, 2, \ldots, j$ consisting of at most i parts, all in the set Q.

Theorem 2: Suppose $n \ge 2$ and Q is a set of n nonnegative integers q_k. Suppose also that $q_1 < q_2 < \cdots < q_n$. Let S_i be the sum of numbers in row i of array $T(i, j)$. Then (S_i) is a linear recurrence sequence of order $q_n + 1$.

Proof:

$$S_i = \sum_{j=0}^{i} T(i, j) = T(i, 0) + T(i, i) + \sum_{j=1}^{i-1} \sum_{k=1}^{n} T(i - q_k - 1, j - q_k)$$

$$= T(i, 0) + T(i, i) + \sum_{k=1}^{n} \sum_{j=1}^{i-1} T(i - q_k - 1, j - q_k),$$

so that, by (5),

$$S_i = T(i, 0) + T(i, i) + \sum_{k=1}^{n} \sum_{j=1}^{i-q_k-1} T(i - q_k - 1, j)$$

$$= 1 + T(i, i) + \begin{cases} -T(i - q_k - 1, 0) + \sum_{k=1}^{n} \sum_{j=0}^{i-q_k-1} T(i - q_k - 1, j) & \text{if } q_1 = 0, \\ \sum_{k=1}^{n} \sum_{j=0}^{i-q_k-1} T(i - q_k - 1, j) & \text{if } q_1 > 0, \end{cases}$$

$$= \begin{cases} T(i, i) + \sum_{k=1}^{n} S_{i-q_k-1} & \text{if } q_i = 0, \\ 1 + T(i, i) + \sum_{k=1}^{n} S_{i-q_k-1} & \text{if } q_1 > 0. \quad \square \end{cases}$$

The proof of Theorem 2 shows that, if $q_1 = 0$ and $T(i, i) = 0$ for all i greater than some i_0, then the linear recurrence is homogeneous for $i > i_0$. This is illustrated by Example 1C.

We turn now to applications of Theorems 1 and 2, in the form of Examples 1A-E, with particular interest in the appearance of Fibonacci and Lucas numbers in row sums or the central column.

Example 1A: A011973 in Sloane [5]

Initial values	$T(i, 0) = 1$ for $i \geq 0$, $T(i, i) = 0$ for $i \geq 1$
Q	$\{0, 1\}$
Recurrence	$T(i, j) = T(i - 1, j) + T(i - 2, j - 1)$ for $1 \leq j \leq i - 1$
Row sums	$1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

This is essentially the triangular array of coefficients of the Fibonacci polynomials [1], having rows (1), (1), (1, 1), (1, 2), (1, 3, 1), ... The two arrays have identical nonzero entries. Note that the southeast diagonals of nonzero entries form Pascal triangle: $T(i, j) = C(i - j, j)$.

```
              1
            1   0
          1   1   0
        1   2   0   0
      1   3   1   0   0
    1   4   3   0   0   0
  1   5   6   1   0   0   0
1   6  10   4   0   0   0   0
```

For example, $T(6, 2) = 6$ counts the compositions of 2 into 4 parts, each a 0 or 1, and it also counts strings of length 5, starting with 0 and ending in 2, with gaps of size 0 or 1:

compositions	0011	0101	0110	1001	1010	1100
strings	00012	00112	00122	01112	01122	01222

Example 1B: A005794 in Sloane [5]

Initial values	$T(i, 0) = 1$ for $i \geq 0$; $T(i, i) = 1$ for $0 \leq i \leq 3$, else $T(i, i) = 0$
Q	$\{0, 1\}$
Recurrence	$T(i, j) = T(i - 1, j) + T(i - 2, j - 1)$ for $1 \leq j \leq i - 1$
Row sums	$1, 2, 4, 7, 11, 18, 29, 47, \ldots$ (Lucas numbers)
Central column	$1, 2, 4, 8, 15, 26, 42, 64, \ldots$ (Cake numbers, A000125 in Sloane [5])

```
              1
            1   1
          1   2   1
        1   3   2   1
      1   4   4   2   0
    1   5   7   4   1   0
  1   6  11   8   3   0   0
1   7  16  15   7   1   0   0
1   8  22  26  15   4   0   0   0
```

For example, $T(7, 4) = 7$ counts the compositions of 1, 2, 3 into 3 parts, each a 0 or 1, and it also counts strings of length 4 starting with 0, 1, 2, or 3 and ending in 4, with gaps of size 0 or 1:

compositions	001	010	100	011	101	110	111
strings	3334	3344	3444	2234	2334	2344	1234

Regarding row sums, for $n \geq 2$, the number of strings $(s_1, s_2, \ldots, s_m)$ having gap sizes 0 or 1 and $m + s_m = n + 1$ is the nth Lucas number; e.g., for $n = 4$, the $L_4 = 11$ strings are as follows:

$$00000; \ 0001, 0111, 1111; \ 012, 112, 122, 222; \ 33, 23.$$

Example 1C: A052509 in Sloane [5]

Initial values	$T(i, 0) = T(i, i) = 1$ for $i \geq 0$
Q	$\{0, 1\}$
Recurrence	$T(i, j) = T(i - 1, j) + T(i - 2, j - 1)$ for $1 \leq j \leq i - 1$
Row sums	$1, 2, 4, 7, 12, \ldots$ (Fibonacci numbers minus 1)
Central column	$1, 1, 2, 4, 8, 16, \ldots$ (powers of 2)

```
                        1
                      1   1
                    1   2   1
                  1   3   2   1
                1   4   4   2   1
              1   5   7   4   2   1
            1   6  11   8   4   2   1
          1   7  16  15   8   4   2   1
        1   8  22  26  16   8   4   2   1
```
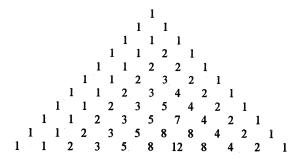
For example, $T(5, 2) = 7$ counts the compositions of 0, 1, 2 into 3 parts, each a 0 or 1, and it also counts strings of length 4 ending in 2 with gaps of size 0 or 1:

compositions	000	001	010	100	011	101	110
strings	2222	1112	1122	1222	0012	0112	0122

By Theorem 2, $S_i = 1 + S_{i-1} + S_{i-2}$ for $i \geq 2$. As a first step in an induction argument, we have $S_0 = F_3 - 1$ and $S_1 = F_4 - 1$. The hypothesis that $S_k = F_{k+3} - 1$ for all $k \leq i - 1$ yields $S_i = 1 + F_{i+2} - 1 + F_{i+1} - 1 = F_{i+3} - 1$.

Example 1D: A055215 in Sloane [5]

Initial values	$T(i, 0) = T(i, i) = 1$ for $i \geq 0$
Q	$\{1, 2\}$
Recurrence	$T(i, j) = T(i - 2, j - 1) + T(i - 3, j - 2)$ for $1 \leq j \leq i - 1$
Central column	$1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

```
                        1
                      1   1
                    1   1   1
                  1   1   2   1
                1   1   2   2   1
              1   1   2   3   2   1
            1   1   2   3   4   2   1
          1   1   2   3   5   4   2   1
        1   1   2   3   5   7   4   2   1
      1   1   2   3   5   8   8   4   2   1
    1   1   2   3   5   8  12   8   4   2   1
```

$T(8, 5) = 7$ counts the compositions of the integers 1, 2, 3, 4, 5 into 3 parts, each a 1 or 2, and it also counts strings of length 4 ending in 5 with gaps of sizes 1 or 2:

compositions	111	112	121	211	122	212	221
strings	2345	1235	1245	1345	0135	0235	0245

In accord with Theorem 2, the sequence (S_i) of row sums satisfies the recurrence $S_i = S_{i-2} + S_{i-3} + 2$.

Example 1E: A055216 in Sloane [5]

Initial values	$T(i, 0) = T(i, i) = 1$ for $i \geq 0$
Q	$\{0, 1, 2\}$
Recurrence	$T(i, j) = T(i-1, j) + T(i-2, j-1) + T(i-3, j-2)$ for $1 \leq j \leq i-1$
Central column	A027914 in Sloane [4]

```
                        1
                     1     1
                  1     2     1
               1     3     3     1
            1     4     6     3     1
         1     5    10     8     3     1
      1     6    15    17     9     3     1
   1     7    21    31    23     9     3     1
1     8    28    51    50    26     9     3     1
1  9   36    78    96    66    27     9     3     1
```

$T(5, 3) = 8$ counts the compositions into 2 parts, each a 0, 1, or 2, of nonnegative integers ≤ 3, and it also counts strings of length 3 ending in 3 with gaps of size 0, 1, or 2:

compositions	00	01	10	11	02	20	12	21
strings	333	223	233	123	133	113	023	013

The array in Example 1E has interesting connections with the array of coefficients of $(1 + x + x^2)^n$ considered by Hoggatt and Bicknell [3]. That array, $U(i, j)$ consists of trinomial coefficients. Written in left-justified form as in Comtet [2], we have

```
1
1   1   1
1   2   3   2   1
1   3   6   7   6   3   1
1   4  10  16  19  16  10   4   1
1   5  15  30  45  51  45  30  15   5   1
```

For example, the partial row-sums, $\sum_{j=1}^{n} U(i, j)$, beginning with 1, 2, 6, 17, 50, form the central column of the preceding array.

Example 2A: A055800 in Sloane [5]

Initial values	$T(i, 0) = 1$ for $i \geq 0$; $T(i, i) = 0$ for $i \geq 1$
Q	$\{1, 3, 5, 7, 9, \dots\}$
Recurrence	$T(i, j) = \sum_{k=1}^{\infty} T(i - 2k, j - 2k + 1)$ for $1 \leq j \leq i - 1$
Row sums	$S_i = 2^{\lfloor i/2 \rfloor}$ (powers of 2)
Central column	$1, 1, 1, 2, 3, 5, 8, \dots$ (Fibonacci numbers)

```
                              1
                           1     0
                        1     1     0
                     1     1     0     0
                  1     1     1     1     0
               1     1     1     1     0     0
            1     1     1     2     2     1     0
         1     1     1     2     2     1     0     0
      1     1     1     2     3     4     3     1     0
   1     1     1     2     3     4     3     1     0     0
1     1     1     2     3     5     7     7     4     1     0
```

For example, $T(10,5) = 5$ counts the compositions of 5 into parts in the set $\{1,3,5\}$, and it also counts strings ending in 5 with gaps of size 1, 3, or 5:

compositions	11111	113	131	311	5
strings	012345	0125	0145	0345	05

Example 2A points toward a more general result.

Theorem 3: The number of compositions of the positive integers $\leq n$ into odd parts is F_n.

Proof: By Corollary 1A, the number of compositions of $0, 1, 2, \ldots, n$ into odd parts is $T(2n, n)$. Therefore, it suffices to prove that $T(2n, n) = F_n$. We shall prove somewhat more: that the first $n + 1$ terms of row $2n$ are $1, F_1, F_2, \ldots, F_{n-2}, F_{n-1}, F_n$ for $n \geq 1$. Assume for arbitrary $n \geq 2$ that this has been established for all $m \leq n - 1$. Then, for row $2n$, we have $T(2n, 0) = 1$ and for $1 \leq j \leq n$,

$$T(2n, j) = \sum_{k=1}^{\infty} T(2n - 2k, j - 2k + 1) = \sum_{h=0}^{\lfloor \frac{j-1}{2} \rfloor} F_{j-2h-1} = F_j. \quad \square$$

Example 2B: **A055801 in Sloane [5]**

Initial values	$T(i,0) = T(i,i) = 1$ for $i \geq 0$
Q	$\{1, 3, 5, 7, 9, \ldots\}$
Recurrence	$T(i,j) = \sum_{k=1}^{\infty} T(i - 2k, j - 2k + 1)$ for $1 \leq j \leq i - 1$
Central column	$1, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

```
                              1
                           1     1
                        1     1     1
                     1     1     1     1
                  1     1     1     2     1
               1     1     1     2     2     1
            1     1     1     2     3     3     1
         1     1     1     2     3     4     3     1
      1     1     1     2     3     5     6     4     1
   1     1     1     2     3     5     7     7     4     1
1     1     1     2     3     5     8     11    10    5     1
```

For example, $T(9,6) = 7$ counts the compositions of numbers ≤ 6 using up to 3 parts, each an odd number, and it also counts strings of length ≤ 4 ending in 6 with no even gap sizes:

compositions	111	113	131	311	33	15	51
strings	3456	1236	1256	1456	036	016	056

3. ARRAYS BASED ON RECTANGULAR SETS $E(i, j)$

In Section 2, the tail-set $E(i, j)$ as defined in Section 1 is of the form $\{(i-1, j+q)\}$. That is to say, all edges into vertex (i, j) emanate from a single column of array $\{R(i, j)\}$. In Section 3, we consider paths for which $E(i, j)$ is a rectangle of more than one column.

Example 3A: A055807 in Sloane [5]

Initial values	$R(i, 0) = 1$ for $i \geq 0$, $R(0, j) = 0$ for $j \geq 1$
Recurrence	$R(i, j) = \sum_{i'=0}^{i-1} \sum_{j'=0}^{j} R(i', j')$ for $i \geq 1$, $j \geq 1$; $T(i, j) = R(i - j, j)$
Row sums	$1, 1, 2, 5, 13, \ldots$ (odd-indexed Fibonacci numbers)

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 0 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 1 & & 3 & & 1 & & 0 & \\
1 & & 7 & & 4 & & 1 & & 0 \\
1 & 15 & & 12 & & 5 & & 1 & & 0 \\
1 & 31 & 32 & & 18 & & 6 & & 1 & & 0 \\
1 & 63 & 80 & 56 & & 25 & & 7 & & 1 & & 0
\end{array}
$$

The array obtained by reflecting this one about its central column appears as A050143 in Sloane [5]. In order to see that the row sums in Example 3A are odd-indexed Fibonacci numbers, we first record an identity having an easy omitted proof:

$$R(i, j) = 2R(i-1, j) + R(i, j-1) - R(i-1, j-1) \text{ for } i \geq 1, \ j \geq 2. \tag{8}$$

$$S_n = \sum_{j=0}^{n} R(n-j, j) = 2^{n-1} + 1 + \sum_{j=2}^{n-1} R(n-j, j)$$

$$= 2^{n-1} + 1 + \sum_{j=2}^{n-1} [2R(n-j-1, j) + r(n-j, j-1) - R(n-j-1, j-1)],$$

by (8), so that

$$S_n = 2\left[2^{n-2} + 1 + \sum_{j=2}^{n-3} R(n-j-1, j) \right] + 1 + R(n-2, 1)$$

$$+ \sum_{j=2}^{n-3} R(n-j-1, j) - \sum_{j=2}^{n-2} R(n-j-1, j-1) \tag{9}$$

$$= (2S_{n-1} + 1) + (2^{n-2} - 1 + S_{n-1} - 2^{n-2} - 1) - \left[2^{n-3} - 1 + \sum_{j=2}^{n-4} R(n-j-2, j) \right]$$

$$= 3S_{n-1} - S_{n-2}.$$

Since both sequences (S_n) and (F_{2n-1}) are uniquely determined by initial values $S_0 = 1$ and $S_1 = 2$ together with the recurrence in (9), we have $S_n = F_{2n-1}$ for $n \geq 0$.

Example 3B: A055818 in Sloane [5]

Initial values	$R(i,0) = R(0,i) = 1$ for $i \geq 0$
Recurrence	$R(i,j) = \sum_{i'=0}^{i-1} \sum_{j'=0}^{j} R(i',j')$ for $i \geq 1$, $j \geq 1$; $T(i,j) = R(i-j,j)$
Row sums	$1, 2, 4, 10, 26, \ldots$ (twice odd-indexed Fibonacci numbers)

```
                         1
                     1       1
                 1       2       1
             1       5       3       1
         1      11       9       4       1
     1      23      24      14       5       1
 1      47      60      43      20       6       1
 1   95     144     122      69      27      7       1
```

The recurrences (8) hold for this array and can be used to prove that the row sums are given by $S_n = 2F_{2n-1}$ for $n \geq 1$.

Next, we break free of the initial values (3). When counting paths into the point $(3, 0)$, for example, rather than counting only the edge $(0,0)$-to-$(3,0)$ as a path, we can treat each of the following as paths:

$(0,0)$-to-$(3,0)$,

$(0,0)$-to-$(2,0)$-to-$(3,0)$,

$(0,0)$-to-$(1,0)$-to-$(3,0)$,

$(0,0)$-to-$(1,0)$-to-$(2,0)$-to-$(3,0)$.

More generally, for this sort of path, the number of paths entering $(i, 0)$ is 2^{i-1} for $i \geq 1$. Using as initial values $R(i, 0) = 2^{i-1}$, we count certain paths over rectangular tail-sets and obtain another array.

Example 3C: A049600 in Sloane [5]

Initial values	$R(0,0) = 1$, $R(i,0) = 2^{i-1}$ and $R(0,i) = 0$ for $i \geq 1$
Recurrence	$R(i,j) = \sum_{i'=0}^{i-1} \sum_{j'=0}^{j} R(i',j')$ for $i \geq 1$, $j \geq 1$; $T(i,j) = R(i-j,j)$
Row sums	$1, 1, 3, 8, 21, 55, \ldots$ (even-indexed Fibonacci numbers)
Alternating row sums	$1, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

```
              1
          1       0
      2       1       0
   4      3      1       0
8      8      4      1       0
```

This array and its connections to compositions are considered in [4]. Again, the recurrences (8) prevail and can be used to prove that the row sums are given by $S_n = F_{2n}$ for $n \geq 1$, and that alternating rows sums defined by

$$A_n = T(n, 0) - T(n, 1) + T(n, 2) - \cdots + (-1)^n T(n, n)$$

satisfy $A_n = F_n$ for $n \geq 1$.

