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A JOURNAL DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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A NOTE ON FIBONACCI QUATERNIONS

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A. F. Horadam has derived in [1] some results regarding Fibonacci and generalized Fibonacci quaternions. The object of this note is to derive some more relations connecting these two quaternions. Following [1] Q_n and P_n are defined as

(1a) $$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

(1b) $$P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}$$

where

(1c) $$i^2 = j^2 = k^2 = -1, \quad ij = -jk = k, \quad jk = -kj = i, \quad ki = -ik = j .$$

Let us now consider the relation

$$P_n + qQ_n = [H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}] + q[F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}].$$

Also from (1) of [1] we have

$$H_n = (p - q)F_n + qF_{n+1} ,$$

so

$$\begin{aligned} P_n + qQ_n = {} & [(p-q)F_n + qF_{n+1}] + i[(p-q)F_{n+1} + qF_{n+2}] \\ & + j[(p-q)F_{n+2} + qF_{n+3}] + k[(p-q)F_{n+3} + qF_{n+4}] \\ & + q[F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}]. \end{aligned}$$

This becomes after some simplifications

$$\begin{aligned} = {} & p(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}) \\ & + q(F_{n+1} + iF_{n+2} + jF_{n+3} + kF_{n+4}) \end{aligned}$$

Hence,

$$P_n + qQ_n = pQ_n + qQ_{n+1}$$

or

$$\begin{aligned} P_n &= pQ_n + q(Q_{n+1} - Q_n) \\ P_n &= pQ_n + qQ_{n-1} \end{aligned} \tag{2}$$

by definition of Q_n. Consider now the quantity

$$P_n\overline{Q}_n - \overline{P}_nQ_n \quad ,$$

where $\overline{P}_n$, $\overline{Q}_n$ are conjugate quaternions respectively of P_n and Q_n.

$$\begin{aligned} P_n\overline{Q}_n - \overline{P}_nQ_n &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3})(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}) \\ &\quad - (H_n - iH_{n+1} - jH_{n+2} - kH_{n+3})(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}) \\ &= -2H_n(Q_n - F_n) + 2F_n(P_n - H_n) \end{aligned}$$

$$P_n\overline{Q}_n - \overline{P}_nQ_n = 2(F_nP_n - H_nQ_n) \tag{3a}$$

Dividing by $P_nQ_n \neq 0$,

$$\begin{aligned} \frac{\overline{Q}_n}{Q_n} - \frac{\overline{P}_n}{P_n} &= 2\left(\frac{F_n}{Q_n} - \frac{H_n}{P_n}\right) \\ \frac{\overline{Q}_n - 2F_n}{Q_n} &= \frac{\overline{P}_n - 2H_n}{P_n} \end{aligned} \tag{3b}$$

Again,

$$\begin{aligned} P_n\overline{Q}_n + \overline{P}_nQ_n &= 2\sum_{i=0}^{3} H_{n+i}F_{n+i} - 2iH_{n+1}(jF_{n+2} + kF_{n+3}) \\ &\quad - 2jH_{n+2}(iF_{n+1} + kF_{n+3}) - 2kH_{n+3}(iF_{n+1} + jF_{n+2}) \end{aligned}$$

Using (1c) and simplifying we have,

$$= 2H_nF_n + 2H_{n+1}(F_{n+1} - kF_{n+2} + jF_{n+3})$$
$$+ 2H_{n+2}(F_{n+2} + kF_{n+1} - iF_{n+3}) + 2H_{n+3}(F_{n+3} - jF_{n+1} + iF_{n+2})$$

Now using

$$i^2 = j^2 = k^2 = -1 \ ,$$

we may write the above relation as

$$P_n\overline{Q}_n + \overline{P}_nQ_n = 2H_nF_n - 2[iH_{n+1} + jH_{n+2} + kH_{n+3}][iF_{n+1} + jF_{n+2} + kF_{n+3}]$$
$$= 2H_nF_n - 2(P_n - H_n)(Q_n - F_n)$$

$$P_n\overline{Q}_n + \overline{P}_nQ_n = -2(P_nQ_n - P_nF_n - Q_nH_n)$$

(4) $$P_n\overline{Q}_n + \overline{P}_nQ_n = 2[P_nF_n + Q_nH_n - P_nQ_n]$$

As P_n and $Q_n \neq 0$, dividing by P_nQ_n,

$$\frac{\overline{Q}_n}{Q_n} + \frac{\overline{P}_n}{P_n} = 2\left[\frac{F_n}{Q_n} + \frac{H_n}{P_n} - 1\right]$$

$$\frac{\overline{Q}_n - 2F_n}{Q_n} + 1 = \frac{2H_n - \overline{P}_n}{P_n} - 1$$

or

(4b) $$\frac{\overline{Q}_n - 2F_n}{Q_n} + 1 = -\left[\frac{\overline{P}_n - 2H_n}{P_n} + 1\right]$$

Also

$$\begin{aligned}
P_nQ_n - \overline{P}_n\overline{Q}_n &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3})(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}) \\
&\quad - (H_n - iH_{n+1} - jH_{n+2} - kH_{n+3})(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}) \\
&= 2F_n(iH_{n+1} + jH_{n+2} + kH_{n+3}) \\
&\quad + 2H_n(iF_{n+1} + jF_{n+2} + kF_{n+3}) \\
&= 2F_n(P_n - H_n) + 2H_n(Q_n - F_n)
\end{aligned}$$

$$P_nQ_n - \overline{P}_n\overline{Q}_n = 2\,[H_nQ_n + F_nP_n - 2H_nF_n] \tag{4}$$

Theorem:

$$Q_{n-1}^2 + Q_n^2 = 2Q_{2n-1} - 3L_{2n+2}$$

Let us consider the left side of the relation.

$$\begin{aligned}
Q_{n-1}^2 + Q_n^2 &= (F_{n-1} + iF_n + jF_{n+1} + kF_{n+2})^2 + (F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})^2 \\
&= [F_{n-1}^2 - F_n^2 - F_{n+1}^2 - F_{n+2}^2 + F_n^2 - F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2] \\
&\quad + 2\,[F_{n-1}(iF_n + jF_{n+1} + kF_{n+2}) + F_n(iF_{n+1} + jF_{n+2} + kF_{n+3})] \\
&\quad + [iF_{n+1}(jF_{n+2} + kF_{n+3}) + jF_{n+2}(iF_{n+1} + kF_{n+3})] \\
&\quad + [iF_n(jF_{n+1} + kF_{n+2}) + jF_{n+1}(iF_n + kF_{n+2})] \\
&\quad + [kF_{n+2}(iF_n + jF_{n+1}) + kF_{n+3}(iF_{n+1} + jF_{n+2})]
\end{aligned}$$

The first term

$$\begin{aligned}
&= F_{n-1}^2 - F_{n+1}^2 - (F_{n+1}^2 + F_{n+2}^2) - (F_{n+2}^2 + F_{n+3}^2) \\
&= -[(F_{n+1}^2 - F_{n-1}^2) + (F_{n+1}^2 + F_{n+2}^2) + (F_{n+2}^2 + F_{n+3}^2)] \\
&= -[(F_{2n} + F_{2n+3} + F_{2n+5}] \\
&= -[12F_{2n} + 7F_{2n-1}]
\end{aligned} \tag{A}$$

Now consider the terms containing i, j, k, namely,

$$\text{(B)} \qquad \begin{aligned} &2i\left[F_nF_{n+1} + F_nF_{n+1}\right] + 2j\left[F_{n-1}F_{n+1} + F_nF_{n+2}\right] \\ &\qquad + 2k\left[F_{n-1}F_{n+2} + F_nF_{n+3}\right] \\ &= 2iF_{2n} + 2jF_{2n+1} + 2kF_{2n+2} \end{aligned}$$

Therefore

$$\begin{aligned} Q_{n-1}^2 + Q_n^2 &= -[12F_{2n} + 7F_{2n-1}] \\ &\quad + 2iF_{2n} + 2jF_{2n+1} + 2kF_{2n+2} \\ &= -[12F_{2n} + 9F_{2n-1} - 2F_{2n-1}] \\ &\quad + 2iF_{2n} + jF_{2n+1} + kF_{2n+2} \\ &= -[3F_{2n+3} + 3F_{2n+1}] + 2Q_{2n-1} \\ &= -3L_{2n+2} + 2Q_{2n=1} \end{aligned}$$

Hence,

$$Q_{n-1}^2 + Q_n^2 = 2Q_{2n-1} - 3L_{2n+2}$$

Hence the theorem.

Other interesting relations will be considered later.

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1. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," Amer. Math. Monthly, 70, 3, 1963.

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REMARK ON A THEOREM BY WAKSMAN

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Let Q denote the set of primes $Q = Q^\star \cup \{1\}$, Z the nonnegative integers and $V = \{K : Q^\star \leq S_K\}$, where $S_K = \{m = Kn + p;\ n \in Z$ and $p = 1$ or $p \in Q$ such that $p \nmid K,\ p < K\} \cup \{p \in Q : p | K\}$. Let $U = \{k : k \in Z$ and each of the $\varphi(k)$ integers $1 = a_1 < a_2 < \cdots < a_{\varphi(k)}$ not greater than k and relatively prime to k, is a member of $Q^\star\}$. We note that $a_2 \in Q$ if $k > 2$.

A. Waksman [1] has shown (with the aid of a computer search) that $V = \{2, 3, 4, 6, 8, 12, 18, 24, 30\}$. Trivially, 1 must also be a member of V. We shall show that $U = V$. It is known that U consists of the integers given above [2, p. 62].

Let $0 < t \in Z$ and let $1 = a_1 < a_2 < \cdots < a_{\varphi(t)}$ be the integers not greater than t and relatively prime to t.

(i) We prove first that $U \subseteq V$. If $t \in U$ (so that $a_i \in Q^\star$) then every positive integer relatively prime to t is a member of the set

$$R = \{tn + a_i : n \in Z,\quad i = 1, 2, \cdots, \varphi(t)\}.$$

Now $1 \in R$ and if q is a prime, then either $q | t$ or $q \in R$. Thus $Q^\star \leq S_t$ and $t \in V$.

(ii) We show now that $V \subseteq U$ (using, in part, a method of Waksman). It is immediate that 1 and $2 \notin V \cap Q$. If $2 < t \in V$ then by the Dirichlet theorem, there is a prime q such that $q = a_2^2 \pmod t$. Since $q \in S_t$ and $q \nmid t$ there is a prime $p < t$ such that $q \equiv p \pmod t$. Thus $p \equiv a_2^2 \pmod t$. If $a_2^2 < t$ then $t \| a_2^2 - p| < t$, which implies $p = a_2^2$, a contradication. Thus $a_2^2 \geq t$. If one of $a_i \notin Q$ $(i = 3, \cdots, \varphi(t))$, then $a_i \geq a_2^2 \geq t$, a contradiction. Thus $a_i \in Q^\star$ $(i = 1, 2, \cdots, \varphi(t))$, and $t \in U$.

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1. A. Waksman, "On the Distribution of Primes," American Mathematical Monthly, 75 (1968), pp. 764-765.
2. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Chelsea, New York, 1953.

* * * * *

RECURRENT SEQUENCES IN THE EQUATION $DQ^2 = R^2 + N$

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INTRODUCTION

The recreational exploration of numbers by the amateur can lead to discovery, or to a different way of looking at problems, because he often does not know the conventional approaches. Sometimes, as a form of amusement, I picked a quadratic expression at random, set it equal to a square and then tried to solve the resulting equation in positive integers. Whenever I was able to solve the problem I noticed that recurrency was evident. One of the most satisfying results came from the solution of $5x^2 \pm 6y + 1 = y^2$ where the recurrent relationships involved Fibonacci and Lucas sequences. However, the method reported [1] for this solution is not general. An improvement in the method resulted from exploring the Pell and Lagrange* equations. As experimental data accumulated I was able to make some conjectures and when I discussed the results with my friend, Professor Burton W. Jones, he urged me to try to prove them. For his encouragement, I am grateful.

The following are some of these conjectures:

a) For any recurrent equation such as $U_{n+1} = cU_n + U_{n-1}$ or $U_{n+1} = cU_n - U_{n-1}$, c constant and even, there exists at least one Pell equation such that the sequence of X's and of Y's follow the given recurrent law.

b) In a Pell equation if $DY_1^2 = X_1^2 + 1$ then the recurrent law for the sequence of X's or Y's is $U_{n+1} = cU_n + U_{n-1}$ and if $DY_1^2 = X_1^2 - 1$ then the governing law is $U_{n+1} = cU_n - U_{n-1}$.

c) In Lagrange equations having the same D as a Pell equation, there exists a recurrent law common to both. (Proof to be offered in another communication.)

---

*The Lagrange equations $Dy^2 = x^2 \pm N$, $N > 1$ will be discussed in another communication.

Since a method of developing the sequence of one of the variables, in a Pell equation, independent of the other is so easy and since the proof justifying such treatment uses only elementary algebra, without the use of continued fractions or convergents, I thought that the method might be of interest. As will be demonstrated, problems, relating to the Pell equations which seem difficult, are solved in an almost trivial fashion by means of the theorems to be developed here. (Before continuing the reader is invited to try solving problems 1-5.)

PART I — THE PELL EQUATIONS $DY_n^2 = X_n^2 - (-1)^n$ and $DY_n^2 = X_n^2 - 1$

For a given $D > 1$ and not a square the complete* Pell equations are either of the forms

$$DY_n^2 = X_n^2 - (-1)^n \tag{1}$$

or

$$DY_n^2 = X_n^2 - 1, \qquad n = 0, 1, 2, 3, \cdots . \tag{2}$$

While both of these equations have the trivial solution $X_0 = 1$, $Y_0 = 0$, the key to the general solution is in finding X_1, Y_1, either by inspection or otherwise. How this may be done by convergents is explained by Burton W. Jones [2], C. D. Olds [3], R. Kortum and G. McNeil [4] and others in books on number theory.