Next, we consider rectangular tail-sets restricted to just two columns. On the x-axis the initial values $R(1, 0) = 1$ and $R(2, 0) = 1$, together with the two-column recurrence, determine the Fibonacci sequence for values of $R(i, 0)$.

Example 3D: A055830 in Sloane [5]

Initial values	$R(0,0) = 1$, $R(i,0) = F_{i+1}$ for $i \geq 1$, $R(0,j) = 0$ for $j \geq 1$
Recurrence	$R(i,j) = \sum\limits_{i'=i-2}^{i-1} \sum\limits_{j'=0}^{j} R(i',j')$ for $i \geq 1$, $j \geq 1$; $T(i,j) = R(i-j,j)$
1st diagonal	$1, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 0 & & & \\
 & & 2 & & 1 & & 0 & & \\
 & 3 & & 3 & & 1 & & 0 & \\
 5 & & 7 & & 4 & & 1 & & 0 \\
 8 & 15 & & 12 & & 5 & & 1 & & 0 \\
13 & 30 & 31 & & 18 & & 6 & & 1 & & 0 \\
21 & 58 & 73 & 54 & & 25 & & 7 & & 1 & & 0
\end{array}
$$

In [4], this sort of array is discussed not only for 2-column tail-sets, but also for m-column tail-sets for $m > 2$.

4. A SYMMETRIC ARRAY

We consider one more array, this one given by one recurrence for points beneath the line $y = x$ and another, symmetric to the first, for the points above the line $y = x$.

Example 4: A038792 in Sloane [5]

Initial values	$T(i, 0) = T(i, i) = 1$ for $i \geq 0$
Recurrence	$T(i,j) = T(i-1,j) + T(i-2,j-1)$ if $i < j/2$, else $T(i,j) = T(i,i-1)$
Central column	$1, 2, 5, 13, 34, \ldots$ (odd-indexed Fibonacci numbers)

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 5 & & 4 & & 1 \\
1 & 5 & & 8 & & 8 & & 5 & & 1 \\
1 & 6 & 12 & & 13 & & 12 & & 6 & & 1 \\
1 & 7 & 17 & 21 & & 21 & & 17 & & 7 & & 1
\end{array}
$$

Note that the recurrence can be written in symmetric form, as follows:

$$
T(i, j) = \begin{cases} T(i-1, j) + T(i-2, j-1) & \text{if } 2i \leq j, \\ T(i-1, j-1) + T(i-2, j-1) & \text{if } 2i > j. \end{cases}
$$

It is easy to prove that the central column of this array is the sequence of odd-indexed Fibonacci numbers, in conjunction with the fact that each column adjacent to the central one is the even-indexed Fibonacci sequence.

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ON THE ALMOST HILBERT-SMITH MATRICES

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1. INTRODUCTION

The study of GCD matrices was initiated by Beslin and Ligh [5]. In that paper the authors investigated GCD matrices in the direction of their structure, determinant, and arithmetic in Z_n. The determinants of GCD matrices were investigated in [6] and [11]. Furthermore, many other results on GCD matrices were established or conjectured (see [2]-[4], [7]-[10], and [12]).

In this paper we define an $n \times n$ matrix $S = (s_{ij})$, where $s_{ij} = \frac{(i,j)}{ij}$, and call S the "almost Hilbert-Smith matrix." In the second section we calculate the determinant and the inverse of the almost Hilbert-Smith matrix. In the last section we consider a generalization of the almost Hilbert-Smith matrix.

2. THE STRUCTURE OF THE ALMOST HILBERT-SMITH MATRIX

The $n \times n$ matrix $S = (s_{ij})$, where $s_{ij} = \frac{(i,j)}{ij}$, is called the almost Hilbert-Smith matrix. In this section we present a structure theorem and then calculate the value of the determinant of the almost Hilbert-Smith matrix. The following theorem describes the structure of the almost Hilbert-Smith matrix.

Theorem 1: Let $S = (s_{ij})$ be the $n \times n$ almost Hilbert-Smith matrix. Define the $n \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} \dfrac{\sqrt{\phi(j)}}{i} & \text{if } j | i, \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ is Euler's totient function. Then $S = AA^T$.

Proof: The ij-entry in AA^T is

$$(AA^T)_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} = \sum_{\substack{k|i \\ k|j}} \frac{\sqrt{\phi(k)}}{i} \frac{\sqrt{\phi(k)}}{j} = \frac{1}{ij} \sum_{k|(i,j)} \phi(k) = \frac{(i,j)}{ij} = s_{ij}. \quad \square$$

Corollary 1: The almost Hilbert-Smith matrix is positive definite, and hence invertible.

Proof: The matrix $A = (a_{ij})$ is a lower triangular matrix and its diagonal is

$$\left(\frac{\sqrt{\phi(1)}}{1}, \frac{\sqrt{\phi(2)}}{2}, ..., \frac{\sqrt{\phi(n)}}{n} \right).$$

It is clear that $\det A = \frac{1}{n!} [\phi(1)\phi(2)...\phi(n)]^{1/2}$ and $\phi(i) > 0$ for $1 \le i \le n$. Since $\det A > 0$, $rank(S) = rank(AA^T) = rank(A) = n$. Thus, S is positive definite. $\square$

Corollary 2: If S is the $n \times n$ almost Hilbert-Smith matrix, then

$$\det S = \frac{1}{(n!)^2} \phi(1)\phi(2) \dots \phi(n).$$

Proof: By Theorem 1, and since the matrix A is a lower triangular matrix, the result is immediate. $\square$

The matrix A in Theorem 1 can be written as $A = E\Lambda^{1/2}$, where the $n \times n$ matrices $E = (e_{ij})$ and $\Lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ are given by

$$e_{ij} = \begin{cases} \dfrac{1}{i} & \text{if } j \mid i, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and $\lambda_j = \phi(j)$. Thus, $S = AA^T = (E\Lambda^{1/2})(E\Lambda^{1/2})^T = E\Lambda E^T$.

Theorem 2: Let $S = (s_{ij})$ be the $n \times n$ almost Hilbert-Smith matrix. Then the inverse of S is the matrix $B = (b_{ij})$ such that

$$b_{ij} = ij \sum_{\substack{i \mid k \\ j \mid k}} \frac{1}{\phi(k)} \mu\left(\frac{k}{i}\right) \mu\left(\frac{k}{j}\right),$$

where μ denotes the Möbius function.

Proof: Let $E = (e_{ij})$ be the matrix defined in (1) and the $n \times n$ matrix $U = (u_{ij})$ be defined as follows:

$$u_{ij} = \begin{cases} j\mu\left(\dfrac{i}{j}\right) & \text{if } j \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the ij-entry of the product EU gives

$$(EU)_{ij} = \sum_{k=1}^{n} e_{ik} u_{kj} = \sum_{\substack{k \mid i \\ j \mid k}} \frac{1}{i} j\mu\left(\frac{k}{j}\right) = \frac{j}{i} \sum_{k \mid \frac{i}{j}} \mu(k) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, $U = E^{-1}$. If $\Lambda = diag(\phi(1), \phi(2), \dots, \phi(n))$, then $S = E\Lambda E^T$. Thus, $S^{-1} = U^T \Lambda^{-1} U = (b_{ij})$, where

$$b_{ij} = (U^T \Lambda^{-1} U)_{ij} = \sum_{k=1}^{n} \frac{1}{\phi(k)} u_{ki} u_{kj} = ij \sum_{\substack{i \mid k \\ j \mid k}} \frac{1}{\phi(k)} \mu\left(\frac{k}{i}\right) \mu\left(\frac{k}{j}\right). \quad \square$$

Example 1: Let $S = (s_{ij})$ be the 4×4 almost Hilbert-Smith matrix,

$$S = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{4} \end{bmatrix}.$$

By Theorem 2, $S^{-1} = (b_{ij})$, where

$$b_{11} = 1 \cdot 1 \cdot \left(\frac{\mu(1)\mu(1)}{\phi(1)} + \frac{\mu(2)\mu(2)}{\phi(2)} + \frac{\mu(3)\mu(3)}{\phi(3)} + \frac{\mu(4)\mu(4)}{\phi(4)} \right) = \frac{5}{2},$$

$$b_{12} = 1 \cdot 2 \cdot \left(\frac{\mu(2)\mu(1)}{\phi(2)} + \frac{\mu(4)\mu(2)}{\phi(4)} \right) = -2, \quad b_{13} = 1 \cdot 3 \cdot \frac{\mu(3)\mu(1)}{\phi(3)} = -\frac{3}{2},$$

$$b_{14} = 1 \cdot 4 \cdot \frac{\mu(4)\mu(1)}{\phi(4)} = 0, \quad b_{22} = 2 \cdot 2 \cdot \left(\frac{\mu(1)\mu(1)}{\phi(2)} + \frac{\mu(2)\mu(2)}{\phi(4)} \right) = 6, \quad b_{23} = 0,$$

$$b_{24} = 2 \cdot 4 \cdot \frac{\mu(2)\mu(1)}{\phi(4)} = -4, \quad b_{33} = 3 \cdot 3 \cdot \frac{\mu(1)\mu(1)}{\phi(3)} = \frac{9}{2}, \quad b_{34} = 0, \quad b_{44} = 4 \cdot 4 \cdot \frac{\mu(1)\mu(1)}{\phi(4)} = 8.$$

Therefore, since S^{-1} is symmetric, we have

$$S^{-1} = \begin{bmatrix} \frac{5}{2} & -2 & -\frac{3}{2} & 0 \\ -2 & 6 & 0 & -4 \\ -\frac{3}{2} & 0 & \frac{9}{2} & 0 \\ 0 & -4 & 0 & 8 \end{bmatrix}.$$

3. GENERALIZATION OF THE ALMOST HILBERT-SMITH MATRIX

In this section we consider an $n \times n$ matrix, the ij-entry of which is the positive m^{th} power of the ij-entry of the almost Hilbert-Smith matrix:

$$s_{ij}^m = \frac{(i, j)^m}{i^m j^m}.$$

Let m be a positive integer and let $S = (s_{ij})$ be the $n \times n$ almost Hilbert-Smith matrix. Define an $n \times n$ matrix S^m, the ij-entry of which is s_{ij}^m. Then

$$s_{ij}^m = \frac{(i, j)^m}{i^m j^m} = \sum_{k|(i, j)} \frac{J_m(k)}{i^m j^m},$$

where J_m is Jordan's generalization of Euler's totient function [1], given by

$$J_m(k) = \sum_{e|k} e^m \mu \left(\frac{k}{e} \right).$$

Theorem 3: Let $C = (c_{ij})$ be an $n \times n$ matrix defined by

$$c_{ij} = \begin{cases} \dfrac{\sqrt{J_m(j)}}{i^m} & \text{if } j|i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S^m = CC^T$.

Proof: The ij-entry in CC^T is

$$(CC^T)_{ij} = \sum_{k=1}^n c_{ik} c_{jk} = \sum_{\substack{k|i \\ k|j}} \frac{\sqrt{J_m(k)}}{i^m} \frac{\sqrt{J_m(k)}}{j^m}$$

$$= \frac{1}{i^m j^m} \sum_{k|(i, j)} J_m(k) = \frac{(i, j)^m}{i^m j^m} = s_{ij}^m. \quad \square$$

Corollary 3: The matrix $S^m = (s_{ij}^m)$ is positive definite, and hence invertible.

Proof: The matrix $C = (c_{ij})$ is a lower triangular matrix and its diagonal is

$$\left(\frac{\sqrt{J_m(1)}}{1^m}, \frac{\sqrt{J_m(2)}}{2^m}, ..., \frac{\sqrt{J_m(n)}}{n^m} \right).$$

It is clear that

$$\det C = \frac{1}{(n!)^m} [J_m(1) J_m(2) ... J_m(n)]^{1/2}$$

and $J_m(i) > 0$ for $1 \le i \le n$. Since $\det C > 0$, $rank(S^m) = rank(CC^T) = rank(C) = n$. Thus, S^m is positive definite. $\square$

Corollary 4: If $S^m = (s_{ij}^m)$ is the $n \times n$ matrix whose ij-entry is $s_{ij}^m = \frac{(i,j)^m}{i^m j^m}$, then

$$\det S^m = \frac{1}{(n!)^{2m}} J_m(1) J_m(2) ... J_m(n).$$

Proof: By Theorem 3, and since the matrix C is a lower triangular matrix, the result is immediate. $\square$

Example 2: Consider S^3, where S is the 5×5 almost Hilbert-Smith matrix. Then

$$S^3 = \begin{bmatrix} 1 & \frac{1}{8} & \frac{1}{27} & \frac{1}{64} & \frac{1}{125} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{216} & \frac{1}{64} & \frac{1}{1000} \\ \frac{1}{27} & \frac{1}{216} & \frac{1}{27} & \frac{1}{1728} & \frac{1}{3375} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{1728} & \frac{1}{64} & \frac{1}{8000} \\ \frac{1}{125} & \frac{1}{1000} & \frac{1}{3375} & \frac{1}{8000} & \frac{1}{125} \end{bmatrix}.$$

By Corollary 4, we have

$$\det S^3 = \frac{1}{(5!)^6} J_3(1) J_3(2) J_3(3) J_3(4) J_3(5) = \frac{19747}{46656000000}. \quad \square$$

We now define the $n \times n$ matrices $D = (d_{ij})$ and $\Omega = diag(\omega_1, \omega_2, ..., \omega_n)$ by

$$d_{ij} = \begin{cases} \frac{1}{i^m} & \text{if } j|i, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and $\omega_j = J_m(j)$. Then the matrix $C = (c_{ij})$ can be written as $C = D\Omega^{1/2}$. Thus, we have

$$S^m = CC^T = (D\Omega^{1/2})(D\Omega^{1/2})^T = D\Omega D^T.$$

Theorem 4: The inverse of the matrix $S^m = (s_{ij}^m)$ is the matrix $G = (g_{ij})$, where

$$g_{ij} = i^m j^m \sum_{\substack{i|k \\ j|k}} \frac{1}{J_m(k)} \mu\left(\frac{k}{i}\right) \mu\left(\frac{k}{j}\right).$$

Proof: Let $D = (d_{ij})$ be the matrix defined in (2) and the $n \times n$ matrix $V = (v_{ij})$ be defined as follows:

$$v_{ij} = \begin{cases} j^m \mu\left(\dfrac{i}{j}\right) & \text{if } j \,|\, i, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the ij-entry of the product DV gives

$$(DV)_{ij} = \sum_{k=1}^{n} d_{ik} v_{kj} = \sum_{\substack{k|i \\ j|k}} \frac{1}{i^m} j^m \mu\left(\frac{k}{j}\right)$$

$$= \frac{j^m}{i^m} \sum_{k|\frac{i}{j}} \mu(k) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, $V = D^{-1}$. If $\Omega = diag(J_m(1), J_m(2), \dots, J_m(n))$, then $S^m = D\Omega D^T$. Therefore, $(S^m)^{-1} = V^T \Omega^{-1} V = G = (g_{ij})$, where

$$g_{ij} = (V^T \Omega^{-1} V)_{ij} = \sum_{k=1}^{n} \frac{1}{J_m(k)} v_{ki} v_{kj} = i^m j^m \sum_{\substack{i|k \\ j|k}} \frac{1}{J_m(k)} \mu\left(\frac{k}{i}\right) \mu\left(\frac{k}{j}\right). \quad \Box$$

Example 3: If S^2 is the 4×4 almost Hilbert-Smith matrix, then

$$S^2 = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{36} & \frac{1}{16} \\ \frac{1}{9} & \frac{1}{36} & \frac{1}{9} & \frac{1}{144} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{144} & \frac{1}{16} \end{bmatrix}.$$

Moreover

$$d_{11} = 1 \cdot 1 \cdot \left(\frac{\mu(1)\mu(1)}{J_2(1)} + \frac{\mu(2)\mu(2)}{J_2(2)} + \frac{\mu(3)\mu(3)}{J_2(3)} + \frac{\mu(4)\mu(4)}{J_2(4)} \right) = \frac{35}{24},$$

$$d_{12} = 1 \cdot 2 \cdot \left(\frac{\mu(2)\mu(1)}{J_2(2)} + \frac{\mu(4)\mu(2)}{J_2(4)} \right) = -\frac{4}{3}, \quad d_{13} = 1 \cdot 3 \cdot \frac{\mu(3)\mu(1)}{J_2(3)} = -\frac{9}{8},$$

$$d_{14} = 1 \cdot 4 \cdot \frac{\mu(4)\mu(1)}{J_2(4)} = 0, \quad d_{22} = 2 \cdot 2 \cdot \left(\frac{\mu(1)\mu(1)}{J_2(2)} + \frac{\mu(2)\mu(2)}{J_2(4)} \right) = \frac{20}{3}, \quad d_{23} = 0,$$

$$d_{24} = 2 \cdot 4 \cdot \frac{\mu(2)\mu(1)}{J_2(4)} = -\frac{16}{3}, \quad d_{33} = 3 \cdot 3 \cdot \frac{\mu(1)\mu(1)}{J_2(3)} = \frac{81}{8}, \quad d_{34} = 0, \quad d_{44} = 4 \cdot 4 \cdot \frac{\mu(1)\mu(1)}{J_2(4)} = \frac{64}{3}.$$