The least positive, non-trivial solution (X_1, Y_1) is variously called the minimal or fundamental or generating solution. Once this solution is found, the general solution is given by

$$X_n + Y_n\sqrt{D} = (X_1 + Y_1\sqrt{D})^n . \tag{3}$$

*The equation $DY_n^2 = X_n^2 - (-1)^n$, $n = 0,1,2,3,\cdots$, is complete. However, it is commonly treated as two equations, e.g., $DY_{2k}^2 = X_{2k}^2 - 1$ and $DY_{2k+1}^2 = X_{2k+1}^2 + 1$, $k = 0,1,2,3,\cdots$. Unless otherwise stated, we will assume that for the given D, the Pell equation is complete and we are dealing with all possible solutions.

The sum of the rational terms in the binomial expansion of $(X_1 + Y_1\sqrt{D})^n$ is X_n and the sum of the irrational terms is $Y_n\sqrt{D}$. That equation (3) gives all of the possible solutions was first shown by Robert D. Carmichael and later explained in his book Diophantine Analysis [5].

When the minimal solution (X_1, Y_1) is substituted in equations (1) and (2) we have respectively the minimal equations

$$DY_1^2 = X_1^2 + 1^* \tag{4}$$

and

$$DY_1^2 = X_1^2 - 1 . \tag{5}$$

In either case, and irrespective of the sign preceding 1, the general solution is given by the single equation (3).

PROOF OF THREE THEOREMS ON RECURRENCY IN THE PELL EQUATIONS

Theorem 1. In the integer solution of a Pell equation, the sequence of X's is recurrent as is the sequence of Y's according to the recurrent law, $U_{n+1} = cU_n \pm U_{n-1}$, $c = 2X_1$. The + sign is used if the minimal equation is $DY_1^2 = X_1^2 + 1$ and the - sign is used if $DY_1^2 = X_1^2 - 1$.

To prove this theorem we combine the minimal equations (4) and (5) so that

$$DY_1^2 = X_1^2 \pm 1 . \tag{6}$$

Then for reference we prepare, from the general solution (3), the following set of equations:

$$(X_1 + Y_1\sqrt{D})^{n-1} = X_{n-1} + Y_{n-1}\sqrt{D} \tag{7a}$$

---

* If the minimal equation for a certain D is $DY_1^2 = X_1^2 + 1$ then there are solutions for $DY^2 = X^2 \pm 1$. If the minimal equation is $DY_1^2 = X_1^2 - 1$ then the only solutions are for $DY^2 = X^2 - 1$. Thus $DY^2 = X^2 + 1$ is not solvable for all D's nor does it have a trivial solution.

(7b) $$(X_1 + Y_1\sqrt{D})^n = X_n + Y_n\sqrt{D}$$

(7c) $$(X_1 + Y_1\sqrt{D})^{n+1} = X_{n+1} + Y_{n+1}\sqrt{D} .$$

When $X_1^2 + 2X_1Y_1\sqrt{D}$ is added to both sides of $DY_1^2 = X_1^2 \pm 1$ we obtain $X_1^2 + 2X_1Y_1\sqrt{D} + DY_1^2 = 2X_1^2 + 2X_1Y_1\sqrt{D} \pm 1$ or

(8) $$(X_1 + Y_1\sqrt{D})^2 = 2X_1(X_1 + Y_1\sqrt{D}) \pm 1 .$$

Multiplying both sides of this equation by $(X_1 + Y_1\sqrt{D})^{n-1}$ we derive

(9) $$(X_1 + Y_1\sqrt{D})^{n+1} = 2X_1(X_1 + Y_1\sqrt{D})^n \pm (X_1 + Y_1\sqrt{D})^{n-1} .$$

Now when the appropriate substitutions are made in this equation from set (7) we get

$$X_{n+1} + Y_{n+1}\sqrt{D} = 2X_1(X_n + Y_n\sqrt{D}) \pm (X_{n-1} + Y_{n-1}\sqrt{D})$$

and rearranging this equation we have

(10) $$X_{n+1} + Y_{n+1}\sqrt{D} = (2X_1X_n \pm X_{n-1}) + (2X_1Y_n \pm Y_{n-1})\sqrt{D} .$$

After equating the rational and then the irrational terms in (10) we finally derive

(11) $$X_{n+1} = 2X_1X_n \pm X_{n-1}$$

and

(12) $$Y_{n+1} = 2X_1Y_n \pm Y_{n-1} .$$

Thus the proof of Theorem 1 is complete and equations (11) and (12) are the equations of the Theorem.*

---

*The equations of the Theorem seem similar to expressions found for the convergents of continued fractions. For instance, the numerator of the nth convergent is $p_n = a_np_{n-1} + p_{n-2}$. This equation seems similar to $X_n = cX_{n-1} \pm X_{n-2}$ but in the equations of Theorem 1, + or - signs are used whereas in the convergent expression <u>only</u> the + sign appears.

As a consequence of Theorem 1 we have

Theorem 2. For every recurrent equation, $U_{n+1} = cU_n + U_{n-1}$ or $U_{n+1} = cU_n - U_{n-1}$ where c is even, there exists at least one Pell equation for which the sequence of either variable is governed by the given recurrent law.

To prove this theorem we note from Theorem 1 that $c = 2X_1$ whence $X_1 = c/2$. When this value of X_1 is substituted in the minimal equations $DY_1^2 = X_1^2 \pm 1$ we have

$$DY_1^2 = \left(\frac{c}{2}\right)^2 \pm 1 .$$

Except for a trivial case,

$$\left(\frac{c}{2}\right)^2 \pm 1 \neq \square ,$$

therefore we can let

$$\left(\frac{c}{2}\right)^2 \pm 1 = D$$

whence $Y_1 = 1$ and thus we have proved Theorem 2. If

$$\left(\frac{c}{2}\right)^2 \pm 1$$

contains a square factor >1 there may be other solutions as demonstrated by problem 1.

In equation (1), $DY_n^2 = X_n^2 - (-1)^n$, we notice that when $n = 2k$ then

$$DY_{2k}^2 = X_{2k}^2 - 1 \tag{13}$$

and when $n = 2k + 1$ then

$$DY_{2k+1}^2 = X_{2k+1}^2 + 1, \quad k = 0,1,2,3,\cdots . \tag{14}$$

In order to study the sequence of every other term in a Pell equation we have

Theorem 3. The sequence of every other X or Y in a Pell equation is recurrent. If the recurrent law for the Pell equation is $U_{n+1} = cU_n + U_{n-1}$ then the sequence of every other X or Y is

$$U_{n+3} = (c^2 + 2)U_{n+1} - U_{n-1}$$

and if the recurrent law is $U_{n+1} = cU_n - U_{n-1}$ then the sequence of every other X or Y is governed by

$$U_{n+3} = (c^2 - 2)\, U_n - U_{n-1} \ .$$

We prove the two parts of Theorem 3 together using the ambiguous $\pm$ sign.

$$U_{n+1} = cU_n \pm U_{n-1}$$

then

$$U_{n+2} = cU_{n+1} \pm U_n$$

and

$$U_{n+3} = cU_{n+2} \pm U_{n+1} \ .$$

But

$$U_{n+2} = cU_{n+1} \pm U_n$$

therefore

$$U_{n+3} = c(cU_{n+1} \pm U_n) \pm U_{n+1}$$

or

$$U_{n+3} = c^2U_{n+1} \pm cU_n \pm U_{n+1}$$

and

$$U_{n+3} = (c^2 \pm 1)U_{n+1} \pm cU_n .$$

But

$$\pm cU_n = \pm U_{n+1} - U_{n-1}$$

therefore

$$U_{n+3} = (c^2 \pm 1)U_{n+1} \pm U_{n+1} - U_{n-1}$$

or

$$U_{n+3} = (c^2 \pm 2)U_{n+1} - U_{n-1} . \tag{15}$$

With the derivation of equation (15) we have proved Theorem 3. For convenience we let $c^2 \pm 2 = c_2$ and then the equations of Theorem 3 become

$$U'_{k+1} = c_2U'_k - U'_{k-1}, \quad U'_0 = U_0, U'_1 = U_2 \tag{16}$$

or

$$U'_1 = U_1, \quad U'_2 = U_3 .$$

The method of proof for Theorem 3 demonstrates that the properties of the sequences of X's or of Y's in the Pell equations are simply the properties to be expected from considerations of the recurrent equations $U_{n+1} = cU_n \pm U_{n-1}$.

EXAMPLES

Example 1. When $D = 2$ the minimal solution is $2Y_1^2 = X_1^2 + 1$, $Y_1 = 1$, $X_1 = 1$. From Theorem 1 we know that we must use the recurrent equation with the + sign and that the constant $c = 2X_1 = 2$. Thus, the sequence of X's develops from $X_{n+1} = 2X_n + X_{n-1}$, $X_0 = 1$, $X_1 = 1$.

$$X_2 = 2X_1 + X_0 = 2 \cdot 1 + 1 = 3$$

$$X_3 = 2X_2 + X_1 = 2 \cdot 3 + 2 = 7$$

$$X_4 = 2X_3 + X_2 = 2 \cdot 7 + 3 = 17 ,$$

etc. Thus

$$X = 1, 1, 3, 7, 17, 41, 99, \cdots .$$

Similarly for Y we have $Y_{n+1} = 2Y_n + Y_{n-1}$, $Y_0 = 0$, $Y_1 = 1$.

$$Y_2 = 2Y_1 + Y_0 = 2 \cdot 1 + 0 = 2$$

$$Y_3 = 2Y_2 + Y_1 = 2 \cdot 2 + 1 = 5$$

$$Y_4 = 2Y_3 + Y_2 = 2 \cdot 5 + 2 = 12 ,$$

etc., and

$$Y = 0, 1, 2, 5, 12, 29, 70, \cdots .$$

Example 2. For $D = 3$ the minimal solution is $X_1 = 2$, $Y_1 = 1$ and the minimal equation is $3Y_1^2 = X_1^2 - 1$, whence the recurrent law for $D = 3$ is

$$U_{n+1} = cU_n - U_{n-1}, \quad c = 2X_1 = 2 \cdot 2 = 4 .$$

Then

$$X_2 = 4X_1 - X_0 = 4 \cdot 2 - 1 = 7$$

$$X_3 = 4X_2 - X_1 = 4 \cdot 7 - 2 = 26$$

$$X_4 = 4X_3 - X_2 = 4 \cdot 26 - 7 = 99,$$

etc., and for the Y's

$$Y_2 = 4Y_1 - Y_0 = 4 \cdot 1 - 0 = 4$$

$$Y_3 = 4Y_2 - Y_1 = 4 \cdot 4 - 1 = 15$$

$$Y_4 = 4Y_3 - Y_2 = 5 \cdot 15 - 4 = 56,$$

etc., and

$$X = 1, 2, 7, 26, 99, \cdots$$

and

$$Y = 0, 1, 4, 15, 56, \cdots .$$

PROBLEMS

The following problems illustrate the use of the theorems developed here. Without knowledge of these theorems, I believe the problems might be difficult to solve.

Problem 1. The numbers 2024 and 32257 are consecutive values of one of the variables in a Pell equation. What are the corresponding values of the other variable? (There are two solutions.)

Problem 2. For $8Y^2 = X^2 - 1$ we have

$$X = 1, 3, 17, 99, \cdots$$

$$Y = 0, 1, 6, 35, \cdots$$

and

$$U_{n+1} = 6U_n - U_{n-1} .$$

Find another Pell equation(s) for which this recurrent law holds.

Problem 3. Prove that

$$X_n = \frac{X_1Y_n \pm Y_{n-1}}{Y_1}$$

and

$$Y_n = \frac{X_1X_n \pm X_{n-1}}{Y_1D} .$$

Use the + sign if $DY_1^2 = X_1^2 + 1$ and the - sign if $DY_1^2 = X_1^2 - 1$. Notice that in this problem the recurrent sequence of one variable is developed in terms of constants and the other variable.

Problem 4. In a Pell equation where $D = a^2 - 1$, $a > 1$, prove that $X_n \pm X_{n-1} \equiv 0 \bmod (X_1 \pm 1)$ using corresponding signs on each side of the congruence.

Problem 5. In Pell equations if $DY_1^2 = X_1^2 + 1$, prove:

$$\sum_{j=1}^{n} X_j = \frac{X_{n+1} + X_n - X_1 - 1}{c}$$

and

$$\sum_{j=1}^{n} Y_j = \frac{Y_{n+1} + Y_n - Y_1}{c}, \quad c = 2X_1 .$$

Note that if $c = 1$ and the X's are Lucas numbers and the Y's are Fibonacci numbers then we have the summation equations for the Lucas and Fibonacci sequences. If $DY_1^2 = X_1^2 - 1$, show that the comparable summations are

$$\sum_{j=1}^{n} X_j = \frac{X_{n+1} - X_n - X_1 - 1}{c - 2}$$

$$\sum_{j=1}^{n} Y_j = \frac{Y_{n+1} - Y_n - Y_1}{c - 2}, \quad c = 2X_1,\ X_1 \neq 1.$$

Problem 6. In each of the following equations find recurrent sequences of rational x's such that y is integral. The ambiguous sign is used to avoid negative roots.

a) $$3x^2 \pm 4x + 1 = y^2$$

b) $$3x^2 \pm 5x + 2 = y^2$$

c) $$2x^2 \pm 6x + 5 = y^2$$

d) $$6x^2 \pm 5x + 1 = y^2$$

EPILOGUE

In this part of the paper, some terms and notations are introduced which were found to be convenient.

a) In the Pell equation, $DY_n^2 = X_n^2 - (-1)^n$, $n = 0,1,2,3,\cdots$, we notice that as n increases, 1 is alternately subtracted and added to the X^2 term. Thus the equation is referred to as an alternating equation. For the equation $DY_n^2 = X_n^2 - 1$, $n = 0,1,2,3,\cdots$, 1 is always subtracted from X^2 and is referred to as non-alternating. The term alternating Pell equation implies the minimal equation $DY_1^2 = X_1^2 + 1$ and the recurrent law $U_{n+1} = cU_n + U_{n-1}$, whereas the term non-alternating Pell equation implies $DY_1^2 = X_1^2 - 1$ and the recurrent law $U_{n+1} = cU_n - U_{n-1}$. In this connection it is interesting to note that in recurrent equations where the n's are negative, the neighboring terms in the sequence developed from $U_{n-1} = U_{n+1} - cU_n$ have opposite signs and thus the signs in the sequence alternate. If $U_{n-1} = cU_n - U_{n+1}$ and $n < 1$, the neighboring terms of the sequence have the same signs and the sequence is non-alternating.

The use of non-positive n's in the equations of Theorem 1 leads to the conjugate solutions of the Pell equations.

b) In the recurrent equation $U_{n+1} = cU_n + U_{n-1}, c \geq 1$ is associated with the + sign preceding the U_{n-1} term and in the equation $U_{n+1} = cU_n - U_{n-1}$ $c \geq 1$ is associated with the - sign preceding the U_{n-1} term. A convenient notation for these recurrent equations is c^+ and c^-. For example 6^+ implies $U_{n+1} = 6U_n + U_{n-1}$ and 4^- implies $U_{n+1} = 4U_n - U_{n-1}$.

Since c^+ or c^- <u>indicates</u> the manner in which the recurrent sequence is developed they are called the indicator, I, of the sequence.

If α and β are the first two terms of a sequence, then the development of the sequence is completely determined by the indicator and the first two terms as $I(\alpha,\beta)$. For example, if $I = 3^+$, $\alpha = 2$, $\beta = 3$ then 3^+, (2,3) defines the sequence and implies $U_{n+1} = 3U_n + U_{n-1}$, $U_0 = 2$, $U_1 = 3$.

Throughout my notes I have used this notation because of its convenience and brevity.

Since each of the Pell equations, (1) and (2) have a unique recurrent law for a given D then it follows that they have a unique indicator but a given indicator does not necessarily determine a Pell equation uniquely.

c) If a sequence is determined by I, (α, β) and α, β have a common factor, f, then all terms of the sequence contain this factor. Let $\alpha = f\alpha_1$ and $\beta = f\beta_1$ then

$$I,\ (\alpha, \beta) = I,\ (f\alpha_1, f\beta_1) = I,\ f(\alpha_1, \beta_1)\ .$$

The nth term of the sequence can be developed from I, (α_1, β_1) to the nth term which is then multiplied by f and by this procedure we can use smaller numbers.

d) Applying these concepts to the Pell equations we have for the general recurrent solution

$$X = I,\ (1, X_1)$$

$$Y = I,\ Y_1(0, 1)$$

where $I = 2X_1^+$ if $DY_1^2 = X_1^2 + 1$ and $I = 2X_1^-$ if $DY_1^2 = X_1^2 - 1$.

We see that in general for any Pell equation $Y_n \equiv 0 \mod Y_1$.

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UNIQUE REPRESENTATIONS OF INTEGERS AS SUMS OF DISTINCT LUCAS NUMBERS

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INTRODUCTION

The Lucas numbers, $\{L_n\}_0^\infty$, are defined by

$$L_0 = 2, \qquad L_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n$$

for $n \geq 0$. Then,

$$L_n = F_{n+1} + F_{n-1}$$

for $n \geq 0$, where

$$F_{-1} = 1, \quad F_0 = 0$$

and

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 1)$$

define the Fibonacci numbers. It is well-known that the Lucas numbers are "complete" [1] in the sense that every positive integer can be expressed as a sum of distinct Lucas numbers. In general, such representations are not unique; for example,

$$4 = L_3 = L_1 + L_2, \qquad 12 = L_1 + L_3 + L_4 = L_0 + L_2 + L_4,$$

etc. Our purpose in this paper is to show, by introducing constraints analogous to those used in obtaining unique expansions of integers in Fibonacci

numbers, that unique representations in terms of Lucas numbers are also possible. We show, as one example, that every positive integer n has a unique representation of the form

$$n = \sum_{0}^{\infty} \alpha_i L_i \tag{1}$$

where $\alpha_i = \alpha_i(n)$ is a binary digit (zero or one) for each $i \geq 0$ and the α_i satisfy the following constraints:

$$\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 0 \tag{2}$$

$$\alpha_0 \alpha_2 = 0 \ . \tag{3}$$

We recall that the constraint $\alpha_i \alpha_{i+1} = 0$, which precludes the use of two successive Lucas numbers in the representation, is essentially the same requirement that gives unique representations in Zeckendorf's theorem for Fibonacci expansions ([2], [3]). The additional condition $\alpha_0 \alpha_2 = 0$ reflects the particularity of the Lucas sequence.

REPRESENTATION THEOREMS

Before stating the main theorems, certain preliminary lemmas will prove useful.

<u>Lemma 1.</u>

$$L_n - 1 = L_{n-1} + L_{n-3} + \cdots + L_{1,2}(n)$$

for $n \geq 2$,

where

$$L_{1,2}(n) = \begin{cases} 2L_1 & \text{if } n \text{ is even} \\ L_2 & \text{if } n \text{ is odd} . \end{cases}$$

Proof. By induction, one easily proves

$$L_{2n+1} - 1 = L_{2n} + L_{2n-2} + \cdots + L_4 + L_2 \quad (n \geq 1)$$

$$L_{2n} - 1 = L_{2n-1} + L_{2n-3} + \cdots + L_3 + 2L_1 \quad (n \geq 1) .$$

The Lemma statement combines these two identities.

Lemma 2.

$$L_{n+2} = 1 + \sum_{i=0}^{n} L_i \quad \text{for} \quad n \geq 0 .$$

Proof. Induction.

Lemma 3. Let

$$n = \sum_{0}^{\infty} \alpha_i L_i ,$$

where each α_i is a binary digit such that

i) $\alpha_i \alpha_{i+1} = 0$ for $i \geq 0$

ii) $\alpha_0 \alpha_2 = 0$.

Such a representation for n is unique.

Proof. Assume n has a competing representation,

$$n = \sum_{0}^{\infty} \gamma_i L_i$$

with γ_i binary, $\gamma_i \gamma_{i+1} = 0$ for $i \geq 0$ and $\gamma_0 \gamma_2 = 0$. Assume, for a proof by contradiction, that the two representations are not identical, that is,

$$\sum_{0}^{\infty} |\gamma_i - \alpha_i| \neq 0 .$$

Then, let k be the largest value of i such that $\alpha_i \neq \gamma_i$. Clearly $k \geq 2$, and since $\alpha_k \neq \gamma_k$, we may assume without loss of generality that $\alpha_k = 1$, $\gamma_k = 0$. It follows that, for some $m \leq n$,

$$m = \sum_{0}^{k} \alpha_i L_i = \sum_{0}^{k-1} \gamma_i L_i ,$$

with $\alpha_k = 1$. Then

$$\sum_{0}^{k} \alpha_i L_i \geq L_k ,$$

while from the coefficient constraints on the $\{\gamma_i\}$,

$$\sum_{0}^{k-1} \gamma_i L_i \leq L_{k-1} + L_{k-3} + \cdots + L_{1,2}(k) = L_k - 1 ,$$

the last equality from Lemma 1. Thus $m \geq L_k$ while $m \leq L_k - 1$, a contradiction.

<u>Lemma 4.</u> Let

$$n = \sum_{0}^{k} \beta_i L_i \quad (k \geq 2) ,$$

where each β_i is a binary digit such that

i) $$\beta_i + \beta_{i+1} \neq 0 \quad \text{for } 0 \leq i \leq k-2$$

ii) $$\beta_0 + \beta_2 \neq 0$$

iii) $$\beta_k = 1 .$$

Such a representation for n is unique.

Proof. Assume n has two representations in the given form; that is,

$$n = \sum_{i=0}^{k} \beta_i L_i = \sum_{i=0}^{m} \gamma_i L_i , \tag{4}$$

where β_i and γ_i are binary digits satisfying

$$\beta_k = \gamma_m = 1, \quad \beta_i + \beta_{i+1} \neq 0$$

for $0 \leq i \leq k - 2$,

$$\beta_0 + \beta_2 \neq 0, \quad \gamma_i + \gamma_{i+1} \neq 0$$

for $0 \leq i \leq m - 2$,

$$\gamma_0 + \gamma_2 \neq 0 .$$

Without loss of generality, we take $m \geq k \geq 2$. If $m > k$, then the right-hand representation in (4), together with the coefficient constraints, implies

$$n \geq \begin{cases} L_m + L_{m-2} + \cdots + L_2 + L_1 = L_{m+1} \geq L_{k+2} & (m \text{ even}) \\ L_m + L_{m-2} + \cdots + L_3 + L_1 + L_0 = L_{m+1} \geq L_{k+2} & (m \text{ odd}) . \end{cases}$$

But

$$n = \sum_{i=0}^{k} \beta_i L_i \leq \sum_{i=0}^{k} L_i = L_{k+2} - 1 ,$$

a contradiction. Hence $m = k$ in (4); that is,

$$n = \sum_0^k \beta_i L_i = \sum_0^k \gamma_i L_i ,$$

or equivalently,

$$\sum_0^k (1 - \beta_i) L_i = \sum_0^k (1 - \gamma_i) L_i .$$

If we now define $\alpha_i = 1 - \beta_i$ and $\delta_i = 1 - \gamma_i$ for $0 \le i \le k$ and $\alpha_i = \delta_i = 0$ for $i \ge k$, then

$$\sum_0^\infty \alpha_i L_i = \sum_0^\infty \delta_i L_i ,$$

with α_i, δ_i binary digits satisfying

$$\alpha_i \alpha_{i+1} = \delta_i \delta_{i+1} = 0$$

for all $i \ge 0$ and

$$\alpha_0 \alpha_2 = \delta_0 \delta_2 = 0 .$$

By Lemma 3, $\alpha_i = \delta_i$ for $i \ge 0$ and thus $\beta_i = \gamma_i$ for $0 \le i \le k$, implying uniqueness of the representation.

<u>Theorem 1.</u> Let n be a nonnegative integer satisfying $0 \le n < L_k$ for some $k \ge 1$. Then

$$n = \sum_0^{k-1} \alpha_i L_i \tag{5}$$

with α_i binary digits satisfying

i) $$\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 0$$

ii) $$\alpha_0 \alpha_2 = 0 \quad .$$

Further, the representation of n in this form is unique. [If $k - 1 < 2$ in (5), we define $\alpha_2 = 0$ so that ii) is automatically satisfied.]

Proof. Uniqueness follows from Lemma 3. It remains to show such a representation exists. For a proof by induction on the index k, we verify directly that the theorem holds for $k = 1$ and $k = 2$. Now, assume as an induction hypothesis that the theorem holds for all $k \leq k_0$ where $k_0 \geq 2$. To show the theorem holds for $k_0 + 1$, it suffices to consider an arbitrary integer n satisfying

$$L_{k_0} \leq n \leq L_{k_0+1} \quad .$$

Then

$$0 \leq n - L_{k_0} < L_{k_0+1} - L_{k_0} = L_{k_0-1} \quad .$$

By the induction hypothesis, there exist binary coefficients γ_i such that

$$n - L_{k_0} = \sum_{0}^{k_0-2} \gamma_i L_i$$

with

$$\gamma_i \gamma_{i+1} = 0 \quad \text{for } i \geq 0, \quad \gamma_0 \gamma_2 = 0 \; .$$

Then

$$n = \sum_{0}^{k_0} \gamma_i L_i$$

where

$$\gamma_{k_0-1} = 0, \quad \gamma_{k_0} = 1 ,$$

so that n is representable in the required form with the given coefficient constraints. q. e. d.

Theorem 2. Let n be a positive integer satisfying

$$\sum_0^{k-1} L_i < n \le \sum_0^k L_i$$

for some $k \ge 2$. Then

$$n = \sum_0^k \beta_i L_i$$

with β_i binary coefficients satisfying

i) $\beta_i + \beta_{i+1} \ne 0$ for $0 \le i \le k-2$

ii) $\beta_0 + \beta_2 \ne 0$

iii) $\beta_k = 1$.

Further, the representation of n in this form is unique.

Proof. Again, uniqueness is a consequence of Lemma 4. To establish the representation, note that

$$\sum_0^{k-1} L_i < n \le \sum_0^k L_i$$

implies

$$0 \le \sum_0^k L_i - n < \sum_0^k L_i - \sum_0^{k-1} L_i = L_k .$$

By Theorem 1, the integer

$$\sum_{0}^{k} L_i - n$$

has a representation

$$\sum_{0}^{k} L_i - n = \sum_{0}^{k-1} \alpha_i L_i ,$$

where the binary coefficients α_i satisfy $\alpha_i \alpha_{i+1} = 0$ for

$$0 \leq i \leq k-2, \quad \alpha_0 \alpha_2 = 0 .$$

Then

$$n = L_k + \sum_{0}^{k-1} (1-\alpha_i) L_i = \sum_{0}^{k} (1-\alpha_i) L_i ,$$

where $\alpha_k = 0$, and the theorem follows on recognizing $\beta_i = 1 - \alpha_i$ $(0 \leq i \leq k)$ as binary coefficients satisfying

$$\beta_i + \beta_{i+1} \neq 0$$

for $0 \leq i \leq k-2$, $\beta_0 + \beta_2 \neq 0$ and $\beta_k = 1$. q. e. d.