Therefore, since $(S^2)^{-1}$ is symmetric, we have

$$(S^2)^{-1} = \begin{bmatrix} \frac{35}{24} & -\frac{4}{3} & -\frac{9}{8} & 0 \\ -\frac{4}{3} & \frac{20}{3} & 0 & -\frac{16}{3} \\ -\frac{9}{8} & 0 & \frac{81}{8} & 0 \\ 0 & -\frac{16}{3} & 0 & \frac{64}{3} \end{bmatrix}.$$

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FIVE CONGRUENCES FOR PRIMES

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1. INTRODUCTION

Let p be an odd prime. In 1988, using the formula for the sum

$$\sum_{k \equiv r \pmod 8} \binom{n}{k},$$

the author proved that (cf. [7], Theorem 2.6)

$$\sum_{1 \le k < \frac{p}{2}} \frac{2^k}{k} \equiv 4(-1)^{\frac{p-1}{2}} \sum_{1 \le k \le \frac{p+1}{4}} \frac{(-1)^{k-1}}{2k-1} \pmod{p}$$

and

$$\sum_{1 \le k < \frac{p}{2}} \frac{1}{k \cdot 2^k} \equiv -4 \sum_{\frac{1+(-1)^{\frac{p-1}{2}}}{2} \le k < \frac{p}{8}} \frac{1}{4k - (-1)^{\frac{p-1}{2}}} \pmod{p}.$$

In 1995, using a similar method, Zhi-Wei Sun [9] proved the author's conjecture,

$$\sum_{1 \le k < \frac{p}{2}} \frac{1}{k \cdot 2^k} \equiv \sum_{1 \le k < \frac{3p}{4}} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

Later, Zun Shan and Edward T. H. Wang [5] gave a simple proof of the above congruence. In [9] and [10], Zhi-Wei Sun also pointed out another congruence,

$$\sum_{1 \le k < \frac{p}{2}} \frac{3^k}{k} \equiv \sum_{1 \le k < \frac{p}{6}} \frac{(-1)^k}{k} \pmod{p}.$$

In this paper, by using the formulas for Fibonacci quotient and Pell quotient, we obtain the following five congruences:

$$\sum_{1 \le k < \frac{p}{2}} \frac{2^k}{k} \equiv 2 \sum_{\frac{p}{4} < k < \frac{p}{2}} \frac{(-1)^{k-1}}{k} \pmod{p}, \tag{1.1}$$

$$\sum_{1 \le k < \frac{p}{2}} \frac{5^k}{k} \equiv 2 \sum_{\frac{p}{5} < k < \frac{p}{2}} \frac{(-1)^{k-1}}{k} \pmod{p}, \tag{1.2}$$

$$\sum_{1 \le k < \frac{p}{2}} \frac{2^k}{k} \equiv - \sum_{\frac{p}{8} < k < \frac{3p}{8}} \frac{1}{k} \pmod{p}, \tag{1.3}$$

$$\sum_{1 \le k < \frac{p}{2}} \frac{1}{k \cdot 2^k} \equiv - \sum_{\frac{p}{4} < k < \frac{3p}{8}} \frac{1}{k} \pmod{p}, \tag{1.4}$$

$$\sum_{1 \le k < \frac{p}{2}} \frac{3^k}{k} \equiv - \sum_{\frac{p}{12} < k < \frac{p}{6}} \frac{1}{k} \pmod{p}, \tag{1.5}$$

where $p > 5$ is a prime.

2. BASIC LEMMAS

The Lucas sequences $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ are defined as follows:

$$u_0(a,b) = 0, \ u_1(a,b) = 1, \ u_{n+1}(a,b) = bu_n(a,b) - au_{n-1}(a,b) \ (n \geq 1),$$

$$v_0(a,b) = 2, \ v_1(a,b) = b, \ v_{n+1}(a,b) = bv_n(a,b) - av_{n-1}(a,b) \ (n \geq 1).$$

It is well known that

$$u_n(a,b) = \frac{1}{\sqrt{b^2-4a}}\left\{\left(\frac{b+\sqrt{b^2-4a}}{2}\right)^n - \left(\frac{b-\sqrt{b^2-4a}}{2}\right)^n\right\} \ (b^2 - 4a \neq 0)$$

and

$$v_n(a,b) = \left(\frac{b+\sqrt{b^2-4a}}{2}\right)^n + \left(\frac{b-\sqrt{b^2-4a}}{2}\right)^n.$$

Let p be an odd prime, and let m be an integer with $m \not\equiv 0 \pmod{p}$. It is evident that

$$2\sum_{\substack{k=1 \\ 2|k}}^{p-1} \binom{p}{k}(\sqrt{m})^k = (1+\sqrt{m})^p - (1-\sqrt{m})^p - 2(\sqrt{m})^p$$

and

$$2\sum_{\substack{k=1 \\ 2\nmid k}}^{p-1} \binom{p}{k}(\sqrt{m})^k = (1+\sqrt{m})^p + (1-\sqrt{m})^p - 2.$$

Since

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \equiv \frac{(-1)^{k-1}}{k}p \pmod{p^2},$$

by the above one can easily prove

Lemma 1 ([7], Lemma 2.4): Suppose that p is an odd prime and that m is an integer such that $p \nmid m$. Then

(a) $\displaystyle\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot m^k} \equiv \frac{m^{p-1}-1}{p} - 2 \cdot \frac{(\frac{m}{p})u_p(1-m,2)-1}{p} \pmod{p}$,

(b) $\displaystyle\sum_{k=1}^{(p-1)/2} \frac{m^k}{k} \equiv \frac{2-v_p(1-m,2)}{p} \pmod{p}$,

where $\left(\frac{m}{p}\right)$ is the Legendre symbol.

For any odd prime p and integer m, set $q_p(m) = \frac{m^{p-1}-1}{p}$. Using Lemma 1, we can prove

Proposition 1: Let m be an integer and let p be an odd prime such that $p \nmid m(m-1)$. Then

$$\frac{u_{p-(\frac{m}{p})}(1-m,2)}{p} \equiv \frac{(m-2)(\frac{m}{p})-m}{4m}\left(\sum_{k=1}^{(p-1)/2} \frac{m^k}{k} + q_p(m-1)\right)$$

$$\equiv \frac{(m-2)(\frac{m}{p})-m}{4}\left(\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot m^k} + q_p(m-1) - q_p(m)\right) \pmod{p}.$$

Proof: Set $u_n = u_n(1-m, 2)$ and $v_n = v_n(1-m, 2)$. From [1], [4], and Lemma 1.7 of [6], we know that

$$v_n^2 - 4mu_n^2 = 4(1-m)^n, \quad v_n = 2u_{n+1} - 2u_n, \quad u_n = \frac{1}{2m}(v_{n+1} - v_n)$$

and

$$u_{p-\left(\frac{m}{p}\right)} \equiv u_p - \left(\frac{m}{p}\right) \equiv 0 \pmod{p}.$$

Thus,

$$v_{p-\left(\frac{m}{p}\right)}^2 \equiv 4(1-m)^{p-\left(\frac{m}{p}\right)} \pmod{p^2},$$

and hence,

$$v_{p-\left(\frac{m}{p}\right)} \equiv \pm 2\left(\frac{1-m}{p}\right)(1-m)^{\left(p-\left(\frac{m}{p}\right)\right)/2} \pmod{p^2}.$$

If $\left(\frac{m}{p}\right) = 1$, then $v_{p-1} = 2u_p - 2u_{p-1} \equiv 2 \pmod{p}$. Hence, by the above, we get

$$v_{p-1} \equiv 2(1-m)^{(p-1)/2}\left(\frac{1-m}{p}\right) \equiv 2 + q_p(m-1)p \pmod{p^2}. \tag{2.1}$$

Now, applying Lemma 1 we find

$$\frac{u_{p-1}}{p} = \frac{1}{2m} \cdot \frac{v_p - v_{p-1}}{p} = \frac{1}{2m}\left(\frac{v_p - 2}{p} - \frac{v_{p-1} - 2}{p}\right)$$

$$\equiv \frac{1}{2m}\left(-\sum_{k=1}^{(p-1)/2} \frac{m^k}{k} - q_p(m-1)\right) \pmod{p}$$

and

$$\frac{u_{p-1}}{p} = \frac{2u_p - v_{p-1}}{2p} = \frac{u_p - 1}{p} + \frac{1}{2} \cdot \frac{2 - v_{p-1}}{p}$$

$$\equiv \frac{1}{2}\left(q_p(m) - \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot m^k} - q_p(m-1)\right) \pmod{p}.$$

This proves the result in the case $\left(\frac{m}{p}\right) = 1$.

If $\left(\frac{m}{p}\right) = -1$, then

$$v_{p+1} = 2u_{p+1} - 2(1-m)u_p \equiv 2(1-m) \pmod{p}.$$

So

$$v_{p+1} \equiv 2(1-m)\left(\frac{1-m}{p}\right)(1-m)^{(p-1)/2} \equiv (1-m)(2 + q_p(m-1)p) \pmod{p^2}. \tag{2.2}$$

Note that

$$u_{p+1} = \frac{1}{2m}(v_{p+1} + (m-1)v_p) = \frac{1}{2}v_{p+1} + (1-m)u_p.$$

Applying (2.2) and Lemma 1, one can easily deduce the desired result. Therefore, the proof is complete.

Corollary 1: Let p be an odd prime and let $\{P_n\}$ denote the Pell sequence given by $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ $(n \geq 1)$. Then

(a) $\quad \displaystyle\sum_{k=1}^{(p-1)/2} \frac{2^k}{k} \equiv -4\frac{P_{p-\left(\frac{2}{p}\right)}}{p} \pmod p.$

(b) $\quad \displaystyle\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv -2\frac{P_{p-\left(\frac{2}{p}\right)}}{p} + q_p(2) \pmod p.$

Proof: Taking $m = 2$ in Proposition 1 gives the result.

Corollary 2: Let $p > 3$ be a prime, $S_0 = 0$, $S_1 = 1$, and $S_{n+1} = 4S_n - S_{n-1}$ $(n \geq 1)$. Then

(a) $\quad \displaystyle\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv -3\left(\tfrac{3}{p}\right)\frac{S_{p-\left(\frac{3}{p}\right)}}{p} - q_p(2) \pmod p.$

(b) $\quad \displaystyle\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 3^k} \equiv -\left(\tfrac{3}{p}\right)\frac{S_{p-\left(\frac{3}{p}\right)}}{p} - q_p(2) + q_p(3) \pmod p.$

Proof: Suppose a and b are integers. From [4] we know that $u_{2n}(a,b) = u_n(a,b)v_n(a,b)$ and

$$u_{p-\left(\frac{b^2-4a}{p}\right)}(a,b) \equiv u_p(a,b) - \left(\frac{b^2-4a}{p}\right) \equiv 0 \pmod p.$$

Thus,

$$v_{p-\left(\frac{3}{p}\right)}(-2,2) = \begin{cases} 2u_p(-2,2) - 2u_{p-1}(-2,2) \equiv 2 \pmod p & \text{if } \left(\tfrac{3}{p}\right) = 1, \\ 2u_{p+1}(-2,2) + 4u_p(-2,2) \equiv -4 \pmod p & \text{if } \left(\tfrac{3}{p}\right) = -1, \end{cases}$$

$$\equiv 3\left(\tfrac{3}{p}\right) - 1 \pmod p.$$

Observing that $S_n = u_n(1,4) = 2^{-n}u_{2n}(-2,2)$, we get

$$S_{p-\left(\frac{3}{p}\right)}/p = 2^{\left(\frac{3}{p}\right)-p} v_{p-\left(\frac{3}{p}\right)}(-2,2)u_{p-\left(\frac{3}{p}\right)}(-2,2)/p$$

$$\equiv 2^{\left(\frac{3}{p}\right)-1}\left(3\left(\tfrac{3}{p}\right) - 1\right)u_{p-\left(\frac{3}{p}\right)}(-2,2)/p$$

$$= \frac{1}{2}\left(1 + 3\left(\tfrac{3}{p}\right)\right)u_{p-\left(\frac{3}{p}\right)}(-2,2)/p \pmod p.$$

This, together with the case $m = 3$ of Proposition 1 gives the result.

Remark 1: The sequence $\{S_n\}$ was first introduced by my brother Zhi-Wei Sun, who gave the formula for the sum $\sum_{k \equiv r \pmod{12}} \binom{n}{k}$ in terms of $\{S_n\}$ (cf. [10]).

Corollary 3: Let $p > 5$ be a prime and let $\{F_n\}$ denote the Fibonacci sequence. Then

(a) $\quad \displaystyle\sum_{k=1}^{(p-1)/2} \frac{5^k}{k} \equiv -5\frac{F_{p-\left(\frac{5}{p}\right)}}{p} - 2q_p(2) \pmod p.$

(b) $\quad \displaystyle\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 5^k} \equiv -\frac{F_{p-\left(\frac{5}{p}\right)}}{p} + q_p(5) - 2q_p(2) \pmod p.$

Proof: It is easily seen that $u_n(-4, 2) = 2^{n-1}F_n$. So we have

$$\frac{F_{p-\left(\frac{5}{p}\right)}}{p} = 2^{1-p+\left(\frac{5}{p}\right)}\frac{u_{p-\left(\frac{5}{p}\right)}(-4, 2)}{p} \equiv 2^{\left(\frac{5}{p}\right)}\frac{u_{p-\left(\frac{5}{p}\right)}(-4, 2)}{p} \pmod{p}.$$

Combining this with the case $m = 5$ of Proposition 1 yields the result.

Let $\{B_n\}$ and $\{B_n(x)\}$ be the Bernoulli numbers and Bernoulli polynomials given by

$$B_0 = 1, \quad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \quad (n \geq 2)$$

and

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

It is well known that (cf. [3])

$$\sum_{x=0}^{n-1}x^m = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1})$$

Lemma 2: Let p be an odd prime and let m be a positive integer such that $p \nmid m$. If $s \in \{1, 2, \ldots, m-1\}$, then

$$\sum_{1\leq k\leq\left[\frac{sp}{m}\right]}\frac{1}{k} \equiv -\left(B_{p-1}\left(\left\{\frac{sp}{m}\right\}\right) - B_{p-1}\right) \pmod{p},$$

where $[x]$ is the greatest integer not exceeding x and $\{x\} = x - [x]$.

Proof: Clearly,

$$\sum_{1\leq k\leq\left[\frac{sp}{m}\right]}\frac{1}{k} \equiv \sum_{1\leq k\leq\left[\frac{sp}{m}\right]}k^{p-2} = \frac{1}{p-1}\left(B_{p-1}\left(\left[\frac{sp}{m}\right]+1\right) - B_{p-1}\right)$$

$$= \frac{1}{p-1}\left(B_{p-1}\left(\frac{sp}{m}+1-\left\{\frac{sp}{m}\right\}\right) - B_{p-1}\right) \pmod{p}.$$

For any rational p-integers x and y, it is evident that (cf. [3])

$$pB_k(x) = \sum_{r=0}^{k}\binom{k}{r}pB_r x^{k-r} \equiv 0 \pmod{p} \quad \text{for } k = 0, 1, \ldots, p-2,$$

and so

$$B_{p-1}(x + py) - B_{p-1}(x) = \sum_{k=0}^{p-2}\binom{p-1}{k}B_k(x)(py)^{p-1-k} \equiv 0 \pmod{p}.$$

Hence, by the above and the relation $B_n(1-x) = (-1)^n B_n(x)$ (cf. [3]), we get

$$\sum_{1\leq k\leq\left[\frac{sp}{m}\right]}\frac{1}{k} \equiv \frac{1}{p-1}\left(B_{p-1}\left(1-\left\{\frac{sp}{m}\right\}\right) - B_{p-1}\right)$$

$$\equiv -\left(B_{p-1}\left(\left\{\frac{sp}{m}\right\}\right) - B_{p-1}\right) \pmod{p}.$$

This proves the lemma.

3. PROOF OF (1.1)-(1.5)

In [8], using the formula for the sum $\sum_{k \equiv r \,(\text{mod } 8)} \binom{n}{k}$, the author proved that

$$\frac{P_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{2} \sum_{\frac{p}{4} < k < \frac{p}{2}} \frac{(-1)^k}{k} \pmod{p} \tag{3.1}$$

and

$$\frac{P_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{4} \sum_{\frac{p}{8} < k < \frac{3p}{8}} \frac{1}{k} \pmod{p}. \tag{3.2}$$

Here, (3.1) was found by Z. W. Sun [10], and (3.2) was also given by Williams [12].

Now, putting (3.1) and (3.2) together with Corollary 1(a) proves (1.1) and (1.3).

To prove (1.2), we note that Williams (see [11]) has shown that

$$\frac{F_{p-(\frac{5}{p})}}{p} \equiv -\frac{2}{5} \sum_{k=1}^{p-1-[p/5]} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

Since Eisenstein, it is well known that (cf. [6])

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^{k-1}}{k} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \equiv q_p(2) \pmod{p}.$$

Thus, by Williams' result,

$$\frac{F_{p-(\frac{5}{p})}}{p} \equiv -\frac{2}{5}\left(2q_p(2) - \sum_{k=1}^{[p/5]} \frac{(-1)^{k-1}}{k}\right) \equiv -\frac{2}{5}\left(q_p(2) + \sum_{\frac{p}{5} < k < \frac{p}{2}} \frac{(-1)^{k-1}}{k}\right) \pmod{p}.$$

Hence, by Corollary 3(a), we have

$$\sum_{1 \le k < \frac{p}{2}} \frac{5^k}{k} \equiv -5\frac{F_{p-(\frac{5}{p})}}{p} - 2q_p(2) \equiv 2 \sum_{\frac{p}{5} < k < \frac{p}{2}} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

This proves (1.2).