Theorem 2 thus guarantees the representation for all positive integers ≥ 4. Representations for the positive integers $1, 2, 3$ are immediate, namely

$$1 = 0 \cdot L_0 + 1 \cdot L_1 , \quad 2 = 1 \cdot L_0, \quad 3 = 1 \cdot L_0 + 1 \cdot L_1 .$$

The constraint $\beta_0 + \beta_2 \neq 0$ is assumed not to be enforced in these three cases where the largest Lucas number appearing in the expansion is less than $L_2 = 3$.

Theorem 2 is a dual to Theorem 1 and corresponds to the dual of the Zeckendorf theorem for Fibonacci numbers [4].

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ASSOCIATION MEETING

The Fibonacci Association held its Fall Meeting on October 18th at San Jose State College. Following was the Program:

MORNING SESSION

9:30 a.m.	SOCIAL GATHERING
10:00 — 10:45	TEST FOR THE PRIMALITY OF MERSENNE NUMBERS Douglas Lind, Stanford University
10:45 — 11:30	WEB SEQUENCES George Ledin, Jr., University of San Francisco
11:30 — 12 Noon	OPPORTUNITY FOR GENERAL DISCUSSION

AFTERNOON SESSION

1:15 — 2:00	FIBONACCI AND RELATED SERIES IN COMBINATORICS Prof. D. H. Lehmer, University of Calif., Berkeley
2:00 — 2:45	MARKOV-FIBONACCI RELATIONS Prof. Gene Gale, San Jose State College
2:45 — 3:30	IT'S GENERALIZED! WHAT'S NEXT? Prof. V. C. Harris, San Diego State College

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COMPOSITIONS AND FIBONACCI NUMBERS

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1. INTRODUCTION

A composition of n is an ordered partition of n; that is, a representation of n as the sum of positive integers with regard to order. For example, 4 has the eight compositions

$$4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1 .$$

Some elementary properties of compositions have been given by Riordan [12, 124-125], and a more extensive study has been made by MacMahon [9, 150-216]. Isolated examples of composition formulas involving Fibonacci numbers have appeared sporadically in the literature (see [11], [13], [14], [15], [16]). In an earlier paper [6] the authors established a general composition formula and its inversion of which the above are particular examples. This formula generalized a result of Moser and Whitney [11], and from it followed a number of further results. In this paper we review the previous results, continue their development, and show how these techniques can be used to prove certain Fibonacci identities.

2. PREVIOUS RESULTS

From direct expansion we find that the enumerator of compositions with exactly k parts is $(x + x^2 + \cdots)^k$. That is, the coefficient of x^n in the resulting series is the number of compositions of n with k parts. If a summand j is given weight w_j, then

$$(w_1x + w_2x^2 + \cdots)^k = [W(x)]^k$$

maybe termed the enumerator of weighted k-part compositions. To obtain an explicit representation, put

$$C(x,y;w) = \sum_{k=0}^{\infty}[W(x)]^k y^k = \frac{1}{1 - yW(x)} = \sum_{k,n=0}^{\infty} c_{nk}(w)x^n y^k , \tag{2.1}$$

where $w \equiv \{w_1, w_2, \cdots\}$. Using the formula for derivatives of composite functions (see [12, p. 36]),

$$c_{nk}(w) \equiv c_{nk} = \sum_{\pi_k(n)} \frac{k!}{k_1! \cdots k_n!} w_1^{k_1} \cdots w_n^{k_n} \qquad (n,k > 0) , \tag{2.2}$$

where the sum is extended over all k-part partitions of n; that is, over all solutions of $k_1 + 2k_2 + \cdots + nk_n = n$ such that $k_1 + \cdots + k_n = k$. Since the number of distinct compositions obtainable from the above partition is the coefficient in (2.2), the omission of the coefficient calls for summation over compositions. We write

$$c_{nk}(w) = \sum_{\gamma_k(n)} w_{a_1} w_{a_2} \cdots w_{a_k} \qquad (n,k > 0) \tag{2.3}$$

where $\gamma_k(n)$ indicates summation over all k-part compositions $a_1 + \cdots + a_k$ of n. Specialize this by letting

$$C(x) \equiv C(x,1;w) = \frac{1}{1 - W(x)} = \sum_{n=0}^{\infty} c_n(w)x^n \tag{2.4}$$

in which

$$c_n(w) \equiv c_n = \sum_{k=1}^{\infty} c_{nk}(w) = \sum_{\gamma(n)} w_{a_1} \cdots w_{a_k} \qquad (n > 0) , \tag{2.5}$$

where $\gamma(n)$ indicates summation over all compositions $a_1 + \cdots + a_k$ of n, the number of summands k in the composition being variable. Equations (2.4) and (2.5) were given by Moser and Whitney [11].

To obtain an inversion formula for (2.5), note that

$$1 - W(x) = C(x)^{-1} = 1 + \sum_{n=1}^{\infty}\left(\sum_{\gamma(n)}(-1)^k c_{a_1} \cdots c_{a_k}\right) x^n .$$

Hence

$$-w_n = \sum_{\gamma(n)}(-1)^k c_{a_1} \cdots c_{a_k} \qquad (n > 0) . \tag{2.6}$$

To help motivate the above, we note that it is shown in [5] and [7] that if a pair of rabbits produces w_n pairs of offspring at the nth time point, and their offspring do likewise, then the total number of pairs born at the nth time point is c_n. We shall see below in example (3d) that our results generalize the famous rabbit reproduction problem which led Leonardo of Pisa to discover the Fibonacci numbers originally.

3. EXAMPLES AND ILLUSTRATIONS

In this section we specialize the above results, obtaining the known instances of Fibonacci related composition formulas appearing in the literature, as well as some other results.

Define the Fibonacci numbers F_n by

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \qquad (n \geq 1) ,$$

and the Lucas numbers L_n by

$$L_1 = 1, \qquad L_2 = 3, \qquad L_{n+2} = L_{n+1} + L_n \qquad (n \geq 1).$$

We make use of several standard generating functions for Fibonacci and Lucas numbers, for which we refer the reader to [2].

(3a) Letting $w_n = 1$ $(n \geq 1)$, so that

$$W(x) = x + x^2 + \cdots = x/(1 - x),$$

and using the convention $\binom{n}{k} = 0$ if $k > n$, we have

$$C(x,y;w) = 1 + \frac{yW(x)}{1 - yW(x)} = \frac{xy}{1 - x(1 + y)}$$

$$= \sum_{n=0}^{\infty} x^n (1 + y)^n xy = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n}{k} x^{n+1} y^{k+1}, \tag{3.1}$$

so that

$$c_{nk}(w) = \sum_{\gamma_k(n)} 1 = \binom{n-1}{k-1} \tag{3.2}$$

is the number of compositions of n into k parts. This appears in [12], and can be verified by combinatorial arguments.

It follows that

$$\sum_{\gamma(n)} 1 = c_n(w) = \sum_{k=0}^{\infty} c_{nk}(w) = \sum_{k=1}^{\infty} \binom{n-1}{k-1} = (1+1)^{n-1} = 2^{n-1} \tag{3.3}$$

is the total number of compositions of n. For example, 4 has the $8 = 2^{4-1}$ compositions mentioned in the Introduction.

(3b) Put $w_n = n$, which gives

$$W(x) = x + 2x^2 + 3x^3 + \cdots = x/(1 - x)^2 .$$

In this case,

$$C(x) - 1 = \frac{x}{1 - 3x + x^2} = \sum_{n=1}^{\infty} F_{2n} x^n .$$

Then (2.5) yields

$$c_n = \sum_{\gamma(n)} a_1 a_2 \cdots a_k = F_{2n} , \tag{3.4}$$

which has been given by Moser and Whitney [11], and proposed as a problem in this Quarterly [16]. As an example, for $n = 4$ we have

$$c_4 = 4 + 2(3 \cdot 1) + 2 \cdot 2 + 3(2 \cdot 1 \cdot 1) + 1 \cdot 1 \cdot 1 \cdot 1 = 21 = F_8.$$

(3c) Set

$$w_1 = w_2 = 1, \quad w_n = 0 \quad (n \geq 3) ,$$

so that $W(x) = x + x^2$. Then

$$C(x) - 1 = \frac{x + x^2}{1 - x - x^2} = \sum_{n=1}^{\infty} F_{n+1} x^n ,$$

and using (2.5) we get

$$c_n = \sum_{\gamma(n); a_j \leq 2} 1 = F_{n+1} , \tag{3.5}$$

since in any composition with $a_j > 2$, $w_{a_j} = 0$ annihilates the summand. Thus the number of compositions of n into 1's and 2's is F_{n+1}. This was proposed by Moser as Problem B-5 [14].

(3d) Let $w_1 = 0$ and $w_n = 1$ $(n \geq 2)$, giving

$$W(x) = x^2(1 + x + \cdots) = x^2/(1 - x) .$$

Then

$$C(x) - 1 = \frac{x^2}{1 - x - x^2} = \sum_{n=1}^{\infty} F_{n-1} x^n ,$$

so that by (2.5) we have

$$c_n = \sum_{\gamma(n);a_j \geq 2} 1 = F_{n-1} . \tag{3.6}$$

Thus the number of compositions of n into parts greater than a unity is F_{n-1}. In this case we have

$$C(x,y;w) - 1 = \frac{x^2y}{1 - x(1 + xy)} = x^2y\sum_{j=0}^{\infty} x^j(1 + xy)^j$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{j}{i}x^{i+j+2}y^{i+1} = \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\binom{n-k-1}{k-1}x^ny^k ,$$

so that by (2.3)

$$c_{nk}(w) = \sum_{\gamma_k(n);a_j>1} 1 = \binom{n-k-1}{k-1} \tag{3.7}$$

is the number of compositions of n into k parts, each of which is greater than one. Then (2.5) shows

$$F_{n-1} = \sum_{k=1}\binom{n-k-1}{k-1} , \tag{3.8}$$

which was first shown by Lucas [8, p. 186].

(3e) If

$$W(x) = x + x^3 + x^5 + \cdots = x/(1 - x^2),$$

then a calculation similar to that in (3d) shows

$$c_{nk}(w) = \sum_{\gamma_k(n);a_j \text{ odd}} 1 = \binom{\frac{1}{2}(n+k) - 1}{k - 1} \tag{3.9}$$

to be the number of k-part compositions of n into odd parts. Since

$$c_n(w) = \sum_{\gamma(n);a_j \text{ odd}} 1 = \sum_{k=1} \binom{\frac{1}{2}(n+k) - 1}{k - 1} = F_n , \tag{3.10}$$

we may state that the number of compositions of n into odd parts is F_n.

(3f) Put

$$W(x) = \frac{x^2}{1 - 3x + x^2} = \sum_{n=1}^{\infty} F_{2n-2}\, x^n ,$$

so that

$$C(x) - 1 = \frac{x^2}{1 - 3x} = \sum_{n=2}^{\infty} 3^{n-2}\, x^n .$$

Then by (2.5)

$$c_n(w) = \sum_{\gamma(n);a_j>1} F_{2a_1-2} \cdots F_{2a_k-2} = 3^{n-2} \quad (n \geq 2) . \tag{3.11}$$

The inverse relation given by (2.6) is

$$-F_{2n-2} = \sum_{\gamma(n);a_j>1} (-1)^k 3^{a_1-2} \cdots 3^{a_k-2} = \sum_{\gamma(n);a_j>1} (-1)^k 3^{n-2k} \qquad (n \geq 2) .$$

Since the summand depends only on the number of integers in the composition, we may use the value of $c_{nk}(w)$ in (3d) to get

$$F_{2n} = \sum_{k=1} (-1)^{k-1} \binom{n-k}{k-1} 3^{n+1-2k} \qquad (n \geq 1) , \tag{3.12}$$

which was proposed as Problem H-83 in this Quarterly [17].

(3g) We shall establish some further Fibonacci identities via composition formulas. Let

$$W(x) = x^2 + 4x^3 + 4^2x^4 + \cdots = x^2/(1-4x) ,$$

so that

$$C(x) - 1 = \frac{x^2}{1-4x-x^2} = \frac{1}{2}\sum_{n=1}^{\infty} F_{3n-3}x^n .$$

Then with (2.5) we get

$$\frac{1}{2} F_{3n-3} = \sum_{\gamma(n);a_j>1} 4^{a_1-2} \cdots 4^{a_k-2} = \sum_{\gamma(n);a_j>1} 4^{n-2k} . \tag{3.13}$$

Again using the value for $c_{nk}(w)$ in (3d), we find

$$\frac{1}{2} F_{3n-3} = \sum_{k=1} \binom{n-k-1}{k-1} 4^{n-2k} . \tag{3.14}$$

We can generalize this as follows. First let s be odd, and set

$$W(x) = \frac{x^2}{1 - L_s x} = \sum_{n=2}^{\infty} L_s^{n-2} x^n .$$

Then

$$C(x) - 1 = \frac{x^2}{1 - L_s x - x^2} = \frac{1}{F_s} \sum_{n=1}^{\infty} F_{s(n-1)} x^n .$$

We then get

$$F_{s(n-1)}/F_s = \sum_{\gamma(n);a_j>1} L_s^{n-2k} ,$$

so using (3d) we have

$$F_{s(n-1)}/F_s = \sum_{k=1} \binom{n-k-1}{k-1} L_s^{n-2k} \qquad (s \text{ odd}) .$$

For even s, a similar calculation with

$$W(x) = \frac{-x^2}{1 - L_s x} = \sum_{n=2}^{\infty} -L_s^{n-2} x^n$$

shows

$$F_{s(n-1)}/F_s = \sum_{k=1} (-1)^{k-1} \binom{n-k-1}{k-1} L_s^{n-2k} \qquad (s \text{ even}) .$$

The even and odd cases can be combined into

$$F_{s(n-1)}/F_s = \sum_{k=1} (-1)^{(k-1)(s-1)} \binom{n-k-1}{k-1} L_s^{n-2k} . \tag{3.15}$$

This result was recently posed as a problem [18].

We conclude this section by noting that Hoggatt [5], in connection with a study of the reproduction patterns of mathematical Fibonacci rabbits, has exhibited a number of generating functions $W(x)$ which have particularly convenient corresponding generating functions $C(x)$. Each of these has the natural combinatorial interpretation provided by (2.5) and (2.6).

4. RELATIONS INVOLVING FIBONACCI GENERALIZATIONS

In this section we consider composition formulas involving three distinct generalizations of the Fibonacci numbers. Most of these reduce to results contained in Section 3.

(4a) Define the Fibonacci polynomials $f_n(t)$ by

$$f_1(t) = 1, \qquad f_2(t) = t, \quad \text{and} \quad f_{n+2}(t) = tf_{n+1}(t) + f_n(t) \quad (n \geq 1) .$$

It follows that $f_n(1) = F_n$. It can also be easily verified that the generating function for these polynomials is

$$\frac{x}{1 - tx - x^2} = \sum_{n=1}^{\infty} f_n(t)\, x^n . \tag{4.1}$$

Letting $W(x)$ equal to (4.1), we find

$$C(x) - 1 = \frac{x}{1 - (t+1)x - x^2} = \sum_{n=0}^{\infty} f_n(t+1)x^n .$$

Then (2.5) yields

$$f_n(t+1) = \sum_{\gamma(n)} f_{a_1}(t) \cdots f_{a_k}(t) \quad . \tag{4.2}$$

As a special case given in [6] we get for $t = 1$ that

$$P_n = \sum_{\gamma(n)} F_{a_1} \cdots F_{a_k} \quad , \tag{4.3}$$

where $P_n = f_n(2)$ is the Pell sequence discussed by Lucas [8].

(4b) Miles [10] has investigated the properties of the r-generalized Fibonacci numbers $f_{n,r}$ defined for $r \geq 1$ by

$$\begin{aligned} f_{n,r} &= 0 \quad (0 \leq n \leq r-2), \quad f_{r-1,r} = 1 \, , \\ f_{n,r} &= \sum_{j=1}^{r} f_{n-j,r} \quad (n \geq r) \quad . \end{aligned} \tag{4.4}$$

It follows that $f_{n,1} = 1$ and $f_{n,2} = F_n$. The numbers $f_{n,3}$ are the so-called Tribonacci numbers studied by Feinberg [1]. It is not difficult to see that the generating function for the $f_{n,r}$ is

$$\frac{x^{r-1}}{1 - x - x^2 - \cdots - x^r} = \sum_{n=r-1}^{\infty} f_{n,r} x^n \quad . \tag{4.5}$$

For our first result, let $W(x) = x + x^2 + \cdots + x^r$. Then

$$C(x) = 1 + \sum_{n=1}^{\infty} f_{n+r-1} x^n \quad .$$

But it follows from (2.5) that $c_n(w)$ is the number of compositions of n into parts not greater than r. Thus we see

$$\sum_{\gamma(n);a_j \leq r} 1 = f_{n+r-1,r} \quad , \tag{4.6}$$

which reduces to (3c) by putting $r = 2$. By letting $r = 3$ we also obtain a partial solution to Problem B-96 in this Quarterly [15].

We may get $f_{n,r}$ in terms of a composition formula involving the $f_{i,r-1}$ in the following manner. Let

$$W(x) = \frac{x^r}{1 - x - \cdots - x^{r-1}} = \sum_{n=2}^{\infty} f_{n-2,r-1} x^n .$$

Then

$$C(x) - 1 = \frac{x^r}{1 - x - \cdots - x^r} = \sum_{n=1}^{\infty} f_{n-1,r-1} x^n .$$

Then from (2.5) we get

$$f_{n,r} = \sum_{\gamma(n+1)} f_{a_1-2,r-1} \cdots f_{a_k-2,r-1} \quad , \tag{4.7}$$

where $f_{n,r} = 0$ if $n < 0$. We note that for $r = 2$, (4.7) becomes (3.6). The inversion relation (2.6) gives

$$-f_{n-1,r-1} = \sum_{\gamma(n+1)} (-1)^k f_{a_1-1,r} \cdots f_{a_k-1,r} , \tag{4.8}$$

giving a formula for $f_{n,r-1}$ in terms of the $f_{i,r}$.

(4c) If $w_j = 0$ $(1 \leq j < r)$, $w_j = 1$ $(j \geq r)$, then $W(x) = x^r/(1 - x)$. Now Hoggatt [4] has shown

$$\frac{1}{1 - x - x^p} = \sum_{n=0}^{\infty} u(n;p-1,1)x^n , \tag{4.9}$$

where the $u(n;p,q)$ are the generalized Fibonacci numbers introduced by Harris and Styles [3] defined by

$$u(n;p,q) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq} \quad (n \geq 0) . \tag{4.10}$$

Then

$$C(x) - 1 = \frac{x^r}{1 - x - x^r} = \sum_{n=r}^{\infty} u(n-r;r-1,1)x^n ,$$

so that

$$c_n(w) = \sum_{\gamma(n);a_j \geq r} 1 = u(n-r;r-1,1) \tag{4.11}$$

is the number of compositions of n into parts greater than or equal to r. It follows from (4.9) that $u(n;1,1) = F_{n+1}$, so that setting $r = 2$ in (4.11) yields (3.6).

On the other hand, letting $W(x) = x + x^p$, $C(x)$ becomes (4.9) and we see

$$\sum_{\gamma(n);a_j=1,p} 1 = u(n;p-1,1) \tag{4.12}$$

is the number of compositions of n into 1's and p's. This reduces to (3.5) by letting $r = 2$.

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ON THE GROWTH OF $d_k(n)$

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1.) Let $d(n)$ denote the number of divisors of n, $\log_k n$ the k-fold iterated logarithm. It was shown by Wigert [1] that ($\exp z = e^z$)

$$d(n) < \exp\left((1+\epsilon)\log^2 \frac{\log n}{\log\log n}\right)$$

for all positive values of ϵ and all sufficiently large values of n, and that

$$d(n) > \exp\left((1-\epsilon)\log^2 \frac{\log n}{\log\log n}\right)$$

for an infinity of values of n.

Let $d_k(n)$ denote the k-fold iterated $d(n)$ (i.e.,

$$d_1(n) = d(n), \ (d_k(n) = d(d_{k-1}(n)), \ k \geq 2).$$

S. Ramanujan remarked in his paper [2] that

$$d_2(n) > 4^{\frac{\sqrt{2\log n}}{\log\log n}},$$

and that

$$d_3(n) > (\log n)^{\log\log\log\log n}$$

for an infinity of values of n.

Let ℓ_k denote the kth element of the Fibonacci sequence (i.e.,

$$\ell_{-1} = 0, \ \ell_0 = 1, \ \ell_k = \ell_{k-1} + \ell_{k-2} \text{ for } k \geq 1).$$

We prove the following:

Theorem 1. We have

$$d_k(n) < \exp (\log n)^{\frac{1}{\ell_k}+\epsilon} \tag{1.1}$$

for all fixed k, all positive ϵ and all sufficiently large values of n, further for every $\epsilon > 0$

$$d_k(n) > \exp\left((\log n)^{\frac{1}{\ell_k}-\epsilon}\right) \tag{1.2}$$

for an infinity of values of n.

It is obvious that $d(n) < n$, if $n > 2$. For a general $n > 1$, let $k(n)$ denote the smallest k for which $d_k(n) = 2$. We shall prove

Theorem 2.

$$0 < \limsup \frac{K(n)}{\log\log\log n} < \infty \quad . \tag{1.3}$$

2.) The letters c, c_1, $c_2, \cdots$ denote positive constants, not the same in every occurrence. The p_i's denote the ith prime number.

3.) First, we prove (1.2). Let r be large. Put $N_1 = 2\cdot 3 \cdots p_r$, where the p's are the consecutive primes. We define $N_2, \cdots, N_k$ by induction. Assume

$$N_j = \prod_{i=1}^{S_j} p_i^{r_2} \quad , \tag{3.1}$$

then

$$N_{j+1} = \left(p_1 \cdots p_{r_1}\right)^{p_1-1}\left(p_{r_1+1} \cdots p_{r_1+r_2}\right)^{p_2-1} \cdots \left(p_{r_1+\cdots+r_{S_j-1}+1} \cdots p_{r_1+\cdots+r_{S_j}}\right)^{p_{S_j}-1} \tag{3.2}$$

From (3.2) $d(N_{j+1}) = N_j$, and thus

$$d_k(N_k) = 2^r . \tag{3.3}$$

Let S_j and Γ_j denote the number of different and all prime factors of N_j, respectively. We have

$$S_1 = \Gamma_1 = r, \quad S_{j+1} = \Gamma_j . \tag{3.4}$$

Furthermore

$$S_{j+2} = \Gamma_{j+1} = \sum_{\nu=1}^{S_j} \gamma_\nu (p_\nu - 1) < p_{S_j} \sum_{\nu=1}^{S_j} \gamma_\nu < c\,\Gamma_j S_j \log S_j , \tag{3.5}$$

since $p_\ell < c_\ell \log \ell$ for $\ell \geq 2$. Hence by (3.4)

$$S_{j+2} < c\, S_{j+1} S_j \log S_j \qquad (j \geq 1) , \tag{3.6}$$

follows.

Using the elementary fact that

$$\sum_{i=1}^{\ell} \log p_i < c p_\ell < c\ell \log \ell ,$$

we obtain from (3.2),

$$\log N_{j+1} \leq p_{S_j} \sum_{i=1}^{\Gamma_j} \log p_i \leq c\, S_j \Gamma_j (\log \Gamma_j)^2 = c S_j S_{j+1} (\log S_{j+1})^2 . \tag{3.7}$$

From (3.3), (3.4) we easily deduce by induction that for every $\epsilon > 0$ and sufficiently large r

$$S_1 = r, \quad \Gamma_1 = r, \quad S_2 = r, \quad \Gamma_2 < r^{2+\epsilon}, \; S_3 < r^{2+\epsilon}, \; \Gamma_3 < r^{3+\epsilon}, \cdots,$$

$$S_k < r^{\ell_{k-1}+\epsilon}, \quad \Gamma_k \leq r^{\ell_k+\epsilon} .$$

Using (3.7), we obtain that

$$\log N_k \leq r^{\ell_k + \epsilon},$$

whence

$$d_k(N_k) = 2^r \geq \exp\left((\log N_k)^{1/\ell_k - \epsilon}\right),$$

which proves (1.2).

4.) Now we prove (1.1). Let $N_0, N_1, \cdots, N_k$ be an arbitrary sequence of natural numbers, such that

$$d(N_{j+1}) = N_j,$$

for $j = 0, 1, \cdots, k-1$.

Let B denote an arbitrary quantity in the interval

$$(\log\log N_k)^{-c} \leq B \leq (\log\log N_k)^c,$$

not necessarily the same at every occurrence.

We prove

$$\log N_k \geq B(\log N_0)^{\ell_k}, \tag{4.1}$$

whence (1.1) immediately follows.

In the proof of (4.1) we may assume that $\log N_0 \geq (\log N_k)$, with a positive constant $\delta < 1/\ell_k$.