Now, consider (1.4). From [2], we know that

$$B_{p-1}\left(\left\{\frac{p}{4}\right\}\right) - B_{p-1} \equiv 3q_p(2) \pmod{p}$$

and

$$B_{p-1}\left(\left\{\frac{3p}{8}\right\}\right) - B_{p-1} \equiv -2\frac{P_{p-(\frac{2}{p})}}{p} + 4q_p(2) \pmod{p}.$$

Thus, by using Lemma 2, we obtain

$$-\sum_{\frac{p}{4} < k < \frac{3p}{8}} \frac{1}{k} = \sum_{1 \le k < \frac{p}{4}} \frac{1}{k} - \sum_{1 \le k < \frac{3p}{8}} \frac{1}{k} \equiv -\left(B_{p-1}\left(\left\{\frac{p}{4}\right\}\right) - B_{p-1}\right) + B_{p-1}\left(\left\{\frac{3p}{8}\right\}\right) - B_{p-1}$$

$$\equiv -3q_p(2) + 4q_p(2) - 2\frac{P_{p-(\frac{2}{p})}}{p} \pmod{p}.$$

This, together with Corollary 1(b) proves (1.4).

Finally, we consider (1.5). By [2],

$$B_{p-1}\left(\left\{\frac{p}{6}\right\}\right) - B_{p-1} \equiv 2q_p(2) + \frac{3}{2}q_p(3) \pmod{p}$$

and

$$B_{p-1}\left(\left\{\frac{p}{12}\right\}\right) - B_{p-1} \equiv 3\left(\frac{3}{p}\right)\frac{S_{p-(\frac{3}{p})}}{p} + 3q_p(2) + \frac{3}{2}q_p(3) \pmod{p}.$$

Thus, by Lemma 2 and Corollary 2(a),

$$-\sum_{\frac{p}{12}<k<\frac{p}{6}}\frac{1}{k} \equiv \left(B_{p-1}\left(\left\{\frac{p}{6}\right\}\right) - B_{p-1}\right) - \left(B_{p-1}\left(\left\{\frac{p}{12}\right\}\right) - B_{p-1}\right)$$

$$\equiv 2q_p(2) + \frac{3}{2}q_p(3) - 3q_p(2) - \frac{3}{2}q_p(3) - 3\left(\frac{3}{p}\right)\frac{S_{p-(\frac{3}{p})}}{p} \equiv \sum_{1\leq k<\frac{p}{2}}\frac{3^k}{k} \pmod{p}.$$

This proves (1.5) and the proof is complete.

Remark 2: The congruences (1.1)-(1.3) can also be proved by using the method in the proof of (1.4) or (1.5).

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FORMULAS FOR CONVOLUTION FIBONACCI NUMBERS AND POLYNOMIALS

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1. INTRODUCTION

The Fibonacci numbers F_n $(n = 0, 1, 2, ...)$ satisfy the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0$, $F_1 = 1$. We denote

$$F(n, k) = \sum_{v_1 + v_2 + \cdots + v_k = n} F_{v_1} F_{v_2} \ldots F_{v_k} \quad (n \geq k), \tag{1}$$

where the summation is over all k-dimension nonnegative integer coordinates $(v_1, v_2, ..., v_k)$ such that $v_1 + v_2 + \cdots + v_k = n$ and k is any positive integer. The numbers $F(n, k)$ are called *convolution Fibonacci numbers* (see [3], [1], [2]). W. Zhang recently studied the convolution Fibonacci numbers $F(n, 2)$, $F(n, 3)$, and $F(n, 4)$ in [4], and the following three identities were obtained:

$$\sum_{a+b=n} F_a F_b = \frac{1}{5}((n-1)F_n + 2nF_{n-1}), \tag{2}$$

$$\sum_{a+b+c=n} F_a F_b F_c = \frac{1}{50}((5n^2 - 9n - 2)F_{n-1} + (5n^2 - 3n - 2)F_{n-2}), \tag{3}$$

$$\sum_{a+b+c+d=n} F_a F_b F_c F_d = \frac{1}{150}((4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3}). \tag{4}$$

The main purpose of this paper is that a recurrence relation and an expression in terms of Fibonacci numbers are given for convolution Fibonacci numbers $F(n, k)$, where n and k are any positive integers with $n \geq k$.

2. DEFINITIONS AND LEMMAS

Definition 1: The kth-order Fibonacci numbers $F_n^{(k)}$ are given by the following expansion formula:

$$\left(\frac{t}{1 - t - t^2}\right)^k = \sum_{n=0}^{\infty} F_n^{(k)} t^n. \tag{5}$$

By (1) and (5), we have $F_n^{(1)} = F_n$, $F(n, k) = F_n^{(k)}$, and $F_n^{(k)} = 0$ $(n < k)$.

Definition 2: The kth-order Fibonacci polynomials $F_n^{(k)}(x; p)$ are given by the following expansion formula:

$$\left(\frac{1}{1 - 2xt - pt^2}\right)^k = \sum_{n=0}^{\infty} F_n^{(k)}(x; p) t^n. \tag{6}$$

By (5) and (6), we have $F_n^{(k)} = F_{n-k}^{(k)}(\frac{1}{2}; 1)$ $(n \geq k)$.

[AUG.

Definition 3: Let n, k, j be three integers with $n \geq k \geq 2$, $0 \leq j \leq k-1$, and

$$M_{k-1-j,\,j} = \left\{ (x_1, x_2, ..., x_{k-1}) \mid x_i = 0 \text{ or } 1 \ (i = 1, 2, ..., k-1) \text{ and } \sum_{i=1}^{k-1} x_i = k-1-j \right\}.$$

For any $(x_1, x_2, ..., x_{k-1}) \in M_{k-1-j,\,j}$, $\lambda(x_1, x_2, ..., x_{k-1}; k, n)$ is defined by

$$\lambda_{k-1-j,\,j}(x_1, x_2, ..., x_{k-1}; k, n) = \left(\frac{y_1}{k-1} + z_1 \right)\left(\frac{y_2}{k-2} + z_2 \right) \cdots \left(\frac{y_{k-1}}{1} + z_{k-1} \right),$$

where $y_1, y_2, ..., y_{k-1}, z_1, z_2, ..., z_{k-1}$ satisfies the following:

(a) If $x_1 = 1$, then $y_1 = n$; if $x_1 = 0$, then $y_1 = n-1$.

(b) $\forall i : 1 \leq i \leq k-1$; if $x_i = 1$, then $z_i = -1$; if $x_i = 0$, then $z_i = 1$.

(c) $\forall i : 1 \leq i \leq k-2$; if $x_i = x_{i+1} = 1$ or $x_i = 0$, $x_{i+1} = 1$, then $y_{i+1} = y_i$; if $x_i = x_{i+1} = 0$ or $x_i = 1$, $x_{i+1} = 0$, then $y_{i+1} = y_i - 1$.

Lemma 1:

(a) $\dfrac{d}{dx} F_n^{(k)}(x; p) = 2k F_{n-1}^{(k+1)}(x; p) \ (n \geq 1)$;   (7)

(b) $(n+1) F_{n+1}^{(k)}(x; p) = 2x(n+k) F_n^{(k)}(x; p) + p(n+2k-1) F_{n-1}^{(k)}(x; p)$;   (8)

(c) $\dfrac{d}{dx} F_{n+1}^{(k)}(x; p) - 2x\dfrac{d}{dx} F_n^{(k)}(x; p) - 2k F_n^{(k)}(x; p) - p\dfrac{d}{dx} F_{n-1}^{(k)}(x; p) = 0$.   (9)

Proof: By Definition 2. $\square$

Lemma 2: For $k \geq 2$, we have:

(a) $x\dfrac{d}{dx} F_n^{(k)}(x; p) + p\dfrac{d}{dx} F_{n-1}^{(k)}(x; p) = n F_n^{(k)}(x; p)$;   (10)

(b) $\dfrac{d}{dx} F_n^{(k)}(x; p) - x\dfrac{d}{dx} F_{n-1}^{(k)}(x; p) = (n-1+2k) F_{n-1}^{(k)}(x; p)$.   (11)

Proof: By Lemma 1(b) and (c), we immediately obtain (10) and (11). $\square$

Lemma 3: We denote

$$s(n, k, j) := \sum_{(x_1, x_2, ..., x_{k-1}) \in M_{k-1-j,\,j}} \lambda_{k-1-j,\,j}(x_1, x_2, ..., x_{k-1}; k, n) \quad (0 \leq j \leq k-1),$$

where the summation is over all $(k-1)$-dimension coordinates $(x_1, x_2, ..., x_{k-1})$ such that $(x_1, x_2, ..., x_{k-1}) \in M_{k-1-j,\,j}$, then:

(a) $\left(\dfrac{n}{k} - 1 \right) s(n, k, 0) = s(n, k+1, 0)$;

(b) $\left(\dfrac{n-1}{k} + 1 \right) s(n-1, k, k-1) = s(n, k+1, k)$;

(c) $\left(\dfrac{n}{k} - 1 \right) s(n, k, j) + \left(\dfrac{n-1}{k} + 1 \right) s(n-1, k, j-1) = s(n, k+1, j) \quad (1 \leq j \leq k-1)$.

Proof:

(a) $\left(\dfrac{n}{k}-1\right)s(n,k,0) = \left(\dfrac{n}{k}-1\right) \displaystyle\sum_{(x_1,x_2,\dots,x_{k-1})\in M_{k-1,0}} \lambda_{k-1,0}(x_1,x_2,\dots,x_{k-1}; k,n)$

$= \left(\dfrac{n}{k}-1\right)\lambda_{k-1,0}(1,1,\dots,1; k,n) = \left(\dfrac{n}{k}-1\right)\left(\dfrac{n}{k-1}-1\right)\left(\dfrac{n}{k-2}-1\right)\cdots\left(\dfrac{n}{1}-1\right)$

$= \lambda_{k,0}(1,1,\dots,1; k+1,n) = \displaystyle\sum_{(x_1,x_2,\dots,x_k)\in M_{k,0}} \lambda_{k,0}(x_1,x_2,\dots,x_k; k+1,n) = s(n,k+1,0).$

(b) $\left(\dfrac{n-1}{k}+1\right)s(n-1,k,k-1) = \left(\dfrac{n-1}{k}+1\right) \displaystyle\sum_{(x_1,x_2,\dots,x_{k-1})\in M_{0,k-1}} \lambda_{0,k-1}(x_1,x_2,\dots,x_{k-1}; k,n-1)$

$= \left(\dfrac{n-1}{k}+1\right)\lambda_{0,k-1}(0,0,\dots,0; k,n-1) = \left(\dfrac{n-1}{k}+1\right)\left(\dfrac{n-2}{k-1}+1\right)\cdots\left(\dfrac{n-k}{1}+1\right)$

$= \lambda_{0,k}(0,0,\dots,0; k+1,n) = \displaystyle\sum_{(x_1,x_2,\dots,x_k)\in M_{0,k}} \lambda_{0,k}(x_1,x_2,\dots,x_k; k+1,n) = s(n,k+1,k).$

(c) $s(n,k+1,j) = \displaystyle\sum_{(x_1,x_2,\dots,x_k)\in M_{k-j,j}} \lambda_{k-j,j}(x_1,x_2,\dots,x_k; k+1,n)$

$= \displaystyle\sum_{(1,x_2,\dots,x_k)\in M_{k-j,j}} \lambda_{k-j,j}(1,x_2,\dots,x_k; k+1,n) + \displaystyle\sum_{(0,x_2,\dots,x_k)\in M_{k-j,j}} \lambda_{k-j,j}(0,x_2,\dots,x_k; k+1,n)$

$= \left(\dfrac{n}{k}-1\right) \displaystyle\sum_{(x_2,\dots,x_k)\in M_{k-1-j,j}} \lambda_{k-1-j,j}(x_2,\dots,x_k; k,n)$

$\qquad\qquad + \left(\dfrac{n-1}{k}+1\right) \displaystyle\sum_{(x_2,\dots,x_k)\in M_{k-j,j-1}} \lambda_{k-j,j-1}(x_2,\dots,x_k; k,n-1)$

$= \left(\dfrac{n}{k}-1\right)s(n,k,j) + \left(\dfrac{n-1}{k}+1\right)s(n-1,k,j-1). \quad\square$

3. MAIN RESULTS

Theorem 1: For $n \geq k \geq 2$, we have:

(a) $F_n^{(k)}(x;p) = \dfrac{x}{2(x^2+p)}\left(\dfrac{n+k}{k-1}-1\right)F_{n+1}^{(k-1)}(x;p) + \dfrac{p}{2(x^2+p)}\left(\dfrac{n+k-1}{k-1}+1\right)F_n^{(k-1)}(x;p);$ (12)

(b) $F_n^{(k)} = \dfrac{1}{5}\left(\dfrac{n}{k-1}-1\right)F_n^{(k-1)} + \dfrac{2}{5}\left(\dfrac{n-1}{k-1}+1\right)F_{n-1}^{(k-1)}.$ (13)

Proof:

(a) By (10) and (11), we have

$$(x^2+p)\dfrac{d}{dx}F_n^{(k)}(x;p) = nxF_n^{(k)}(x;p) + p(n-1+2k)F_{n-1}^{(k)}(x;p).$$ (14)

By (14) and (7), we immediately obtain (12).

(b) Taking $x = \frac{1}{2}$ and $p = 1$ in (12) and noting that

$$F_n^{(k)} = F_{n-k}^{(k)}\left(\frac{1}{2}; 1\right),$$

we immediately obtain (13). $\square$

Theorem 2: For $n \geq k \geq 2$, we have

$$F_{n-k}^{(k)}(x; p) = \sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-1-j}\left(\frac{p}{2(x^2+p)}\right)^j s(n, k, j)F_{n-1-j}(x; p), \tag{15}$$

where $s(n, k, j)$ is defined as in Lemma 3.

Proof (using mathematical induction):

$1°$ When $k = 2$, by Theorem 1, we have

$$F_{n-2}^{(2)}(x; p) = \frac{x}{2(x^2+p)}(n-1)F_{n-1}(x; p) + \frac{p}{2(x^2+p)}nF_{n-2}(x; p)$$

$$= \frac{x}{2(x^2+p)}\lambda_{1,0}(1; 2, n)F_{n-1}(x; p) + \frac{p}{2(x^2+p)}\lambda_{0,1}(0; 2, n)F_{n-2}(x; p)$$

$$= \sum_{j=0}^{1}\left(\frac{x}{2(x^2+p)}\right)^{1-j}\left(\frac{p}{2(x^2+p)}\right)^j \sum_{(x_1)\in M_{1-j,j}} \lambda_{1-j,j}(x_1; 2, n)F_{n-1-j}(x; p)$$

$$= \sum_{j=0}^{1}\left(\frac{x}{2(x^2+p)}\right)^{1-j}\left(\frac{p}{2(x^2+p)}\right)^j s(n, 2, j)F_{n-1-j}(x; p). \tag{16}$$

(16) shows that (15) is true for the natural number 2.

$2°$ Suppose that (15) is true for some natural number k. By the supposition, Theorem 1, and Lemma 3, we have

$$F_{n-(k+1)}^{(k+1)}(x; p) = \frac{x}{2(x^2+p)}\left(\frac{n}{k}-1\right)F_{n-k}^{(k)}(x; p) + \frac{p}{2(x^2+p)}\left(\frac{n-1}{k}+1\right)F_{n-1-k}^{(k)}(x; p)$$

$$= \sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^j\left(\frac{n}{k}-1\right)s(n, k, j)F_{n-1-j}(x; p)$$

$$+ \sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-1-j}\left(\frac{p}{2(x^2+p)}\right)^{j+1}\left(\frac{n-1}{k}+1\right)s(n-1, k, j)F_{n-2-j}(x; p)$$

$$= \sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^j\left(\frac{n}{k}-1\right)s(n, k, j)F_{n-1-j}(x; p)$$

$$+ \sum_{j=1}^{k}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^j\left(\frac{n-1}{k}+1\right)s(n-1, k, j-1)F_{n-1-j}(x; p)$$

$$= \left(\frac{x}{2(x^2+p)}\right)^k\left(\frac{n}{k}-1\right)s(n, k, 0)F_{n-1}(x; p) +$$

$$+\sum_{j=1}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}\left(\left(\frac{n}{k}-1\right)s(n,k,j)+\left(\frac{n-1}{k}+1\right)s(n-1,k,j-1)\right)F_{n-1-j}(x;p)$$

$$+\left(\frac{p}{2(x^2+p)}\right)^{k}\left(\frac{n-1}{k}+1\right)s(n-1,k,k-1)F_{n-1-k}(x;p)$$

$$=\left(\frac{x}{2(x^2+p)}\right)^{k}s(n,k+1,0)F_{n-1}(x;p)$$

$$+\sum_{j=1}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}s(n,k+1,j)F_{n-1-j}(x;p)$$

$$+\left(\frac{p}{2(x^2+p)}\right)^{k}s(n,k+1,k)F_{n-1-k}(x;p)$$

$$=\sum_{j=0}^{k}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}s(n,k+1,j)F_{n-1-j}(x;p). \tag{17}$$

(17) shows that (15) is also true for the natural number $k+1$. $\square$

From 1° and 2°, we know that (15) is true.

Theorem 3: For $n\geq k\geq 2$, we have

$$F(n,k)=F_n^{(k)}=\left(\frac{1}{5}\right)^{k-1}\sum_{j=0}^{k-1}2^j s(n,k,j)F_{n-j}, \tag{18}$$

where $s(n,k,j)$ is defined as in Lemma 3.

Proof: Taking $x=\frac{1}{2}$ and $p=1$ in Theorem 2, and noting that

$$F(n,k)=F_n^{(k)}=F_{n-k}^{(k)}\left(\frac{1}{2},1\right) \quad\text{and}\quad F_{n-j}=F_{n-1-j}\left(\frac{1}{2};1\right),$$

we immediately obtain (18). $\square$

Corollary 1: For $n\geq k\geq 2$, we have

(a) $F(n,2)=\frac{1}{5}((n-1)F_n+2nF_{n-1})$;

(b) $F(n,3)=\frac{1}{50}((n^2-3n+2)F_n+(4n^2-6n-4)F_{n-1}+(4n^2-4)F_{n-2})$;

(c) $F(n,4)=\frac{1}{750}((n^3-6n^2+11n-6)F_n+(6n^3-24n^2+6n+36)F_{n-1}$
$+(12n^3-24n^2-48n+36)F_{n-2}+(8n^3-32n)F_{n-3})$;

(d) $F(n,5)=\frac{1}{15000}((n^4-10n^3+35n^2-50n+24)F_n+(8n^4-60n^3+100n^2+120n-288)F_{n-1}$
$+(24n^4-120n^3-60n^2+660n-144)F_{n-2}+(32n^4-80n^3-320n^2+440n+288)F_{n-3}$
$+(16n^4-160n^2+144)F_{n-4})$.

Remark: By Corollary 1(a)-(c) and $F_n=F_{n-1}+F_{n-2}$ $(n\geq 2)$, we immediately obtain (2), (3), and (4) (see Zhang [4]).

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A NOTE ON HORADAM'S SEQUENCE

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1. INTRODUCTION

Horadam's sequence $\{w_n(a,b;p,q)\}$, or briefly $\{w_n\}$, is defined by the recurrence relation $w_0 = a$, $w_1 = b$, $w_n = pw_{n-1} - qw_{n-2}$ $(n \geq 2)$ (see, e.g., [1], [2], [3]). The sequence $\{u_n(p,q)\}$, or briefly $\{u_n\}$, is defined as $u_n = w_n(1, p; p, q)$, and the sequence $\{v_n(p,q)\}$, or briefly $\{v_n\}$, is defined as $v_n = w_n(2, p; p, q)$. The sequences $\{u_n\}$ and $\{v_n\}$, respectively, are generalized Fibonacci and Lucas sequences.

In this note we study linear combinations of Horadam's sequences and the generating function of the ordinary product of two of Horadam's sequences. Similar results for the generalized Fibonacci sequence $\{u_n\}$ are given by McCarthy and Sivaramakrishnan [4]. In [1], Horadam studied the generating function of powers of $\{w_n\}$. The main results are in Sections 2 and 3 below

2. LINEAR COMBINATIONS

Let $p_1, p_2, ..., p_k$ be distinct complex numbers and let $w_n^{(j)} = w_n(a, b; p_j, q)$ and $u_n^{(j)} = u_n(p_j, q)$ for $j = 1, 2, ..., k$. McCarthy and Sivaramakrishnan show that the sequences $\{u_n^{(j)}\}$ are linearly independent and that if $u_n = c_1 u_n^{(1)} + c_2 u_n^{(2)} + \cdots + c_k u_n^{(k)}$ for all $n \geq 0$ then, for some h with $1 \leq h \leq k$, we have $c_h = 1$, $c_j = 0$ for $j \neq h$ and $p = p_h$ (see Theorems 3 and 4 in [4]).

In this section we show that these results hold for the more general sequences $\{w_n\}$ and $\{w_n^{(j)}\}$. In the proofs, we need the identities

$$w_n = a \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} p^{n-2k} q^k + (b-pa) \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-1-k}{k} p^{n-1-2k} q^k \qquad (2.1)$$

(see eq. (1.7) in [3]) and

$$w_n = bu_{n-1} - qau_{n-2} \qquad (2.2)$$

(see eq. (2.14) in [2]).

We now give the generalizations.

Theorem 2.1: Let $a \neq 0$. The sequences $\{w_n^{(j)}\}$ are linearly independent.

Proof: Suppose that

$$\sum_{j=1}^{k} c_j w_n^{(j)} = 0 \quad \text{for } n \geq 0.$$

Then the first k of these equations form a system of k linear equations with $c_1, c_2, ..., c_k$ as unknowns. The matrix of coefficients is $[w_i^{(j)}]$, $i = 0, 1, ..., k-1$; $j = 1, 2, ..., k$. The row with $i = 0$ is $a, a, ..., a$. The other rows can be obtained by equation (2.1). Thus, if $i \geq 1$, then by adding appropriate multiples of the rows $0, 1, ..., i-1$ to row i, the matrix can be transformed into one having $2ap_1^i, 2ap_2^i, ..., 2ap_k^i$ as its row i. Thus,

$$\det[w_i^{(j)}] = a(2a)^{k-1} \prod_{1 \le i < j \le k} (p_j - p_i) \ne 0,$$

and hence $c_j = 0$, $j = 1, 2, \dots, k$. $\square$

Theorem 2.2: Let $a, b \ne 0$. If, for complex numbers $c_1, c_2, \dots, c_k$,

$$\sum_{j=1}^{k} c_j w_n^{(j)} = w_n, \quad n \ge 0, \tag{2.3}$$

then for some h with $1 \le h \le k$ we have $c_h = 1$, $c_j = 0$ for $j \ne h$ and $p = p_h$.

Proof: We prove that

$$\sum_{j=1}^{k} c_j u_n^{(j)} = u_n, \quad n \ge 0. \tag{2.4}$$

Assume that $n = 0$. Then $\sum_{j=1}^{k} c_j w_0^{(j)} = w_0$ or $\sum_{j=1}^{k} c_j a = a$. Since $a \ne 0$, we have $\sum_{j=1}^{k} c_j = 1$ or $\sum_{j=1}^{k} c_j u_0^{(j)} = u_0$. Thus, (2.4) holds for $n = 0$. Assume that (2.4) holds for $n < m$. Then, with the aid of (2.2) and the induction assumption,

$$\sum_{j=1}^{k} c_j w_{m+1}^{(j)} = b\sum_{j=1}^{k} c_j u_m^{(j)} - qa\sum_{j=1}^{k} c_j u_{m-1}^{(j)} = b\sum_{j=1}^{k} c_j u_m^{(j)} - qau_{m-1}.$$

On the other hand, with the aid of (2.3) and (2.2),

$$\sum_{j=1}^{k} c_j w_{m+1}^{(j)} = w_{m+1} = bu_m - qau_{m-1}.$$

Since $b \ne 0$, we see that (2.4) holds for $n = m$. This completes the proof of (2.4). Now, applying the result of McCarthy and Sivaramakrishnan [4] referred to at the beginning of this section to (2.4), we have $c_h = 1$, $c_j = 0$ for $j \ne h$ and $p = p_h$ for some h with $1 \le h \le k$. This completes the proof of Theorem 2.2. $\square$

Remark 2.1: Considering the sequence $\{w_n / a\}$, $a \ne 0$, and using the results of [4], it can be shown that Theorems 2.1 and 2.2 hold for the sequences $\{w_n(a, p; p, q)\}$ and $\{w_n(a, p_j; p_j, q)\}$. Thus, Theorems 2.1 and 2.2 hold for the generalized Lucas sequence $\{v_n\}$.

3. GENERATING FUNCTIONS

Let the sequence $\{w_n'(a', b'; p', q')\}$, or briefly $\{w_n'\}$, be defined by the recurrence relation $w_0' = a'$, $w_1' = b'$, $w_n' = p'w_{n-1}' - q'w_{n-2}'$ $(n \ge 2)$. In this section we evaluate the generating function of the ordinary product $\{w_n w_n'\}$ of two of Horadam's sequences. For the sake of brevity, we denote $W_2(x) = \sum_{n=0}^{\infty} w_n w_n' x^n$. We thus evaluate $W_2(x)$. As special cases, we obtain the generating functions of $\{u_n u_n'\}$ and $\{w_n^2\}$ evaluated by McCarthy and Sivaramakrishnan [4] and by Horadam [1], respectively.

Lemma 3.1:

$$\sum_{n=0}^{\infty} w_{n+1} w_n' x^n = \frac{a'b - paa' + (p'qaa' - qab')x + (p - p'qx)W_2(x)}{1 - qq'x^2}.$$

Proof: It is clear that

$$(1 - qq'x^2)\sum_{n=0}^{\infty} w_{n+1}w_n'x^n = \sum_{n=0}^{\infty} w_{n+1}w_n'x^n - qq'\sum_{n=0}^{\infty} w_{n+1}w_n'x^{n+2}$$

$$= a'b + \sum_{n=1}^{\infty} (pw_n - qw_{n-1})w_n'x^n + q\sum_{n=0}^{\infty} w_{n+1}(w_{n+2}' - p'w_{n+1}')x^{n+2}$$

$$= a'b + pW_2(x) - paa' - q\sum_{n=1}^{\infty} w_{n-1}w_n'x^n + q\sum_{n=2}^{\infty} w_{n-1}w_n'x^n - p'qx(W_2(x) - aa')$$

$$= a'b - paa' - qab'x + p'qaa'x + pW_2(x) - p'qxW_2(x).$$

The proof of Lemma 3.1 is complete. □

Lemma 3.2:

$$\sum_{n=0}^{\infty} w_n w_{n+1}'x^n = \frac{ab' - p'aa' + (pq'aa' - q'a'b)x + (p' - pq'x)W_2(x)}{1 - qq'x^2}.$$

Lemma 3.2 follows from Lemma 3.1 by replacing a, b, p, and q with a', b', p', and q', respectively.

Theorem 3.1:

$$W_2(x) = \frac{A(x)}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2q'^2x^4},$$

where

$$A(x) = aa' + (bb' - aa'pp')x + (aa'p^2q' + aa'p'^2q - aa'qq' - ab'p'q - a'bpq')x^2$$
$$+ (ab'pqq' + a'bp'qq' - aa'pp'qq' - bb'qq')x^3.$$

Proof: We have

$$W_2(x) = \sum_{n=0}^{\infty} w_n w_n'x^n = aa' + bb'x + \sum_{n=2}^{\infty} (pw_{n-1} - qw_{n-2})(p'w_{n-1}' - q'w_{n-2}')x^n$$

$$= aa' + bb'x + pp'\sum_{n=2}^{\infty} w_{n-1}w_{n-1}'x^n - pq'\sum_{n=2}^{\infty} w_{n-1}w_{n-2}'x^n$$

$$- p'q\sum_{n=2}^{\infty} w_{n-2}w_{n-1}'x^n + qq'\sum_{n=2}^{\infty} w_{n-2}w_{n-2}'x^n$$

$$= aa' + bb'x + pp'x\left(\sum_{n=0}^{\infty} w_n w_n'x^n - aa'\right) - pq'x^2\sum_{n=0}^{\infty} w_{n+1}w_n'x^n$$

$$- p'qx^2\sum_{n=0}^{\infty} w_n w_{n+1}'x^n + qq'x^2\sum_{n=0}^{\infty} w_n w_n'x^n.$$

Applying Lemmas 3.1 and 3.2, we obtain

$$W_2(x) = aa' + bb'x - aa'pp'x + pp'xW_2(x) - pq'x^2\frac{a'b - paa' + p'qaa'x - qab'x + (p - p'qx)W_2(x)}{1 - qq'x^2}$$

$$- p'qx^2\frac{ab' - p'aa' + pq'aa'x - q'a'bx + (p' - pq'x)W_2(x)}{1 - qq'x^2} + qq'x^2W_2(x).$$

Solving for $W_2(x)$, we get Theorem 3.1. □

Corollary 3.1 [4]:

$$\sum_{n=0}^{\infty} u_n u_n' x^n = \frac{1 - qq'x^2}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2 q'^2 x^4}.$$

Corollary 3.2:

$$\sum_{n=0}^{\infty} v_n v_n' x^n = \frac{4 - 3pp'x + (2p^2 q' + 2p'^2 q - 4qq')x^2 - pp'qq'x^3}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2 q'^2 x^4}.$$

Corollary 3.3:

$$\sum_{n=0}^{\infty} u_n v_n' x^n = \frac{2 - pp'x + (p'^2 q - 2qq')x^2}{1 - pp'x + [(p^2 - q)q' + (p'^2 - q')q]x^2 - pp'qq'x^3 + q^2 q'^2 x^4}.$$

Corollary 3.4 [1]:

$$\sum_{n=0}^{\infty} w_n^2 x^n = \frac{a^2 + [b^2 - a^2(p^2 - q)]x + q(b - pa)^2 x^2}{(1 - qx)[1 - (p^2 - 2q)x + q^2 x^2]}.$$

Corollary 3.5 [4]:

$$\sum_{n=0}^{\infty} u_n^2 x^n = \frac{1 + qx}{(1 - qx)[1 - (p^2 - 2q)x + q^2 x^2]}.$$

Corollary 3.6:

$$\sum_{n=0}^{\infty} v_n^2 x^n = \frac{4 + (4q - 3p^2)x + p^2 qx^2}{(1 - qx)[1 - (p^2 - 2q)x + q^2 x^2]}.$$

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CLOSED FORMULA FOR POLY-BERNOULLI NUMBERS

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1. INTRODUCTION AND BACKGROUND

In the present note we shall give two proofs of a property of the poly-Bernoulli numbers, the closed formula for negative index poly-Bernoulli numbers given by Arakawa and Kaneko [1]. The first proof uses weighted Stirling numbers of the second kind (see [2], [3]). The second, much simpler, proof is due to Zeilberger.

In Kaneko's paper, "On Poly-Bernoulli Numbers" [5], the poly-Bernoulli numbers, which generalize the classical Bernoulli numbers, are defined and studied. For every integer k, called the index, we define a sequence of rational numbers B_n^k $(n = 0, 1, 2, ...)$, which we refer to as poly-Bernoulli numbers, by

$$\frac{1}{z} \mathrm{Li}_k(z)\big|_{z=1-e^{-x}} = \sum_{n=0}^{\infty} B_n^k \frac{x^n}{n!}. \tag{1}$$

Here, for any integer k, $\mathrm{Li}_k(z)$ denotes the formal power series $\sum_{m=1}^{\infty} z^m / m^k$, which is the kth polylogarithm if $k \geq 1$ and a rational function if $k \leq 0$. When $k = 1$, B_n^1 is the usual Bernoulli number (with $B_1^1 = 1/2$). In [4] Kaneko obtained an explicit formula for B_n^k:

$$B_n^k = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{(m+1)^k} \left\{ {n \atop m} \right\}, \tag{2}$$

where $\left\{ {n \atop m} \right\}$ is an integer referred to as a Stirling number of the second kind [6].

2. CLOSED FORMULA

Theorem 2.1 (Closed Formula): For any $n, k \geq 0$, we have

$$B_n^{-k} = \sum_{j=0}^{} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}. \tag{3}$$

We need two lemmas. We use the notation and numeration of the equations in Carlitz's paper [3].

Lemma 2.1:

$$\sum_{m=0}^{n} (-1)^m m! \binom{m}{\ell} \left\{ {n \atop m} \right\} = (-1)^n \ell! \left\{ {n+1 \atop \ell+1} \right\} = (-1)^n \ell! \, R(n, \ell, 1), \tag{4}$$

where

$$R(n, k, \lambda) = \sum_{m=0}^{n-k} \binom{n}{m} \left\{ {n-m \atop k} \right\} \lambda^m.$$

Proof: In order to prove this lemma, we calculate the generating function:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m m! \binom{m}{\ell} \left\{ {n \atop m} \right\} \frac{z^n}{n!} = \sum_{m=0}^{\infty} (-1)^m \binom{m}{\ell} m! \sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{z^n}{n!}$$

$$= \sum_{m=0}^{\infty} (-1)^m \binom{m}{\ell} m! \frac{(e^z-1)^m}{m!} = \frac{(1-e^z)^{\ell}}{(1-(1-e^z))^{\ell+1}}, \quad \text{by the generalized binomial theorem,}$$

$$= e^{-z}(e^{-z}-1)^{\ell} = \sum_{n=0}^{\infty} \ell! \, R(n, \ell, 1)(-1)^n \frac{z^n}{n!}, \quad \text{by [3], (3.9).}$$

Lemma 2.2:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{-k} x^n y^k = \sum_{j=0}^{\infty} p_j(x) p_j(y) \tag{5}$$

where $p_j(x) = j! \sum_{n=0}^{\infty} R(n, j, 1)x^n$.