Let

$$N_1 = \prod_{i=4}^{S_1} q_i^{\alpha_i - 1}.$$

Then

$$N_0 = \prod_{i=1}^{S_1} \alpha_i \ .$$

Since

$$2^{\alpha_i - 1} \leq q_i^{\alpha_i - 1} \Big| N_1 \ ,$$

we have

$$\alpha_i \leq c \log N_1 \ .$$

Hence

$$(\log 2) S_1 \leq \log N_0 = \Sigma \log \alpha_i \leq (\log \log N_1 + c) S_1 \ ,$$

i.e.,

$$\log N_0 = BS_1 \ .$$

We need the following:

<u>Lemma.</u> Suppose that for some integer j, $1 \leq j \leq k - 1$,

$$Q_1^{\gamma_1 - 1} \cdots Q_A^{\gamma_A - 1} \Big| N_j \ , \tag{4.2}$$

where $Q_1, \cdots, Q_A$ are different prime numbers and

$$A \geq BS_1^{\ell_{j-1}} ; \ Q_i \geq BS_1^{\ell_{j-1}} , \ \gamma_i \geq BS_1^{\ell_{j-2}} \quad (i = 1, \cdots, A) . \tag{4.3}$$

Then either

$$\log N_{j+1} \geq (\log N_0)^{\ell_k} , \tag{4.4}$$

or

$$r_1^{\beta_1 - 1} \cdots r_C^{\beta_C - 1} \Big| N_{j+1} \ , \tag{4.5}$$

where $r_1, \cdots, r_C$ are different primes and

$$C \geq BS_1^{\ell_j}, \quad r_i \geq BS_1^{\ell_j}, \quad \beta_i \geq BS_1^{\ell_{j-1}} \quad (i = 1, \cdots, C) . \tag{4.6}$$

To prove the lemma, let

$$N_{j+1} = \prod^{S_{j+1}} t_i^{\delta_i - 1} , \quad t_i \text{ primes} .$$

Since $d(N_{j+1}) = N_j$, by (4.2),

$$\prod_{i=1}^{A} Q_i^{\gamma_i - 1} \Big| \prod^{S_{j+1}} \delta_i = N_j \ . \tag{4.7}$$

Assume first that there is a δ_i which has at least $2\ell_k$ (not necessarily distinct) prime divisors amongst the Q_i. We then have

$$\log N_{j+1} \geq \frac{1}{2}\delta_i \log t_i \geq \frac{\log 2}{2}\,\delta_i \geq \left((BS_1^{\ell_{j-1}})\right)^{2\ell_k} \geq$$
$$\geq (BS_1)^{2\ell_k} \geq (\log N_0)^{\ell_k} \ ,$$

if N_0 is sufficiently large, i.e. (4.4) holds. Then by (4.2), the number D of δ's, each of which contains a prime divisor amongst the Q's satisfies the inequality

$$D \geq \frac{1}{2_k}\sum_{i=1}^{A} (\gamma_i - 1) \geq \frac{A}{4_k}\min \gamma_2 \geq ABS_1^{\ell_{j-2}} \geq BS_1^{\ell_{j-2}+\ell_{j-1}} = BS_1^{\ell_j} \ . \tag{4.8}$$

Without loss of generality, we assume that these δ's are $\delta_1, \cdots, \delta_D$ and $t_1 > t_2 > \cdots > t_D$ in (4.7). Since at least one Q divides $\delta_i (i \leq D)$, by (4.3), we have

$$\delta_i > BS_1^{\ell_{j-1}} \ .$$

Furthermore it is obvious that $t_{[D/2]} > D$. By choosing

$$C = D - \frac{D}{2}\ , \quad r_i = t_i, \quad \beta_i = \delta_i \quad (i = 1, \cdots, C)\ ,$$

we obtain (4.5) and (4.6).

This completes the proof of the Lemma.

Now (4.1) rapidly follows. Indeed, the validity of (4.4) for some j, $1 \leq j \leq k - 1$, immediately implies (4.1). So we may assume that (4.4) does not hold for $j = 1, \cdots, k - 1$. Now we use the Lemma for $j = 1, \cdots, k - 1$. Since N_1 has S_1 different prime divisors ($[1/2\, S_1]$ of these is greater than S_1)

the conditions (4.2), (4.3) are satisfied for $j = 1$. Hence (4.5)-(4.6) holds, i.e., the conditions (4.2)-(4.3) hold for $j = 2$. By induction we obtain that N_k has at least

$$BS_1^{\ell_{k-1}}$$

distinct prime factors each with the exponent greater than $BS_1^{\ell_{k-2}}$. Let

$$N_k = \Pi P_i^{\rho_i - 1} .$$

Since

$$\log N_k > \frac{1}{4} \sum \rho_i ,$$

we have

$$\log N_k > BS_1^{\ell_{k-1} + \ell_{k-2}} = B(\log N_0)^{\ell_k} .$$

Consequently (4.1) holds.

5.) Proof of Theorem 2. Using (1.1) in the form

$$d_2(n) < \exp\left((\log n)^{2/3}\right)$$

for $n \geq c$, and applying this k times, we have

$$\log d_{2k}(n) < (\log n)^{(2/3)^k}, \text{ when } d_{2k-2}(n) \geq c. \tag{5.1}$$

Equation (5.1) implies the upper bound in (1.3) by a simple computation.

For the proof of the lower bound we use the construction as in 3). Let r be so large that

$$c S_{j+1} (\log S_{j+1})^2 < S_{j+1}^{1+\epsilon}$$

in (3.6). Using that

$$\log N_{j+1} \leq (\log N_j)^{2+\epsilon} .$$

Thus

$$\log N_k \leq (\log N_1)^{(2+\epsilon)^k} ,$$

hence by taking logarithms twice,

$$K(N_k) \geq k \geq c_1 \log_3 N_k ,$$

which completes the proof of (1.3).

Denote by $L(n)$ the smallest integer for which $\log n_{L(n)} < 1$. We conjecture that

$$\frac{1}{n} \sum_{m=1}^{n} K(m)$$

increases about like $L(n)$, but we have not been able to prove this.

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2. S. Ramanujan, "Highly Composite Numbers," Proc. London Math. Soc., 2(194), 1915, 347-409, see p. 409.

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CORRECTION

On p. 113 of Volume 7, No. 2, April, 1969, please make the following changes:

Change the author's name to read George E. Andrews. Also, change the name "Einstein," fourth line from the bottom of p. 113, to "Eisenstein."

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Π IN THE FORM OF A CONTINUED FRACTION WITH INFINITE TERMS

N. A. DRAIM
Ventura, California

$$\frac{\Pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\cdots \qquad \text{(Wallis)}$$

$$\therefore\ \Pi = \frac{2}{1}\cdot\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots \quad ,$$

for which the successive products, as $n \to \infty$, are:

$$\frac{4}{1}, \frac{8}{3}, \frac{32}{9}, \frac{128}{45}, \frac{768}{225}, \cdots \frac{p_n}{q_n}\cdots \quad ,$$

for which the ordinal numbers are $1, 2, 3, \cdots, n, \cdots$.

These products are identical with the convergents, after the first two, of the following continued fraction with infinite terms, for which the ordinal numbers are $-1, 0, 1, 2, 3, \cdots, n$.

$$\Pi = \cfrac{4}{1 + \cfrac{1}{1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{2\cdot 3}{1 + \cfrac{3\cdot 4}{1 + \cdots \cfrac{(n-1)n}{1\cdots}}}}}}} \ .$$

These C. F. convergents are, when spun out in ordinal succession:

$$4, \frac{4}{2}, \frac{4}{1}, \frac{8}{3}, \frac{32}{9}, \frac{128}{45}, \frac{768}{225}, \cdots \frac{p_n}{q_n}\cdots \quad ,$$

The series corresponding to the infinite C. F. is:

$$\Pi = 4 - 2 + 2 - \frac{4}{3} + \frac{8}{9} - \frac{32}{45} + \frac{128}{225} - \frac{768}{1575} + \cdots + (-1)^{n-1}\frac{p_{n-1}}{q_n}\cdots$$

the p_{n-1} and q_n being the products as found above in the successive products for Π.

(The author acknowledges with appreciation the help of Lavar Rigby, Instructor for Computers at Ventura College, who checked the convergence trend of the subject continued fraction for Π on an IBM 1620.)

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(Continued from p. 336)

ELEMENTARY PROBLEMS AND SOLUTIONS

B-153 Proposed by Klaus-Gunther Recke, Gottingen, Germany.

Prove that

$$F_1F_3 + F_2F_6 + F_3F_8 + \cdots + F_nF_{3n} = F_nF_{n+1}F_{2n+1}.$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

Since $F_1F_3 = F_1F_2F_3$, we need only show that when we add one to n, the increase on the left side of the equation is the same as that on the right. The increase on the left side is F_nF_{3n}; and, using the solution to B-152 with $m = 2n$,

$$\begin{aligned} F_nF_{3n} &= F_nF_{2n+n} \\ &= F_n(F_{2n+1}F_{n+1} - F_{2n-1}F_{n-1}) \\ &= F_nF_{n+1}F_{2n+1} - F_{n-1}F_nF_{2n-1} \end{aligned}$$

which is just the increase on the right side of the equation.

Also solved by Clyde A. Bridger, Herta T. Freitag, Serge Hamelin (Canada), John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, C. C. Yalavigi (Mercara, India), David Zeitlin, and the Proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-158 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

If $f_n(x)$ be the Fibonacci polynomial as defined in H-127, show that

(a) for integral values of x, $f_n(x)$ and $f_{n+1}(x)$ are prime to each other

(b) $$\left\{1 + \sum_1^n (1/f_{2n-1}f_{2n+1})\right\}\left\{1 - x^2 \sum_1^n (1/f_{2n}f_{2n+2})\right\} = 1 .$$

H-159 Proposed by Clyde Bridger, Springfield College, Springfield, Illinois.

Let

$$D_k = \frac{c^k - d^k}{c - d}$$

and

$$E_k = c^k + d^k ,$$

where c and d are the roots of $z^2 = az + b$. Consider the four numbers e, f, x, y, where $e = c^k$ and $f = d^k$ are the roots of $z^2 - z\,E_k + (-b)^k = 0$ and y is the harmonic conjugate of x with respect to e and f. Find y when

$$x = \frac{D_{nk+k}}{D_{nk}} . \qquad (k \neq 0) .$$

H-160 Proposed by D. and E. Lehmer, University of California, Berkeley, California.

Find the roots and the discriminant of

$$x^3 - (-1)^k 3x - L_{3k} = 0 .$$

H-161 Proposed by David Klarner, University of Alberta, Edmonton, Alberta, Canada.

Let

$$b(n) = \sum_{a_1+a_2+\cdots+a_i=n} \binom{a_i + a_2}{a_2} \binom{a_2 + a_3}{a_3} \cdots \binom{a_{i-1} + a_i}{a_i} ,$$

where the sum is extended over all compositions of n and the contribution to the sum is 1 when there is only one part in the composition. Find an asymptotic estimate for $b(n)$.

SOLUTIONS

MULTI-VARIABLE SERIES

H-126 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Sum the series

$$\sum_{m,n=0}^{\infty} F_m F_n F_{m+n} x^m y^n , \qquad \sum_{m,n=0}^{\infty} F_m F_n L_{m+n} x^m y^n .$$

Sum the series

$$\sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} F_m F_n x^m y^n , \qquad \sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} L_m L_n x^m y^n .$$

Sum the series

$$S = \sum_{m,n,p=0}^{\infty} F_{n+p} F_{p+m} F_{m+n} x^m y^n z^p .$$

Solution by the Proposer.

Put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = \tfrac{1}{2}(1 - \sqrt{5}),$$

so that

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} .$$

Then

$$S_1 = \sum_{m,n=0}^{\infty} F_m F_n F_{m+n} x^m y^n$$

$$= \frac{1}{\alpha - \beta}\left\{\sum_{m,n=0}^{\infty} F_m F_n \alpha^{m+n} x^m y^n - \sum_{m,n=0}^{\infty} F_m F_n \beta^{m+n} x^m y^n\right\}$$

$$= \frac{1}{\alpha - \beta}\left\{\frac{\alpha x}{1 - \alpha x - \alpha^2 x^2} \frac{\alpha y}{1 - \alpha y - \alpha^2 y^2} - \frac{\beta x}{1 - \beta x - \beta^2 x^2} \frac{\beta y}{1 - \beta y - \beta^2 y^2}\right\} .$$

Since $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$, it follows that

$$1 - \alpha x - \alpha^2 x^2 = (1 - \alpha^2 x)(1 + x) ,$$
$$1 - \beta x - \beta^2 x^2 = (1 - \beta^2 x)(1 + x) ,$$

so that

$$S_1 = \frac{1}{(\alpha - \beta)(1 + x)(1 + y)} \left\{ \frac{\alpha x}{1 - \alpha^2 x} \frac{\alpha y}{1 - \alpha^2 y} - \frac{\beta x}{1 - \beta^2 x} \frac{\beta y}{1 - \beta^2 y} \right\}$$

$$= \frac{xy}{(\alpha - \beta)(1+x)(1+y)} \frac{\alpha^2[1 - \beta^2(x + y) + \beta^4 xy] - \beta^2[1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 - 3x + x^2)(1 - 3y + y^2)}.$$

This reduces to

$$S_1 = \frac{xy - x^2y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

$$S_2 = \sum_{m,n=0}^{\infty} F_m F_n L_{m+n} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} F_m F_n \alpha^{m+n} x^m y^n + \sum_{m,n=0}^{\infty} F_m F_n \beta^{m+n} x^m y^n$$

$$= \frac{\alpha x}{1 - \alpha x - \alpha^2 x^2} \frac{\alpha y}{1 - \alpha y - \alpha^2 y^2} + \frac{\beta x}{1 - \beta x - \beta^2 x^2} \frac{\beta y}{1 - \beta y - \beta^2 y^2}$$

$$= \frac{1}{(1 + x)(1 + y)} \left\{ \frac{\alpha x}{1 - \alpha^2 x} \frac{\alpha y}{1 - \alpha^2 y} + \frac{\beta x}{1 - \beta^2 x} \frac{\beta y}{1 - \beta^2 y} \right\}$$

$$= \frac{xy}{(1 + x)(1 + y)} \frac{\alpha^2 [1 - \beta^2(x + y) + \beta^4 xy] + \beta^2 [1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 - 3x + x^2)(1 - 3y + y^2)}$$

$$= \frac{xy}{(1 + x)(1 + y)} \frac{L_2 - 2(x + y) + L_2 xy}{(1 - 3x + x^2)(1 - 3y + y^2)}$$

and therefore

$$S_2 = \frac{3xy - 2xy(x + y) + 3x^2y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Clearly

$$\sum_{\substack{m,n=0\\ m+n \text{ even}}}^{\infty} F_mF_nx^my^n = \frac{1}{2}\left\{\sum_{m,n=0}^{\infty} F_mF_nx^my^n + \sum_{m,n=0}^{\infty} (-1)^{m+n}F_mF_nx^my^n\right\}$$

$$= \frac{1}{2}\left\{\frac{xy}{(1-x-x^2)(1-y-y^2)} + \frac{xy}{(1+x-x^2)(1+y-y^2)}\right\}$$

$$= \frac{xy}{2}\,\frac{(1+x-x^2)(1+y-y^2)+(1-x-x^2)(1-y-y^2)}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)}$$

$$= \frac{xy-(x^2+y^2)xy+x^2y^2+x^3y^3}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)}.$$

Similarly,

$$\sum_{\substack{m,n=0\\ m+n \text{ even}}}^{\infty} L_mL_nx^my^n = \frac{1}{2}\left\{\sum_{m,n=0}^{\infty} L_mL_nx^my^n + \sum_{m,n=0}^{\infty} (-1)^{m+n}L_mL_nx^my^n\right\}$$

$$= \frac{1}{2}\,\frac{(2-x)(2-y)}{(1-x-x^2)(1-y-y^2)} + \frac{(2+x)(2+y)}{(1+x-x^2)(1+y-y^2)}$$

$$= \frac{1}{2}\,\frac{(2-x)(2-y)(1+x-x^2)(1+y-y^2)+(2+x)(2+y)(1-x-x^2)(1-y-y^2)}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)}$$

$$= \frac{4+xy-6(x^2+y^2)+5x^2y^2+x^3y^3}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)}.$$

Now for the last series, we have

$$S = \frac{1}{(\alpha-\beta)^3}\sum_{m,n,p=o}^{\infty}(\alpha^{n+p}-\beta^{n+p})(\alpha^{p+m}-\beta^{p+m})(\alpha^{m+n}-\beta^{m+n})x^my^nz^p$$

$$= \frac{1}{(\alpha-\beta)^3}\sum_{m,n,p=o}^{\infty}(\alpha^{2m+2n+2p}-\beta^{2m+2n+2p})x^my^nz^p$$

$$-\frac{1}{(\alpha-\beta)^3}\sum_{m,n,p=0}^{\infty}\sum_{m,n,p}(\alpha^{2m+n+p}\beta^{n+p}-\alpha^{n+p}\beta^{2m+n+p})x^my^nz^p$$

$$= \frac{1}{(\alpha-\beta)^3}\left\{\frac{1}{(1-\alpha^2x)(1-\alpha^2y)(1-\alpha^2z)} - \frac{1}{(1-\beta^2x)(1-\beta^2y)(1-\beta^2z)}\right\}$$

$$-\frac{1}{(\alpha-\beta)^3}\left\{\sum_{x,y,z}\frac{1}{(1-\alpha^2x)(1+y)(1+z)} - \sum_{x,y,z}\frac{1}{(1-\beta^2x)(1+y)(1+z)}\right\}$$

$$= \frac{1}{5}\,\frac{\Sigma x - 3\Sigma xy + 8xyz}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)} - \frac{1}{5}\sum_{x,y,z}\frac{x}{(1-3x+x^2)(1+y)(1+z)}.$$

The sums

$$\sum_{m,n,p}, \sum_{x,y,z}$$

indicate summation over all permutations of the letters indicated.

We find, after some computation, that

$$S = \frac{b - 5c - 2ac + 2bc - b^2 + c^2}{(1+x)(1+y)(1+z)(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)},$$

where

$$a = x + y + z, \quad b = yz + zx + xy, \quad c = xyz .$$

For $z = 0$ the above result reduces to

$$\sum_{m,n=0}^{\infty} F_{m+n} F_m F_n x^m y^n = \frac{xy - x^2y^2}{(1+x)(1+y)(1-3x+x^2)(1-3y+y^2)}$$

in agreement with the first result, where $0^0 = 1$, by convention.

Also solved by A. Shannon, Boroko, T. P. N. G.

Note: Due to an editorial error, problem H-120 was also listed as H-127.

MOD SQUAD

H-128 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively. Show that

$$F_n \equiv 2^{2n+3} - 2^{3n+3} \pmod{11},$$
$$L_n \equiv 2^{2n} + 2^{3n} \pmod{11} .$$

Generalize.

Solution by the Proposer.

Let p be any prime $\equiv \pm 1 \pmod{10}$. Then it is known (Dmitri Thoro, "An Application of Unimodular Transformations," Fibonacci Quarterly, 2(1964), 291-295) that 5 is a quadratic residue (mod p), so let x_0 be a solution of $x^2 \equiv 5 \pmod{p}$. Since x_0 and $p - x_0$ are both solutions of this, one of which is odd, we may assume x_0 is odd, say $x_0 = 2a - 1$. Then

$$2a - 1 \equiv \sqrt{5}, \qquad a \equiv (1 + \sqrt{5})/2 ,$$

so that $a^2 - a - 1 \equiv 0 \pmod{p}$. Hence $x - a$ divides $x^2 - x - 1 \pmod{p}$, showing that

$$x^2 - x - 1 \equiv (x - a)(x - b) \pmod{p}$$

for some integer b. It follows that $u_n = c_1a^n + c_2b^n$ obeys

$$u_{n+2} \equiv u_{n+1} + u_n \pmod{p} ,$$

where c_1 and c_2 are arbitrary constants.

We first evaluate c_1 and c_2 when $u_n \equiv F_n \pmod{p}$. When $n = 0,1$, we find

$$c_1 + c_2 \equiv 0 \pmod{p}$$

$$c_1a + c_2b \equiv 1 \pmod{p},$$

which has a solution if and only if $a \not\equiv b \pmod{p}$, which is clearly the case here. We see that then $c_1 = 1/(a - b)$, $c_2 = -1/(a - b)$, so

$$F_n \equiv \frac{a^n - b^n}{a - b} \pmod{p}.$$

Similarly,

$$L_n \equiv a^n + b^n \pmod{p}.$$

These may be considered the Binet forms for the Fibonacci and Lucas numbers in the integers modulo p.

The above problem follows from this by noting

$$x^2 - x - 1 \equiv (x - 4)(x - 8) \pmod{11},$$

and that $1/(4 - 8) \equiv 8 \pmod{11}$.

Also solved by L. Carlitz, Duke University, and A. Shannon, Boroko, T. P. N. G.

RADICAL TSCHEBYSHEV

H-129 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define the Fibonacci polynomials by $f_1(x) = 1$, $f_2(x) = x$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, $n > 0$. Solve the equation

$$(x^2 + 4)f_n^2(x) = 4k(-1)^{n-1}$$

in terms of radicals, where k is a constant.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

It is stated in The American Mathematical Monthly, 1968, p. 295, that

$$i^{n-1} f_n(x) = U_{n-1}\left(\frac{1}{2}\, ix\right),$$

where

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}.$$

Thus

$$\begin{aligned}(-1)^{n-1}f_n^2(x) &= U_{n-1}^2(\tfrac{1}{2}\, ix) = \sin^2(n\cos^{-1}\tfrac{1}{2}\, ix)/\sin^2(\cos^{-1}\tfrac{1}{2}\, ix)\\ &= 4\sin^2(n\cos^{-1}\tfrac{1}{2}\, ix)/(4 + x^2)\end{aligned}$$

Thus

$$(x^2 + 4)f_n^2(x) = (-1)^{n-1}4 \sin^2(n \cos^{-1} \tfrac{1}{2} ix) .$$

By comparison,

$$k = \sin^2 (n \cos^{-1} \tfrac{1}{2} ix)$$

whence

$$x = -2i \cos\left(\frac{1}{n} \sin^{-1}\sqrt{k}\right) .$$

<u>Note</u>: The proposer obtained the solution,

$$x = i \left(1 - 2k + 2\sqrt{k^2 - k}\right)^{\frac{1}{2}n} + \left(1 - 2k - 2\sqrt{k^2 - k}\right)^{\frac{1}{2}n} ,$$

where any $(2n)^{th}$ root may be taken in the first radical and the $(2n)^{th}$ root of the second radical is chosen so that their product is unity.

Also solved by A. Shannon, Boroko, T. P. N. G.

GAUCHE PASCAL

H-131 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Consider the left-adjusted Pascal triangle. Denote the left-most column of ones as the zeroth column. If we take sums along the rising diagonals, we get Fibonacci numbers. Multiply each column by its column number and again take sums, C_n, along these same diagonals. Show $C_1 = 0$ and

$$C_{n+1} = \sum_{j=0}^{n} F_{n-j} F_j$$

Solution by L. Carlitz, Duke University.

Clearly,

$$C_n = \sum_{2j \le n} j \binom{n-j}{j}, \quad C_0 = C_1 = 0 .$$

Hence

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} x^n \sum_{2j \le n} j \binom{n-j}{j}$$

$$= \sum_{j=0}^{\infty} j\, x^{2j} \sum_{n=0}^{\infty} \binom{n+j}{j} x^n$$

$$= \sum_{j=0}^{\infty} j\, x^{2j} (1-x)^{-j-1}$$

$$= \frac{x^2}{(1-x)^2} \sum_{j=0}^{\infty} (j+1) x^{2j} (1-x)^{-j}$$

$$= \frac{x^2}{(1-x)^2} \left(1 - \frac{x^2}{1-x}\right)^{-2}$$

$$= x^2 (1 - x - x^2)^{-2} \quad .$$

Since

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad ,$$

it follows at once that

$$C_n = \sum_{j=0}^{n} F_j F_{n-j} \quad .$$

Also solved by D. Zeitlin, Minneapolis, Minnesota; A. Shannon, Boroko, T. P. N. G.; and E. Frankel.

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ASSOCIATED ADDITIVE DECIMAL DIGITAL BRACELETS*

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A bracelet is one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Distances between terms may be measured in degrees or in spaces. A bracelet may be regenerated by starting at any arbitrary point to apply the generating law. A bracelet may be cut at any arbitrary point for straight line representation without loss of any properties.

A digital bracelet may be constructed by starting with a pair of digits, affixing the units' digit of their sum, again affixing the units' digit of the sum of the last two digits, and continuing the process. (Some bracelets generated from a sequence of four digits have been discussed previously [1], [2]). This is equivalent to using the recurrence formula $u_{n+2} = u_n + u_{n+1}$ and, in the decimal system, reducing each sum modulo 10. When all operations of addition and multiplication are reduced modulo 10, the computations are in a modular arithmetic dealing with individual digits. In it 1, 7, 9, 3 and 2, 6, 8, 4 and their reverses are cyclic geometric progressions (G. P.) which themselves are bracelets.

The 60-digit [3] Fibonacci bracelet (F), so called since it is one period of the units' digits of the Fibonacci series, is generated by $u_1 = 0$, $u_2 = 1$. Five associated bracelets [4] result from other generating pairs, as follows:

A(1): 0 0 ' 0 0

B(3): 0 5 5 ' 0 5

C(4): 2 6 8 4 ' 2 6

D(12): 1 3 4
7 1 8
9 7 6
3 9 2 ' 13

E(20): 0 2 2 4 6
0 6 6 2 8
0 8 8 6 4
0 4 4 8 2 ' 0 2

F(60): 0 1 1 2 3 5 8 3 1 4 5 9 4 3 7
0 7 7 4 1 5 6 1 7 8 5 3 8 1 9
0 9 9 8 7 5 2 7 9 6 5 1 6 7 3
0 3 3 6 9 5 4 9 3 2 5 7 2 9 1 ' 0 1

---

*Presented at the March 11, 1967 meeting of the Mathematical Association of America, Southern California Section.

The number in parentheses following the identifying letter is the number of ordered digit pairs in the bracelet. Except in A, this is the same as the number of digits in the bracelet — the length of the bracelet. Together, the six associated bracelets contain all 10^2 ordered digit pairs.

PROPERTIES OF ASSOCIATED BRACELETS

1. The number of digits in each bracelet is a factor of 60. This is in accord with the observation by Wall [5] that the length of any digital period resulting from $u_0 \neq 0$, $u_1 \neq 1$ divides the length of the Fibonacci digital period.

2. The sum of diametrically opposite digits is zero. Hence, the sum of the digits at the vertices of any inscribed polygon with an even number of sides is zero. The sum of the digits in every bracelet is zero.

3. The digits of C form a cyclic G. P. As the bracelets are arranged, the same G. P. appears in four of the digit columns of E and in the sums of its four rows. The reverse G. P. appears in 1 column of D, 4 columns of F, the sums of the four rows of D, the sums of the four rows of F, the sums of the rows of the last pentad column of F, and when the rows of F are broken up into triads in the sums of the rows of four of the five triad columns.

The cyclic G. P., 1, 7, 9, 3, appears in 2 columns of D, 8 columns of F, and the sums of the rows of the first two pentad columns of F.

The trivial G. P. of 0's which matches A, appears in the remaining digit column of E, and in the sums of the rows of the second column of triads in F. The remaining digit columns leading the pentad columns in F are in trivial G. P.'s of 0's, 5's, and 5's, which match B horizontally.

4. Bracelets D and F may be written in the terms:

```
D: 1 3 4 7
   1 8 9 7
   6 3 9 2 ' 1 3


F: 0 1 1 2 3    5 8 3 1 4    5 9 4 3 7    0 7 7 4 1
   5 6 1 7 8    5 3 8 1 9    0 9 9 8 7    5 2 7 9 6
   5 1 6 7 3    0 3 3 6 9    5 4 9 3 2    5 7 2 9 1 ' 0 1
```

The sums of successive rows of D and of F are 5, 5, 0, a cyclic permutation (c.p.) of 8. The digit columns of D and F and the sums of the rows of the pentad columns in F are c.p.'s of the sets obtained by separately adding the digits of C to the digits of B. The digits in each of these sets are congruent modulo 5.

5. Bracelets E and F may be written in the forms:

E:	0	2	2	4				F:	0	1	1	2		3	5	8	3		1	4	5	9
	6	0	6	6					4	3	7	0		7	7	4	1		5	6	1	7
	2	8	0	8					8	5	3	8		1	9	0	9		9	8	7	5
	8	6	4	0					2	7	9	6		5	1	6	7		3	0	3	3
	4	4	8	2	'	0	2		6	9	5	4		9	3	2	5		7	2	9	1 ' 0 1

Each row of E sums to 8 and each row of F sums to 2. Each quartet in the first column of F sums to 4, the other quartets sum to 9.