Proof: By (2), we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{-k} x^n y^k = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left((-1)^n (-1)^m m! \begin{Bmatrix} n \\ m \end{Bmatrix} (m+1)^k \right) x^n y^k, \quad \text{by [3], (3.4),}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \ell! \binom{m}{\ell} R(k, \ell, 1) \left((-1)^n (-1)^m m! \begin{Bmatrix} n \\ m \end{Bmatrix} \right) x^n y^k$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \ell! \binom{m}{\ell} \left((-1)^n (-1)^m m! \begin{Bmatrix} n \\ m \end{Bmatrix} \right) \frac{p_{\ell}(y)}{\ell!} x^n$$

$$= \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} p_{\ell}(y)(-1)^n \sum_{m=0}^{\infty} \binom{m}{\ell} (-1)^m m! \begin{Bmatrix} n \\ m \end{Bmatrix} x^n, \quad \text{by Lemma 2.1,}$$

$$= \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} p_{\ell}(y)(-1)^n (-1)^n \ell! \, R(n, \ell, 1) x^n = \sum_{\ell=0}^{\infty} p_{\ell}(y) \ell! \sum_{n=0}^{\infty} R(n, \ell, 1) x^n = \sum_{\ell=0}^{\infty} p_{\ell}(x) p_{\ell}(y).$$

Proof of (3): To prove (3), we compare the coefficients on both sides of (5). In the course of Arakawa and Kaneko's proof they prove the following proposition.

Proposition 2.1: For $n > 0$,

$$\sum_{\ell=0}^{n} (-1)^{\ell} B_{n-\ell}^{-\ell} = 0.$$

Proof: We offer a more direct proof:

$$\sum_{\ell=0}^{n} (-1)^{\ell} B_{n-\ell}^{-\ell} = \sum_{\ell=0}^{n} (-1)^{\ell} (-1)^{n-\ell} \sum_{m=0}^{n-\ell} (-1)^m m! (m+1)^{\ell} \begin{Bmatrix} n-\ell \\ m \end{Bmatrix}$$

$$= (-1)^n \sum_{m=0}^{n} \sum_{\ell=0}^{n} (-1)^m m! (m+1)^{\ell} \begin{Bmatrix} n-\ell \\ m \end{Bmatrix}, \quad \text{by [4], (6.20),}$$

$$= (-1)^n \sum_{m=0}^{n} (-1)^m m! \begin{Bmatrix} n+1 \\ m+1 \end{Bmatrix} = (-1)^n \delta_{1\,n+1} = 0.$$

3. ANOTHER PROOF

In Kaneko's paper [4], he obtained the *symmetric* formula:

$$\sum_{k \geq 0} \sum_{n \geq 0} B_n^{-k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \tag{6}$$

By using (6), D. Zeilberger gives a much simpler proof of (3) as follows:

$$\sum_{k \geq 0} \sum_{n \geq 0} B_n^{-k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}} = e^{x+y} \sum_{j \geq 0} (1-e^x)^j (1-e^y)^j$$

$$= \sum_{j \geq 0} \frac{1}{(1+j)^2} (j+1)(1-e^x)^j (-e^x)(j+1)(1-e^y)^j (-e^y)$$

$$= \sum_{j \geq 0} \frac{1}{(j+1)^2} D_x[(1-e^x)^{j+1}] D_y[(1-e^y)^{j+1}].$$

Now using the usual generating function for the Stirling numbers of the second kind $\left\{\begin{matrix} n \\ k \end{matrix}\right\}$, i.e.,

$$\sum_{n \geq k} \left\{\begin{matrix} n \\ k \end{matrix}\right\} \frac{u^n}{n!} = \frac{(e^u - 1)^k}{k!},$$

he obtains:

$$\sum_{n \geq 0} \sum_{k \geq 0} B_n^{-k} \frac{x^n}{n!} \frac{y^k}{k!} = \sum_{j \geq 0} \frac{1}{(j+1)^2} D_x \left[(-1)^{j+1}(j+1)! \sum_{n \geq j+1} \left\{\begin{matrix} n \\ j+1 \end{matrix}\right\} \frac{x^n}{n!} \right]$$

$$\times D_y \left[(-1)^{j+1}(j+1)! \sum_{k \geq j+1} \left\{\begin{matrix} k \\ j+1 \end{matrix}\right\} \frac{y^k}{k!} \right]$$

$$= \sum_{j \geq 0} j!^2 \sum_{n \geq j} \left\{\begin{matrix} n+1 \\ j+1 \end{matrix}\right\} \frac{x^n}{n!} \sum_{k \geq j} \left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\} \frac{y^k}{k!}$$

$$= \sum_{n \geq 0} \sum_{k \geq 0} \frac{x^n}{n!} \frac{y^k}{k!} \sum_{j \geq 0} j!^2 \left\{\begin{matrix} n+1 \\ j+1 \end{matrix}\right\} \left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\}.$$

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SOME CONSEQUENCES OF GAUSS' TRIANGULAR NUMBER THEOREM

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INTRODUCTION

If n is a nonnegative integer, let $t(n) = n(n+1)/2$ denote the nth triangular number. Gauss' triangular number theorem states that if x is a complex variable such that $|x| < 1$, then

$$\prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}} = \sum_{n \geq 0} x^{t(n)}$$

(see [1], p. 326, Ex. 5b, or [3], Theorem 354, p. 284).

In this note, we make use of this Gaussian formula to derive several apparently new identities concerning $q_0(n)$, the number of self-conjugate partitions of n.

PRELIMINARIES

Definition 1: Let $p(n)$ denote the number of unrestricted partitions of n.

Definition 2: Let $q_0(n)$ denote the number of partitions of n into distinct odd parts (or the number of self-conjugate partitions of n).

Definition 3: If $r \geq 1$, let $q_r(n)$ denote the number of partitions of n into distinct parts in r colors.

Remark: If $f(n)$ is any of the above partition functions, we define $f(0) = 1$, $f(\alpha) = 0$ if α is not a nonnegative integer.

Definition 4 (Pentagonal numbers): If $k \in \mathbb{Z}$, then

$$\omega(k) = \frac{k(3k - 1)}{2}.$$

IDENTITIES

Let x be a complex variable such that $|x| < 1$. Let $r \geq 1$. Let $j \geq 1$. Then we have:

$$\prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}} = \sum_{n \geq 0} x^{t(n)}, \tag{1}$$

$$\prod_{n \geq 1} (1 - x^{jn})^{-1} = \sum_{n \geq 0} p\left(\frac{n}{j}\right) x^n, \tag{2}$$

$$\prod_{n \geq 1} (1 - x^{jn}) = 1 + \sum_{k \geq 1} (-1)^k (x^{j\omega(k)} + x^{j\omega(-k)}), \tag{3}$$

$$\prod_{n \geq 1} (1 + x^{2n-1}) = \sum_{n \geq 0} q_0(n) x^n, \tag{4}$$

$$\prod_{n\geq 1}(1-x^{2n-1})^{-1} = \prod_{n\geq 1}(1+x^n) = \sum_{n\geq 0}q(n)x^n, \tag{5}$$

$$\prod_{n\geq 1}(1+x^n)^r = \sum_{n\geq 0}q_r(n)x^n, \tag{6}$$

$$\left(\sum_{n\geq 0}a_n x^n\right)\left(\sum_{n\geq 0}b_n x^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}a_{n-k}b_k\right)x^n. \tag{7}$$

Remark: For proofs, see Chapter 19 of [3].

THE MAIN RESULTS

Theorem 1: Let the integer $n \geq 0$. Then

$$q_0(n) + \sum_{k\geq 1}(-1)^k\left(q_0(n-4\omega(k)) + q_0(n-4\omega(-k))\right) = \begin{cases}1 & \text{if } n = t(j) \text{ for some } j \geq 0, \\ 0 & \text{otherwise.}\end{cases}$$

Proof: By (1) and (5), we have:

$$\sum_{n\geq 0}x^{t(n)} = \prod_{n\geq 1}(1+x^n)(1-x^{2n}) = \prod_{n\geq 1}(1+x^{2n-1})(1+x^{2n})(1-x^{2n})$$

$$= \prod_{n\geq 1}(1+x^{2n-1})(1-x^{4n}) = \prod_{n\geq 1}(1+x^{2n-1})\prod_{n\geq 1}(1-x^{4n}) = \left(\sum_{n\geq 0}q_0(n)x^n\right)\prod_{n\geq 1}(1-x^{4n})$$

$$= \sum_{n\geq 0}\left(q_0(n) + \sum_{k\geq 1}(q_0(n-4\omega(k)) + q_0(n-4\omega(-k)))\right)x^n.$$

The last few steps required the use of (4), (3), and (7). The conclusion now follows by matching coefficients of like powers of x.

Remark: We earlier proved similar recurrences concerning $q_0(n)$, namely:

$$q_0(n) + \sum_{k\geq 1}(-1)^k\left(q_0(n-\omega(k)) + q_0(n-\omega(-k))\right) = \begin{cases}2(-1)^m & \text{if } n = 2m^2, \\ 0 & \text{otherwise.}\end{cases}$$

$$q_0(n) + \sum_{k\geq 1}(-1)^k\left(q_0(n-2\omega(k)) + q_0(n-2\omega(-k))\right) = \begin{cases}(-1)^{\left[\frac{1\mp m}{2}\right]} & \text{if } n = \omega(\pm m), \\ 0 & \text{otherwise.}\end{cases}$$

(See Theorem 2 in each of [4] and [5], respectively.)

Theorem 2: Let the integer $n \geq 0$. Then

$$q_0(n) + \sum_{k\geq 1}(q_0(n-8\omega(k)) + q_0(n-8\omega(-k))) = \sum_{j\geq 0}q\left(\frac{n-t(j)}{4}\right).$$

Proof: In the proof of Theorem 1, we encountered the identity:

$$\prod_{n\geq 1}(1+x^{2n-1})(1-x^{4n}) = \sum_{n\geq 0}x^{t(n)}.$$

Therefore,

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$$\prod_{n\geq 1} (1+x^{2n-1})(1-x^{4n})(1+x^{4n}) = \prod_{n\geq 1}(1+x^{4n})\sum_{n\geq 0} x^{t(n)},$$

$$\prod_{n\geq 1}(1+x^{2n-1})\prod_{n\geq 1}(1-x^{8n}) = \left(\sum_{n\geq 0} q\left(\frac{n}{4}\right)x^n\right)\left(\sum_{n\geq 0} x^{t(n)}\right),$$

$$\left(\sum_{n\geq 0} q_0(n)x^n\right)\prod_{n\geq 1}(1-x^{8n}) = \left(\sum_{n\geq 0} q\left(\frac{n}{4}\right)x^n\right)\left(\sum_{n\geq 0} x^{t(n)}\right).$$

The conclusion now follows by invoking (4), (3), and (7), and matching coefficients of like powers of x.

The following theorem regarding $q_0(n)$ is not a recurrence; it expresses $q_0(n)$ in terms of $p(n)$.

Theorem 3:

$$q_0(n) = \sum_{j\geq 0} p\left(\frac{n-t(j)}{4}\right).$$

Proof:

$$\sum_{n\geq 0} q_0(n)x^n = \prod_{n\geq 1}(1+x^{2n-1}) = \prod_{n\geq 1}\frac{1+x^n}{1+x^{2n}} = \prod_{n\geq 1}\frac{1+x^{2n}}{(1-x^{4n})(1-x^{2n-1})}$$

$$= \prod_{n\geq 1}(1-x^{4n})^{-1}\prod_{n\geq 1}\frac{1-x^{2n}}{1-x^{2n-1}} = \left(\sum_{n\geq 0} p\left(\frac{n}{4}\right)x^n\right)\left(\sum_{n\geq 0} x^{t(n)}\right)$$

by (4), (3), and (1). The conclusion now follows if one invokes (7) and matches coefficients of like powers of x.

Remark: Theorem 3 is essentially Watson's identity:

$$\chi(x) = \left(\sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}}\right)\left(\sum_{n=0}^{\infty} p(n)x^{4n}\right)$$

(see [6], p. 551).

The content of Theorem 3 may be stated more explicitly as Theorem 3a below.

Theorem 3a:

$$q_0(4n) = p(n) + p(n-7) + p(n-9) + p(n-30) + p(n-34) + \cdots,$$
$$q_0(4n+1) = p(n) + p(n-5) + p(n-11) + p(n-26) + p(n-38) + \cdots,$$
$$q_0(4n+2) = p(n-1) + p(n-2) + p(n-16) + p(n-19) + p(n-47) + \cdots,$$
$$q_0(4n+3) = p(n) + p(n-3) + p(n-13) + p(n-22) + p(n-42) + \cdots.$$

Corollary: $q_0(n) \geq p([n/4])$.

Proof: This follows from Theorem 3.

Remark: In [2], J. Ewell proved a theorem similar to Theorem 3, namely:

$$q(n) = \sum_{j\geq 0} p\left(\frac{n-t(j)}{2}\right).$$

Using similar reasoning, it follows that

$$q_2(n) = \sum_{j \geq 0} p(n - t(j)).$$

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A NEW LARGEST SMITH NUMBER

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1. INTRODUCTION

In 1982, Albert Wilansky, a mathematics professor at Lehigh University wrote a short article in the *Two-Year College Mathematics Journal* [6]. In that article he identified a new subset of the composite numbers. He defined a **Smith number** to be a composite number where the sum of the digits in its prime factorization is equal to the digit sum of the number. The set was named in honor of Wilansky's brother-in-law, Dr. Harold Smith, whose telephone number 493-7775 when written as a single number 4,937,775 possessed this interesting characteristic. Adding the digits in the number and the digits of its prime factors 3, 5, 5 and 65,837 resulted in identical sums of 42.

Wilansky provided two other examples of numbers with this characteristic: 9,985 and 6,036. Since that time, many things have been discovered about Smith numbers including the fact that there are infinitely many Smith numbers [4]. The largest Smith numbers were produced by Samuel Yates. Using a large repunit and large palindromic prime, Yates was able to produce Smith numbers having ten million digits and thirteen million digits. Using the same large repunit and a new large palindromic prime, the author is able to find a Smith number with over thirty-two million digits.

2. NOTATIONS AND BASIC FACTS

For any positive integer n, we let $S(n)$ denote the sum of the digits of n. For any positive integer n, we let $S_p(n)$ denote the sum of the digits of the prime factorization of n. For example, $S(27) = 2 + 7 = 9$ and $S_p(27) = S_p(3 \cdot 3 \cdot 3) = 3 + 3 + 3 = 9$. Hence, 27 is a Smith number.

A *repunit*, denoted R_n, is a number consisting of a string of n one's. For example, $R_4 = 1111$. Currently, the largest known prime repunit is R_{1031}, which was shown to be prime by Hugh Williams and Harvey Dubner in 1985.

The following facts are used in constructions of very large Smith numbers.

Fact 1: If you multiply $9R_n$ by any natural number less than $9R_n$, then the digit sum is $9n$, i.e., $S(M * 9R_n) = 9n = S(9R_n)$ when $M < 9R_n$ (for a proof, see [3]).

Keith Wayland and Sham Oltikar in [5] provided the following.

Fact 2: If $S(u) > S_p(u)$ and $S(u) \equiv S_p(u)$ (mod 7), then $10^k \cdot u$ is a Smith number, where $k = (S(u) - S_p(u))/7$.

3. PRIOR LARGEST SMITH NUMBERS

In 1987, Dubner discovered the large palindromic prime $M = 10^{4594} + 3 * 10^{2297} + 1$. When this prime is raised to a power t, the digit sum will be the sum of the digits of the numbers in front of each power of 10^{2297}. As long as each coefficient of a power of 10^{2297} is less than $9R_{1031}$,

when it is multiplied by $9R_{1031}$ that coefficient has a digit sum of $9*1031$. Since the largest coefficient occurs in the middle, it is sufficient to bound it by $9R_{1031}$.

Suppose that $N = 9R_{1031}*M^t$ with each coefficient of a power of 10^{2297} being less than $9R_{1031}$, then each of the $2t+1$ powers of 10^{2297} contributes $9*1031$ to the digit sum. Hence,

$$S(N) = (2t+1)*9*1031.$$

On the other hand, the prime factorization of N is simply $3*3*R_{1031}*M^t$ and so

$$S_p(N) = 3+3+1031+5t$$

because $S_p(M) = 5$. For any positive t, we have $S(N) > S_p(N)$ and

$$\begin{aligned} S(N) - S_p(N) &= 18553t + 8242 \\ &= 3t+3 \ (\mathrm{mod}\ 7) \\ &= 3(t+1) \ (\mathrm{mod}\ 7). \end{aligned}$$

This result will be 0 (mod 7) when $t \equiv 6$ (mod 7). Yates [7] was able to find the optimal t value that is congruent to 6 (mod 7) and has a coefficient of 10^{2297*t} less than $9R_{1031}$ was 1476. In this case, the coefficient of $10^{2297*1476}$ is $7.85*10^{1029}$ and increasing t by 7 causes the middle coefficient to be greater than $9R_{1031}$. Finally, the computation

$$(S(N) - S_p(N))/7 = (18553*1476+8242)/7 = 3913210$$

and Oltikar and Wayland's fact say that

$$9R_{1031}*(10^{4594}+3*10^{2297}+1)^{1476}*10^{3913210}$$

is a Smith number. This number has $1031+1476*4594+1+3913210 = 10,694,986$ digits.

This number did not remain as the largest Smith number for very long because Dubner found a new larger palindromic prime in 1990. Yates used Dubner's prime, $10^{6572}+3*10^{3286}+1$, to produce the Smith number

$$9R_{1031}*(10^{6572}+3*10^{3286}+1)^{1476}*10^{3913210}$$

which has 13,614,514 digits. Yates published his finding in a poem serving as a tribute to Martin Gardner [8].

4. NEW LARGEST SMITH NUMBER

Chris Caldwell is keeping on the World Wide Web a list of the 5000 largest primes [1] which is changing monthly. He also has available for retrieval a list of the largest palindromic primes [2]. In his list, we found a new very large palindromic prime with a small middle. The list credits Daniel Heuer for using the program Primeform in 2001 to discover that $M = 10^{28572}+8*10^{14286}+1$ is prime.

Suppose $N = 9R_{1031}*M^t$ with each coefficient of a power of 10^{14286} being less than $9R_{1031}$. Since Heuer's new palindromic prime has a middle digit 8, $S(N) - S_p(N)$ is now $18548t + 8242 \equiv 5t+3$ (mod 7).. This will be 0 (mod 7) when $t \equiv 5$ (mod 7). The optimal t value that is congruent to 5 (mod 7) and has a coefficient of $10^{14286*t}$ less than $9R_{1031}$ turns out to be 1027. Then, the exponent to use on 10 is $(S(N) - S_p(N)/7 = (18548*1027+8242)/7 = 2722434$. Hence the new Smith number is

$$9R_{1031} * (10^{28572} + 8 * 10^{14286} + 1)^{1027} * 10^{2722434}$$

which has $1031 + 1027 * 28572 + 1 + 2722434 = 32,066,910$ digits.

While there are larger palindomic primes in Caldwell's list, the larger ones have middle terms that are not single-digit numbers. Then you must use a much smaller t value on the palindromic prime so that the middle coefficient in the trinomial expansion is bounded by $9R_{1031}$. This limitation forces the number of digits in the resulting Smith number to be much smaller than 32 million. In fact, using the larger palindromic prime,

$$M = 10^{35352} + 2049402 * 10^{17673} + 1,$$

the optimal t value is $t = 157$ and yields a Smith number having only $5,968,187$ digits.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-940 *Proposed by Gabriela Stănică & Pantelimon Stănică, Auburn Univ. Montgomery, Montgomery, AL*

How many perfect squares are in the sequence

$$x_n = 1 + \sum_{k=0}^{n} F_k! \text{ for } n \geq 0?$$

B-941 *Proposed by Walther Janous, Innsbruck, Austria*

Show that

$$\frac{nF_{n+6}}{2^{n+1}} + \frac{F_{n+8}}{2^n} - F_8 < 0.$$

B-942 *Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA*

(a) For $n > 3$, find the Fibonacci number closest to L_n.

(b) For $n > 3$, find the Fibonacci number closest to L_n^2.

B-943 *Proposed by José Luis Diaz & Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{L_k^2}{F_k} \geq \frac{(L_{n+2}-3)^2}{F_{n+2}-1}.$$

When does equality occur?

B-944 *Proposed by Paul S. Bruckman, Berkeley, CA*
For all odd primes p, prove that

$$L_p \equiv 1 - \frac{p}{2} \sum_{k=1}^{p-1} \frac{L_k}{k} \pmod{p^2}$$

where $\frac{1}{k}$ represents the residue $k^{-1} \pmod{p}$.

B-945 *Proposed by N. Gauthier, Royal Military College of Canada*
For $n \geq 0$, $q > 0$, s integers, show that

$$\sum_{l=0}^{n} \binom{n}{l} F_{q-1}^l F_{(q+1)(n-l)+s} = F_{q+1}^n F_{2n+s}.$$

SOLUTIONS

Some Sum Divides Another

B-925 *Proposed by José Luis Diaz & Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain*
(Vol. 39, no. 5, November 2001)
Prove that $\sum_{k=0}^{n} F_{k+1}^2$ divides

$$\sum_{k=0}^{n} F_{k+1}^2 [F_{k+2} + (-1)^k F_k] \quad \text{for } n \geq 0.$$

Solution by H.-J. Seiffert, Berlin, Germany
From (I_3) of [1], we know that

$$\sum_{k=0}^{n} F_{k+1}^2 = F_{n+1} F_{n+2},$$

so it suffices to prove that, for all $n \geq 0$,

$$S_n := \sum_{k=0}^{n} F_{k+1}^2 [F_{k+2} + (-1)^k F_k] = \left(\frac{1-(-1)^n}{2} F_{n+3} + (-1)^n F_{n+2} \right) F_{n+1} F_{n+2}. \tag{1}$$

Direct computation shows that this is true for $n = 0$. Assuming that (1) holds for $n-1$, $n \geq 1$, we obtain

$$S_n = S_{n-1} + F_{n+1}^2 [F_{n+2} + (-1)^n F_n]$$

$$= \left(\frac{1+(-1)^n}{2} F_{n+2} - (-1)^n F_{n+1} \right) F_n F_{n+1} + F_{n+1}^2 [F_{n+2} + (-1)^n F_n]$$

$$= \left(\frac{1+(-1)^n}{2} F_n + F_{n+1} \right) F_{n+1} F_{n+2} = \left(\frac{1-(-1)^n}{2} F_{n+3} + (-1)^n F_{n+2} \right) F_{n+1} F_{n+2},$$

where the latter equality is easily established by considering the cases in which n is even and n is odd. This completes the induction proof of (1).

Reference

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Almost all solvers used essentially a similar method and provided the same reference.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles Cook, Kenneth B. Davenport, L. A. G. Dresel, Russell J. Hendel, Walther Janous, and the proposers.

Find the Limit

B-926 *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan (Vol. 39, no. 5, November 2001)*

If $1 < a < \alpha$, evaluate

$$\lim_{n \to \infty} \left(a^{\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_n}} - a^{\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{n-1}}} \right).$$

Solution by Gurdial Arora & Vlazko Kocic, New Orleans

To find the above limit, we use the following result from [1]:

$$\lim_{n \to \infty} \left(\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_n} \right) = 2 + \alpha \approx 3.6180339\ldots.$$

Therefore,

$$\lim_{n \to \infty} \left(a^{\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_n}} - a^{\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{n-1}}} \right) = 0.$$

Reference

1. Zdzislaw W. Trzaska. "On Factorial Fibonacci Numbers." *The Math. Gazette* (1998):82-85.

Almost all solvers noted that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{F_k}$$

exists and gave several references. In fact, the result is not difficult to prove.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Russell J. Hendel, Walther Janous, H.-J. Seiffert, Naim Tugler, and the proposer.

A More General Identity

B-927 *Proposed by R. S. Melham, University of Technology, Sydney, Australia (Vol. 39, no. 5, November 2001)*

G. Candido ["A Relationship between the Fourth Powers of the Terms of the Fibonacci Series," *Scripta Mathematica* **17.3-4** (1951):230] gave the following fourth-power relation:

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) = (F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2.$$

Generalize this relation to the sequence defined for all integers n by

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \ W_1 = b.$$

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

We find $2(q^4 W_n^4 + p^4 W_{n+1}^4 + W_{n+2}^4) = (q^2 W_n^2 + p^2 W_{n+1}^2 + W_{n+2}^2)^2$, because

$$(q^2 W_n^2 + p^2 W_{n+1}^2 + W_{n+2}^2)^2 = q^4 W_n^4 + p^4 W_{n+1}^4 + W_{n+2}^4 + q^2 W_n^2 (p^2 W_{n+1}^2 + W_{n+2}^2)$$
$$+ p^2 W_{n+1}^2 (q^2 W_n^2 + W_{n+2}^2) + W_{n+2}^2 (q^2 W_n^2 + p^2 W_{n+1}^2),$$

and we complete the proof by noting that

$$q^2 W_n^2 (p^2 W_{n+1}^2 + W_{n+2}^2) = q^2 W_n^2 [(W_{n+2} - p W_{n+1})^2 + 2 p W_{n+1} W_{n+2}]$$
$$= q^4 W_n^4 + 2 p q^2 W_n^2 W_{n+1} W_{n+2},$$

$$p^2 W_{n+1}^2 (q^2 W_n^2 + W_{n+2}^2) = p^2 W_{n+1}^2 [(W_{n+2} + q W_n)^2 - 2 q W_n W_{n+2}]$$
$$= p^4 W_{n+1}^4 - 2 p^2 q W_n W_{n+1}^2 W_{n+2},$$

$$W_{n+2}^2 (q^2 W_n^2 + p^2 W_{n+1}^2) = W_{n+2}^2 [(p W_{n+1} - q W_n)^2 + 2 p q W_n W_{n+1}]$$
$$= W_{n+2}^4 + 2 p q W_n W_{n+1} W_{n+2}^2,$$

and

$$2 p q^2 W_n^2 W_{n+1} W_{n+2} - 2 p^2 q W_n W_{n+1}^2 W_{n+2} + 2 p q W_n W_{n+1} W_{n+2}^2$$
$$= 2 p q W_n W_{n+1} W_{n+2} (q W_n - p W_{n+1} + W_{n+2}) = 0.$$

Also solved by Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, Russell J. Hendel, Walther Janous, and the proposer.

A Complex Fibonacci Polynomial

B-928 *Proposed by H.-J. Seiffert, Berlin, Germany*
(Vol. 39, no. 5, November 2001)

The Fibonacci polynomials are defined by $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+1}(x) = x F_{n+1}(x) + F_n(x)$ for $n \geq 0$. Show that, for all complex numbers x and all nonnegative integers n,

$$F_{2n+1}(x) = \sum_{k=0}^{n} (-1)^{\lceil k/2 \rceil} \binom{n - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} (x^2 + 2)^{n-k},$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor- and ceiling-function, respectively.

Solution by Paul S. Bruckman, Berkeley, CA

We may restate Seiffert's putative identity as follows:

$$F_{2n+1}(x) = G_{2n+1}(x), \tag{1}$$

where

$$G_{2n+1}(x) = \sum_{k=0}^{n} (-1)^{[(k+1)/2]} \binom{n - [(k+1)/2]}{[k/2]} (x^2 + 2)^{n-k}. \tag{2}$$

Our proof of (1) uses a modified form of induction. First, however, we derive the recurrence satisfied by the $F_{2n+1}(x)$'s. Note that the basic recurrence satisfied by the $F_n(x)$'s has the characteristic equation, $z^2 - xz - 1 = 0$, which has the solutions $u = u(x) = x + \theta$ and $v = v(x) = x - \theta$, where $\theta = (x^2 + 4)^{1/2}$. Note that $u + v = x$ and $uv = -1$. Therefore, the characteristic equation of the $F_{2n+1}(x)$'s is as follows: $(z - u^2)(z - v^2) = 0$, i.e., $z^2 - (x^2 + 2)z + 1 = 0$. In other words, the $F_{2n+1}(x)$'s satisfy the following recurrence:

$$F_{2n+5}(x) = (x^2+2)F_{2n+3}(x) - F_{2n+1}(x), \quad n=0,1,\dots. \tag{3}$$

Note that $F_1(x) = 1 = G_1(x)$. Also, $F_3(x) = x^2+1 = (x^2+2)-1 = G_3(x)$. Now

$$(x^2+2)G_{2n+3}(x) - G_{2n+1}(x) = \sum_{k=0}^{n+1}(-1)^{[(k+1)/2]}\binom{n+1-[(k+1)/2]}{[k/2]}(x^2+2)^{n+2-k}$$

$$-\sum_{k=0}^{n}(-1)^{[(k+1)/2]}\binom{n-[(k+1)/2]}{[k/2]}(x^2+2)^{n-k}.$$

In the last sum, we replace k by $k-2$; thus, this sum becomes

$$+\sum_{k=2}^{n+2}(-1)^{[(k+1)/2]}\binom{n+1-[(k+1)/2]}{[k/2]-1}(x^2+2)^{n+2-k}.$$

We may extend this last sum by including the terms for $k=0$ and $k=1$, since the combinatorial term vanishes for such values. Similarly, the first sum may be extended to include the term for $k=n+2$, for the same reason. We also note that

$$\binom{n+1-[(k+1)/2]}{[k/2]} + \binom{n+1-[(k+1)/2]}{[k/2]-1} = \binom{n+2-[(k+1)/2]}{[k/2]}.$$

Accordingly, we obtain the following result:

$$(x^2+2)G_{2n+3}(x) - G_{2n+1}(x) = \sum_{k=0}^{n+2}(-1)^{[(k+1)/2]}\binom{n+2-[(k+1)/2]}{[k/2]}(x^2+2)^{n+2-k},$$

which we recognize to equal $G_{2n+5}(x)$. Therefore, the $F_{2n+1}(x)$'s and the $G_{2n+1}(x)$'s satisfy the same recurrence, and also have the same initial values. It follows that

$$F_{2n+1}(x) = G_{2n+1}(x), \quad n=0,1,\dots, \text{ for all } x. \text{ Q.E.D.}$$

Also solved by Walther Janous and the proposer.

Between Fibonacci, Lucas, and Legendre

B-929 *Proposed by Harvey J. Hindin, Huntington Station, NY*
(Vol. 39, no. 5, November 2001)

Prove that

A) $F_{2N} = (1/5^{1/2})\sum\limits_{K=0}^{2N-1} P_K(5^{1/2}/2)P_{2N-1-K}(5^{1/2}/2)$ for $N \ge 1$

and

B) $L_{2N+1} = \sum\limits_{K=0}^{2N} P_K(5^{1/2}/2)P_{2N-K}(5^{1/2}/2)$ for $N \ge 0$,

where $P_K(x)$ is the Legendre polynomial given by $P_0(x) = 1$, $P_1(x) = x$, and the recurrence relation $(K+1)P_{K+1}(x) = (2K+1)xP_K(x) - KP_{K-1}(x)$.

Solution by H.-J. Seiffert, Berlin, Germany

The sequences of Fibonacci and Lucas polynomials are defined by

$$F_0(x) = 0, \quad F_1(x) = 1, \text{ and } F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad n \ge 0,$$

and

$$L_0(x) = 2, \ L_1(x) = x, \ \text{and} \ L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \ n \geq 0,$$

respectively. We shall prove that, for all real numbers x,

A) $\quad F_{2N}(x) = \dfrac{x}{\sqrt{x^2+4}} \displaystyle\sum_{K=0}^{2N-1} P_K(\sqrt{x^2+4}/2) P_{2N-1-K}(\sqrt{x^2+4}/2) \ \text{for} \ N \geq 1,$

and

B) $\quad L_{2N+1}(x) = x \displaystyle\sum_{K=0}^{2N} P_K(\sqrt{x^2+4}/2) P_{2N-K}(\sqrt{x^2+4}/2) \ \text{for} \ N \geq 0.$

It is known from equations (1.7) and (1.8) of [1] that

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\sqrt{x^2+4}} \quad \text{and} \quad L_n(x) = \alpha(x)^n + \beta(x)^n, \ n \geq 0, \tag{1}$$

where $\alpha(x) = (x + \sqrt{x^2+4})/2$ and $\beta(x) = (x - \sqrt{x^2+4})/2$. For sufficiently small $|z|$, let

$$G(z) = \sum_{L=1}^{\infty} (\alpha(x)^L - (-1)^L \beta(x)^L) z^{L-1}.$$

Then, by (1),

$$G(z) = \sqrt{x^2+4} \sum_{N=1}^{\infty} F_{2N}(x) z^{2N-1} + \sum_{N=0}^{\infty} L_{2N+1}(x) z^{2N}. \tag{2}$$

On the other hand, the known closed form expression for infinite geometric sums gives

$$G(z) = \frac{\alpha(x)}{1 - \alpha(x)z} + \frac{\beta(x)}{1 + \beta(x)z},$$

so that, by $\alpha(x) + \beta(x) = x$, $\beta(x) = -1$, $\alpha(x)\beta(x) = -1$, and $\alpha(x) - \beta(x) = \sqrt{x^2+4}$,

$$G(z) = \frac{x}{1 - \sqrt{x^2+4}\,z + z^2}. \tag{3}$$

The Legendre polynomials have the generating function (see [2], p. 190)

$$\sum_{L=0}^{\infty} P_L(x) z^L = \frac{1}{\sqrt{1 - 2xz + z^2}} \quad \text{for small} \ |z|.$$

Squaring, replacing x by $\sqrt{x^2+4}/2$, and multiplying the obtained identity by x, in view of (3) we get

$$\sum_{L=0}^{\infty} \left(x \sum_{K=0}^{L} P_K(\sqrt{x^2+4}/2) P_{L-K}(\sqrt{x^2+4}/2) \right) z^L = G(z). \tag{4}$$

The above stated identities A) and B) now follow from (2) and (4) by comparing coefficients.

Taking $x = 1$ solves the present proposal.

Remarks: By analytic continuation, the identities A) and B) remain valid for all complex numbers x. Other identities involving Fibonacci and Lucas numbers can be obtained by taking $x = \sqrt{5}$, 4, $1/\sqrt{5}$, $3i$, etc. For example, since $F_{2N}(\sqrt{5}) = \sqrt{5}F_{4N}/3$, from A) with $x = \sqrt{5}$, we find

$$F_{4N} = \sum_{K=0}^{2N-1} P_K(3/2)P_{2N-1-K}(3/2) \quad \text{for } N \geq 1.$$

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1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
2. F. G. Tricomi. *Vorlesungen über Orthogonalreihen.* 2. Auflage, Springer, 1970.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Ovidiu Furdui, Walther Janous, and the proposer.

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Author and Title Index

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-585 *Proposed by Herrmann Ernst, Siegburg, Germany*

Let (d_n) denote a sequence of positive integers d_n with $d_1 \geq 3$ and $d_{n+1} - d_n \geq 1$, $n = 1, 2, \ldots$.
We introduce the following sets of sequences (d_n):

$$A = \left\{ (d_n): \sum_{k=1}^{\infty} \frac{1}{F_{d_k}} \leq 1 \right\};$$

$$B = \left\{ (d_n): \frac{1}{F_{d_n}} < \sum_{k=n}^{\infty} \frac{1}{F_{d_k}} < \frac{1}{F_{d_n-1}} \text{ for all } n \in N \right\};$$

$$C = \left\{ (d_n): 0 \leq \frac{1}{F_{d_n-1}} - \frac{1}{F_{d_n}} - \frac{1}{F_{d_{n+1}-1}} \text{ for all } n \in N \right\}.$$

Show that:

(a) there is a bijection $f:]0, 1] \to B$, $f(x) = (d_n(x))_{n=1}^{\infty}$;

(b) B is a subset of A with $A \setminus B \neq \emptyset$;

(c) C is a subset of B with $B \setminus C \neq \emptyset$.

H-586 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the sequence of Fibonacci and Lucas polynomials by

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \in N,$$
$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad n \in N,$$

respectively. Show that, for all complex numbers x and all positive integers n,

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k F_{3k}(x) = \frac{xF_{2n+1}(x) - F_{2n}(x) + (-x)^{n+2} F_n(x) + (-x)^{n+1} F_{n-1}(x)}{2x^2 - 1}$$

and

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k L_{3k}(x) = \frac{xL_{2n+1}(x) - L_{2n}(x) + (-x)^{n+2} L_n(x) + (-x)^{n+1} L_{n-1}(x)}{2x^2 - 1}.$$

H-587 *Proposed by N. Gauthier & J. R. Gosselin, Royal Military College of Canada*

Let x and y be indeterminates and let

$$\alpha \equiv \alpha(x, y) = \frac{1}{2}(x + \sqrt{x^2 + 4y}), \quad \beta \equiv \beta(x, y) = \frac{1}{2}(x - \sqrt{x^2 + 4y})$$

be the distinct roots of the characteristic equation for the generalized Fibonacci sequence $\{H_n(x, y)\}_{n=0}^{n=\infty}$, where

$$H_{n+2}(x, y) = xH_{n+1}(x, y) + yH_n(x, y).$$

If the initial conditions are taken as $H_0(x, y) = 0$, $H_1(x, y) = 1$, then the sequence gives the generalized Fibonacci polynomials $\{F_n(x, y)\}_{n=0}^{n=\infty}$. On the other hand, if $H_0(x, y) = 2$, $H_1(x, y) = x$, then the sequence gives the generalized Lucas polynomials $\{L_n(x, y)\}_{n=0}^{n=\infty}$.

Consider the following 2×2 matrices,

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}, \quad C = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}, \quad D = \begin{pmatrix} \beta & 1 \\ 0 & \alpha \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let n and m be nonnegative integers. [By definition, a matrix raised to the power zero is equal to the unit matrix I.]

    a. Express $f_{n,m}(x, y) \equiv [(A - B)^{-1}(A^n - B^n)]^m$ in closed form, in terms of the Fibonacci polynomials.

    b. Express $g_{n,m}(x, y) \equiv [A^n + B^n]^m$ in closed form, in terms of the Lucas polynomials.

    c. Express $h_{n,m}(x, y) \equiv [C^n + D^n]^m$ in closed form, in terms of the Fibonacci and Lucas polynomials.

H-588 *Proposed by José Luiz Díaz-Barrero & Juan José Egozcue, Barcelona, Spain*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} F_{k+2} \geq \frac{n^{n+1}}{(n+1)^n} \prod_{k=1}^{n} \left\{ \frac{L_{k+1}^{\frac{n+1}{n}} - F_{k+1}^{\frac{n+1}{n}}}{L_{k+1} - F_{k+1}} \right\},$$

where F_n and L_n are, respectively, the nth Fibonacci and Lucas numbers.

SOLUTIONS

A Fractional Problem

H-574 *Proposed by J. L. Diaz-Barrero, Barcelona, Spain*
       (Vol. 39, no. 4, August 2001)

Let n be a positive integer greater than or equal to 2. Determine

$$\frac{F_n + L_n P_n}{(F_n - L_n)(F_n - P_n)} + \frac{L_n + F_n P_n}{(L_n - F_n)(L_n - P_n)} + \frac{P_n + F_n L_n}{(P_n - F_n)(P_n - L_n)},$$

where F_n, L_n, and P_n are, respectively, the nth Fibonacci, Lucas, and Pell numbers.

Solution by Paul S. Bruckman, Berkeley, CA

We employ certain well-known results from finite difference theory. For any well-defined, complex-valued function $f(x)$ with complex domain D, and for any three distinct values

$x_i \in D$, $i = 1, 2, 3$, define the "second-order" divided difference of f, valued at (x_1, x_2, x_3), as follows:

$$\Delta^2(f(x)) \mid (x_1, x_2, x_3) \equiv \theta_1 f(x_1) + \theta_2 f(x_2) + \theta_3 f(x_3), \tag{1}$$

where $\theta_1 = 1/\{(x_1 - x_2)(x_1 - x_3)\}$, $\theta_2 = 1/\{(x_2 - x_1)(x_2 - x_3)\}$, $\theta_3 = 1/\{(x_3 - x_1)(x_3 - x_2)\}$.

For brevity, we may also denote the left member of (1) as $\Delta^2(f(x))$ when no confusion is likely to arise. If f is a polynomial, the second-order divided difference has the following properties:

$$\Delta^2(f(x)) = 0 \text{ if degree}(f) = 0 \text{ or } 1; \quad \Delta^2(f(x)) = 1 \text{ if degree}(f) = 2. \tag{2}$$

Given distinct values x_1, x_2, x_3, let $\sigma_1 = x_1 + x_2 + x_3$, $\sigma_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$. Consider the following expression:

$$U(x_1, x_1, x_3) \equiv \sigma_2 \Delta^2(1) + (1 - \sigma_1)\Delta^2(x) + \Delta^2(x^2). \tag{3}$$

Using (1), this becomes

$$U(x_1, x_2, x_3) = \sigma_2(\theta_1 + \theta_2 + \theta_3) + (1 - \sigma_1)(x_1\theta_1 + x_2\theta_2 + x_3\theta_3) + (x_1^2\theta_1 + x_2^2\theta_2 + x_3^2\theta_3).$$

After expansion (using the definitions of σ_1 and σ_2), this simplifies to

$$U(x_1, x_1, x_3) = (x_1 + x_2 x_3)\theta_1 + (x_2 + x_1 x_3)\theta_2 + (x_3 + x_1 x_2)\theta_3. \tag{4}$$

On the other hand, since $\Delta^2(1) = \Delta^2(x) = 0$ and $\Delta^2(x^2) = 1$, we see from (3) that $U(x_1, x_1, x_3) = 1$. This yields the following general identity,

$$(x_1 + x_2 x_3)\theta_1 + (x_2 + x_1 x_3)\theta_2 + (x_3 + x_1 x_2)\theta_3 = 1 \tag{5}$$

which is true for any distinct values x_1, x_2, and x_3.

We now need to show that P_n, L_n, and F_n are distinct if $n \geq 2$. Note that $P_1 = L_1 = F_1 = 1$; $P_2 = 2, L_2 = 3, F_2 = 1$; $P_3 = 5, L_3 = 4, F_3 = 2$. Since $P_{n+2} = 2P_{n+1} + P_n$, $L_{n+2} = L_{n+1} + L_n$, and $F_{n+2} = F_{n+1} + F_n$, it follows by an easy inductive proof that, if $n \geq 3$, $P_n > L_n > F_n$, while $L_2 > P_2 > F_2$. Therefore, if $n \geq 2$, we may let $x_1 = F_n$, $x_2 = L_n$, $x_3 = P_n$ in (5), proving that the given expression simplifies to 1.

Also solved by G. Arora, D. Iannucci, H.-J. Seiffert, P. & G. Stănică, and the proposer.

A Remarkable Problem

<u>H-575</u> *Proposed by N. Gauthier, Royal Military College of Canada*
     (Vol. 39, no. 4, August 2001)

Problem Statement: "Four Remarkable Identities for the Fibonacci-Lucas Polynomials"

For n a nonnegative integer, the following Fibonacci-Lucas identities are known to hold:

$$L_{2n+2} = 5F_{2n+1} - L_{2n}; \quad F_{2n+3} = L_{2n+2} - F_{2n+1}.$$

The corresponding identities for the Fibonacci $\{F_n(u)\}_{n=0}^{\infty}$ and the Lucas $\{L_n(u)\}_{n=0}^{\infty}$ polynomials, defined by

$$F_0(u) = 0, \quad F_1(u) = 1, \quad F_{n+2}(u) = uF_{n+1}(u) + F_n(u),$$

$$L_0(u) = 2, \quad L_1(u) = u, \quad L_{n+2}(u) = uL_{n+1}(u) + L_n(u),$$

respectively, are:

$$L_{2n+2}(u) = (u^2 + 4)F_{2n+1}(u) - L_{2n}(u); \quad F_{2n+3}(u) = L_{2n+2}(u) = F_{2n+1}(u). \tag{1}$$

For m, n nonnegative integers, with the convention that a discrete sum with a negative upper limit is identically zero, prove the following generalizations of (1).

Case a: $\quad (2n+2)^{2m}L_{2n+2}(u) = (u^2+4)\left[\sum_{l=0}^{m}\binom{2m}{2l}(2n+1)^{2l}\right]F_{2n+1}(u)$

$$+u\left[\sum_{l=0}^{m-1}\binom{2m}{2l+1}(2n+1)^{2l+1}\right]L_{2n+1}(u) - [(2n)^{2m}]L_{2n}(u).$$

Case b: $\quad (2n+3)^{2m}F_{2n+3}(u) = \left[\sum_{l=0}^{m}\binom{2m}{2l}(2n+2)^{2l}\right]L_{2n+2}(u)$

$$+u\left[\sum_{l=0}^{m-1}\binom{2m}{2l+1}(2n+2)^{2l+1}\right]F_{2n+2}(u) - [(2n+1)^{2m}]F_{2n+1}(u).$$

Case c: $\quad (2n+2)^{2m+1}F_{2n+2}(u) = u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}(2n+1)^{2l}\right]F_{2n+1}(u)$

$$+\left[\sum_{l=0}^{m}\binom{2m+1}{2l+1}(2n+1)^{2l+1}\right]L_{2n+1}(u) - [(2n)^{2m+1}]F_{2n}(u).$$

Case d: $\quad (2n+3)^{2m+1}L_{2n+3}(u) = u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}(2n+2)^{2l}\right]L_{2n+2}(u)$

$$+(u^2+4)\left[\sum_{l=0}^{m}\binom{2m+1}{2l+1}(2n+2)^{2l+1}\right]F_{2n+2}(u)$$

$$-[(2n+1)^{2m+1}]L_{2n+1}(u).$$

Solution by the proposer

Start from the identity

$$x^{n+2} + x^{-(n+2)} = (x+x^{-1})(x^{n+1}+x^{-(n+1)}) - (x^n + x^{-n}),$$

which is valid for any variable x and number n. Next, introduce the differential operator $D \equiv x\frac{d}{dx}$ and note that $D^m x^\lambda = \lambda^m x^\lambda$ for m a nonnegative integer and λ an arbitrary number. Acting on the identity with D^m then gives

$$(n+2)^m(x^{n+2} + (-1)^m x^{-(n+2)}) = D^m[(x+x^{-1})(x^{n+1}+x^{-(n+1)})] - n^m(x^n + (-1)^m x^{-n}). \qquad (*)$$

Now let f and g be two arbitrary differentiable functions of x and note that

$$D(fg) = (Df)g + f(Dg); \quad D^2(fg) = (D^2f)g + 2(Df)(Dg) + f(D^2g);$$
$$D^3(fg) = (D^3f)g + 3(D^2f)(Dg) + 3(Df)(D^2g) + f(D^3g); \quad etc. \ldots$$

The general term is

$$D^m(fg) = \sum_{l=0}^{m}\binom{m}{l}(D^{m-l}f)(D^l g), \qquad (**)$$

as can easily be established by induction on m, so we skip the details.

Insertion of $(**)$ in $(*)$ with $f = (x+x^{-1})$ and $g = (x^{n+1}+x^{-(n+1)})$ gives

$$(n+2)^m(x^{n+2} + (-1)^m x^{-(n+2)})$$

$$= \sum_{l=0}^{m} \binom{m}{l}(x + (-1)^{m-l}x^{-1})(n+1)^l(x^{n+1} + (-1)^l x^{-(n+1)}) - n^m(x^n + (-1)^m x^{-n}). \qquad (***)$$

It is well known that the Fibonacci and Lucas polynomials can be represented in Binet form as follows:

$$F_n(u) = \frac{\alpha^n(u) - \beta^n(u)}{\alpha(u) - \beta(u)}; \quad L_n(u) = \alpha^n(u) + \beta^n(u);$$

$$\alpha(u) = \tfrac{1}{2}(u + \sqrt{u^2 + 4}); \quad \beta(u) = \tfrac{1}{2}(u - \sqrt{u^2 + 4}).$$

We now set $x = \alpha(u) \equiv \alpha$ in $(***)$ and invoke the property $\alpha^{-1} = -\beta$ to get

$$(n+2)^m(\alpha^{n+2} + (-1)^{n+m+2}\beta^{n+2})$$

$$= \sum_{l=0}^{m} \binom{m}{l}(\alpha + (-1)^{m+l+1}\beta)(n+1)^l(\alpha^{n+1} + (-1)^{m+l+1}\beta^{n+1}) - n^m(\alpha^n + (-1)^{n+m}\beta^n).$$

Next, separate the sum over all l in the right-hand member of the above into a sum over even values $(2l)$ and one over odd values $(2l+1)$ and make the following substitutions to obtain the four cases given in the problem statement.

Case a: $m \to 2m; n \to 2n;$ **Case c:** $m \to 2m+1; \; n \to 2n;$

Case b: $m \to 2m; \; n \to 2n+1;$ **Case d:** $m \to 2m+1; \; n \to 2n+1.$

The algebra is straightforward and we skip the details. This completes the solution.

Also solved by P. S. Bruckman and H.-J. Seiffert.

General IZE

H-576   *Proposed by Paul S. Bruckman, Berkeley, CA*
          (Vol. 39, no. 4, August 2001)

Define the following constant, $C_2 \equiv \prod_{p>2}\{1 - 1/(p-1)^2\}$, as an infinite product over all odd primes p.

(A) Show that $C_2 = \sum_{n=1}^{\infty} \mu(2n-1)/\{\phi(2n-1)\}^2$, where $\mu(n)$ and $\phi(n)$ are the Möbius and Euler functions, respectively.

(B) Let $\sum_{d|n} \mu(n/d)2^d$. Show that $C_2 = \prod_{n=2}^{\infty}\{\zeta^*(n)\}^{-R(n)/n}$, where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann Zeta function (with $n > 1$) and $\zeta^*(n) = \sum_{k=1}^{\infty}(2k-1)^{-n} = (1 - 2^{-n})\zeta(n)$.

Note: C_2 is the "twin-primes" constant that enters into Hardy and Littlewood's "extended" conjectures regarding the distribution of twin primes and Goldbach's Conjecture.

Solution by the proposer

Solution to Part (A): C_2 is easily shown to be a well-defined constant in $(0, 1)$. We may express the product defining C_2 as a Euler product:

$$C_2 = \prod_{p>2}\{1 - 1/(p-1)^2\} = \prod_{p>2}[1 + \mu(p)/\{\phi(p)\}^2]$$

$$= \prod_{p>2}[1 + \mu(p)/\{\phi(p)\}^2 + \mu(p^2)/\{\phi(p^2)\}^2 + \mu(p^3)/\{\phi(p^3)\}^3 + \cdots]$$

$$= \sum_{n=1,\, n\,\text{odd}}^{\infty} \mu(n) / \{\phi(n)\}^2 = \sum_{n=1}^{\infty} \mu(2n-1) / \{\phi(2n-1)\}^2.$$

Solution to Part (B): From the expression for C_2,

$$-\log C_2 = \sum_{p>2} \{2\log(1-p^{-1}) - \log(1-2p^{-1})\}$$

$$= \sum_{n=1} \sum_{p>2} (2^n - 2)p^{-n} / n = \sum_{n=2} (2^n - 2)g*(n) / n.$$

Taking the logarithm of the Euler product for the "modified" Zeta function, we obtain

$$\log \zeta^*(s) = -\sum_{p>2} \log(1-p^{-s})$$

valid for all s with $\text{Re}(s) > 1$. Then

$$\log \zeta^*(s) = \sum_{m=1}^{\infty} \sum_{p>2} p^{-ms} / m = \sum_{m=1}^{\infty} g*(ms) / m.$$

By a variant form of Möbius inversion, we obtain

$$g*(s) = \sum_{m=1}^{\infty} \mu(m) \log \zeta^*(ms) / m. \qquad (*)$$

Then

$$-\log C_2 = \sum_{n=2}^{\infty} (2^n - 2) / n \sum_{m=1}^{\infty} \mu(m) \log \zeta^*(mn) / m$$

$$= \sum_{N=2}^{\infty} \zeta^*(N) / N \sum_{d|N} \mu(N/d)(2^d - 2) = \sum_{N=2}^{\infty} \log \zeta^*(N) R(N) / N$$

since

$$\sum_{d|N} \mu(N/d) = 0 \quad (\text{for } N > 1).$$

Now, taking the antilogarithm leads to the expression given in (B).

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Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. $23.00

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

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Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman. $20.00

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