Each digit column consists of distinct even digits or of distinct odd digits. It follows that every even digit occurs with the same frequency in E. Also, in F the frequency of every even digit is the same, and the frequency of every odd digit is the same (and twice that of the even digits).

In E the digits in each column are in arithmetic progression with the successive common differences being 6, 8, 4, 2, a c.p. of C. In F the digits in each column are in arithmetic progression with the successive common differences being 4, 2, 6, 8, 4, 2, 6, 8, 4, 2, 6, 8, again involving a c.p. of C.

ASSOCIATION BY ADDITION

Each of the bracelets A, B, C, D, E, F is produced by the same recurrence formula. Consequently, proceeding clockwise, if the digits of the two bracelets are consecutively matched and added, the resulting sequence must follow the same formative law. But the six bracelets exhaust the 100-pair field, so the sequence created by the addition must duplicate all or part of one of the six bracelets.

The digits of each bracelet, as tabulated, are numbered consecutively from the left. In the additions a <u>selected</u> bracelet will be considered to be <u>operated on</u> by itself or by a shorter bracelet repeated. If the length of the

selected bracelet is not a multiple of the length of the operator bracelet, then each is repeated to a total length equal to the l. c. m. of the two lengths.

The digits of the operator bracelet are successively matched with the initial digit of the selected bracelet in a series of additions. A particular addition is identified by the sequence number of the matching digit (m. d.) of the operator. The addition may produce a clockwise rotation (r.) of the selected bracelet or a change to another bracelet (b. c.).

Thus, when the third digit of D is matched with the initial digit of another D, the addition produces a b. c. into a series of 4 B's.

```
1 3 4 7    1 8 9 7    6 3 9 2    1 3 4 7    1 8 9 7    6 3 9 2
4 7 1 8    9 7 6 3    9 2 1 3    7 8 4 2    6 8 4 2    6 8 4 2
5'0 5 5 '  0 5 5'0    5 5 '0 0   7 1 8 9    7 6 3 9    2'1 3 4
```

and when C operates on D with the second digit of C matching the initial digit of D, the result of the addition is an r. of D through nine spaces or 270° from the position of the selected D.

Any bracelet operated on by A is not changed, which is equivalent to a rotation through 360°.

B operated on by B produces two rotations of B and one conversion to A's. Thus

B m. d.	r. or b. c.	B r.	Operator
1	A's	120°	B
2	120°	240°	B
3	240°	360°	A

C operated on by A, B, and C gives the following results:

C m. d.	r. or b. c.	C r.	Operator
1	90°	90°	C
2	180°	180°	C
3	A's	270°	C
4	270°	360°	A

When the 3-digit B operates on the 4-digit C, four B's and 3 C's are involved. The additions produce three D bracelets, with their initial digits 120° apart.

D operated on by A, B, C, and D produces:

B m.d.	r. or b.c.	D m.d.	r. or b.c.	D r.	Operator
1	240°	1	3 C's	30°	D
2	3 C's	2	300°	60°	D
3	120°	3	4 B's	90°	C
		4	3 C's	120°	B
		5	30°	150°	D
		6	60°	180°	C
		7	A's	210°	D
C m.d.	r. or b.c.	8	210°	240°	B
1	90°	9	150°	270°	C
2	270°	10	3 C's	300°	D
3	180°	11	4 B's	330°	D
4	4 B's	12	330°	360°	A

With reference to a fixed D, the initial digits of the C's generated by the D and B operators occur at 30° intervals, as do the initial digits of the B's produced by the D and C operators.

E operated on by A, B, C, D, and E gives:

C m.d.	r. or b.c.	E m.d.	r. or b.c.	E r.	Operator
1	216°	1	90°	18°	E
2	288°	2	324°	36°	E
3	144°	3	5 C's	54°	E
4	72°	4	54°	72°	C
		5	234°	90°	E
		6	180°	108°	E
		7	5 C's	126°	E
		8	126°	144°	C
		9	18°	162°	E

(continued on next p.)

E m.d.	r. or b.c.	E r.	Operator
10	36°	180°	E
11	A's	198°	E
12	198°	216°	C
13	162°	234°	E
14	252°	252°	E
15	5 C's	270°	E
16	270°	288°	C
17	306°	306°	E
18	108°	324°	E
19	5 C's	342°	E
20	342°	360°	A

With reference to a fixed E, the initial digits of the C's produced by the E operator are 18° apart. When D repeated five times operates on E repeated three times, F's in twelve different positions are produced. When B repeated twenty times operates on E repeated three times, F's in three different positions are produced. With reference to a fixed E, the F's produced by operator B are 120° apart, and those produced by operators B and D considered together are 24° apart.

F operated on by A, B, C, D, E, and F produces:

B m.d.	r. or b.c.
1	3 E's
2	240°
3	120°

C m.d.	r. or b.c.
1	144°
2	72°
3	216°
4	288°

D m.d.	r. or b.c.
1	312°
2	336°
3	3 E's
4	24°
5	192°
6	3 E's
7	168°
8	264°
9	3 E's
10	96°
11	48°
12	3 E's

E m.d.	r. or b.c.
1	90°
2	342°
3	36°
4	5 D's
5	234°
6	270°
7	108°
8	5 D's
9	18°
10	198°
11	180°
12	5 D's
13	162°

(continued on next p.)

E m. d.	r. or b. c.
14	126°
15	252°
16	20 B's
17	306°
18	54°
19	324°
20	5 D's

F operating on F produces thirty rotations of F, one A's, two 20 B's, four 15 C's, eight 5 D's, and three 15 E's. The rotations of F produced by all the operators neatly drop in at 6° intervals.

Beginning at 6°, the successive operators were:

F F E D F E F D E F F C F F E D F E F B E F F C F F E D F E
F D E F F C F F E B F E F D E F F C F F E D F E F D E F F A .

With reference to a fixed F, the initial digits of the B's produced by the various operators are 6° apart, as are the initial digits of the C's, the D's, and the E's produced by the various operators.

In general any one of the six associated bracelets with length p may be rotated through any desired multiple of 360°/p by operating on it with the proper associated bracelet of the same or shorter length and using the appropriate m. d.

Each of the operators produces rotations symmetrically distributed about 180°, so that when all the operator letters of the successive rotations of a selected bracelet are listed, a palindromic sequence is formed.

When the various operators produce bracelets other than the one operated on, the initial digits of the derived bracelets of the same type are equally spaced, when referred to the initial digit of the selected bracelet.

When the length of a shorter bracelet does not divide that of a longer one (as in the cases of B and C, B and E, D and E), then the operation of the shorter bracelet on the longer one always produces a bracelet of length equal to the l. c. m. of the lengths of the two bracelets involved. With reference to a fixed position of the longer bracelet, initial digits of the bracelets produced by the addition are equally distributed around a circle.

ASSOCIATION BY MULTIPLICATION

Since multiplication is repeated addition, it follows that multiplication of any one of these digital bracelets by a positive digit will either rotate the bracelet or convert it into another bracelet. The rotations and conversions of the bracelets in the first column upon multiplication by the digits at the heads of the columns are shown in the body of the following table.

	8	4	2	6	3	9	7	1	5
B	A	A	A	A	B	B	B	B	B
C	90°	180°	270°	360°	90°	180°	270°	360°	A
D	C	C	C	C	270°	180°	90°	0°	B
E	90°	180°	270°	360°	90°	180°	270°	360°	A
F	E	E	E	E	270°	180°	90°	0°	B

The C's produced by multiplying D by 8, 4, 2, 6 in order go into each other by counterclockwise 90° rotations. The E's produced from F behave similarly.

To indicate that multiplication by 9 rotates a bracelet through 180° is equivalent to saying that the diametrically opposite digits sum to zero.

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LINEAR RECURSION RELATIONS — LESSON FIVE
RECURSION RELATIONS OF HIGHER ORDER

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The considerations applied to linear recursion relations of the second order form a pattern for dealing with relations of higher order. Given a linear recursion relation of the kth order:

$$T_{n+1} = a_1 T_n + a_2 T_{n-1} + \cdots + a_k T_{n-k+1} , \tag{1}$$

where the quantities a_i and T_i are real, there would be an auxiliary equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} \cdots - a_k = 0 , \tag{2}$$

for which there could be real and distinct roots, multiple real roots or complex roots conjugate in pairs. The major difficulty that arises in a relation of this type is the problem of determining the roots which ordinarily would be approximate in value.

As an example, consider one extension of the Fibonacci sequences, namely, adding the last three terms, or adding the last four terms, and so on. The recursion relations and corresponding auxiliary equations would be:

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad \text{and} \quad x^3 - x^2 - 1 = 0 \tag{3}$$

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} + T_{n-3} \quad \text{and} \quad x^4 - x^3 - x^2 - x - 1 = 0 . \tag{4}$$

If we look at the general type of this equation:

$$x^k = x^{k-1} + x^{k-2} + x^{k-3} + \cdots + x + 1, \tag{5}$$

it appears that since

$$2^k - 1 = 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2 + 1 , \tag{6}$$

there should be a root near 2. The following table gives an approximation to this root for various values of k.

k	Approximation to Root near 2
3	1.83928676
4	1.92756198
5	1.96594824
6	1.98358285
7	1.99196420
8	1.99603118
9	1.99802948

Approximations, such as these, to real or complex roots can be determined, but expressing T_n in terms of them does not seem very satisfying. Nevertheless, as will be seen in a subsequent lesson, such evaluations of roots of the auxiliary equation provide interesting information regarding the generated sequence.

MULTIPLE ROOTS

The case of multiple roots calls for additional consideration. If a polynomial equation

$$a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k = 0 \tag{7}$$

has a root of multiplicity s, then (7) can be written:

$$(x - r)^s F(x) = 0 , \tag{8}$$

where $F(x)$ is a polynomial of degree $k - s$. Clearly, this equation and the equations formed by setting the first $s - 1$ derivatives equal to zero are all satisfied by r. This provides a clue for dealing with roots of any multiplicity when found in the auxiliary equation of a recursion relation. For concreteness, let us consider a root r of multiplicity 3.

Let the equation having this multiple root be:

$$x^3 - ax^2 - bx - c = 0 . \tag{9}$$

Multiply both sides of the equation by x^n to obtain:

$$x^{n+3} - ax^{n+2} - bx^{n+1} - cx^n = 0 . \tag{10}$$

Take the derivative and set the resulting polynomial equal to zero.

$$(n + 3)x^{n+2} - a(n + 2)x^{n+1} - b(n + 1)x^n - cnx^{n-1} = 0 . \tag{11}$$

Repeat this operation on (11).

$$(n + 3)(n + 2)x^{n+1} - a(n + 2)(n + 1)x^n - b(n + 1)nx^{n-1} - cn(n - 1)x^{n-2} = 0. \tag{12}$$

The multiple root r must satisfy the relations (10), (11), and (12) so that on replacing x by r and multiplying (11) by r and (12) by r^2 we have the following three recursion relations for r.

$$r^{n+3} = ar^{n+2} + br^{n+1} + cr^n , \tag{13}$$

$$(n + 3)r^{n+3} = a(n + 2)r^{n+2} + b(n + 1)r^{n+1} + cnr^n , \tag{14}$$

$$(n + 3)(n + 2)r^{n+3} = a(n + 2)(n + 1)r^{n+2} + b(n + 1)nr^{n+1} + cn(n - 1)r^n . \tag{15}$$

On the basis of these recursion relations the indicated expression for T_n is:

$$T_n = An(n - 1)r^n + Bnr^n + Cr^n . \tag{16}$$

We show first that this relation continues to hold for succeeding values of n if it is true for three consecutive values. For if

$$T_{n+1} = A(n + 1)nr^{n+1} + B(n + 1)r^{n+1} + Cr^{n+1} , \tag{17}$$

and

$$T_{n+2} = A(n+2)(n+1)r^{n+2} + B(n+2)r^{n+2} + C\,r^{n+2}\,, \tag{18}$$

then

$$T_{n+3} = a\,T_{n+2} + b\,T_{n+1} + c\,T_n$$

is equal to:

$$T_{n+3} = A(n+3)(n+2)r^{n+3} + B(n+3)r^{n+3} + C\,r^{n+3} \tag{19}$$

on the basis of relations (13), (14), and (15).

Given three initial values T_1, T_2, and T_3, the relations they should satisfy on the basis of (16) would be:

$$\begin{aligned} T_1 &= \qquad\quad Br + Cr \\ T_2 &= 2Ar^2 + 2Br^2 + Cr^2 \\ T_3 &= 6Ar^3 + 3Br^3 + Cr^3 \quad . \end{aligned} \tag{20}$$

The determinant of the coefficients of the unknowns A, B, C has a value of $-2r^6$, so that if r is not zero, there are unique solutions for A, B, and C. Thus three initial values T_1, T_2 and T_3 can be expressed in the form given by (16). It follows that this form will continue to hold for all values of n.

It may be noted in passing that if the multiple root has a value of 1, T_n reduces to a polynomial in n.

Example. Express the terms of the recursion relation

$$T_{n+1} = 7T_n - 17T_{n-1} + 14T_{n-2} + 4T_{n-3} - 8T_{n-4}$$

in terms of the roots of the auxiliary equation:

$$x^5 - 7x^4 + 17x^3 - 14x^2 - 4x + 8 = 0.$$

By synthetic division three equal roots, 2, are found and the residual quadratic has the roots

$$\frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{1 - \sqrt{5}}{2} \quad .$$

Accordingly,

$$T_{n+1} = An(n - 1)x2^n + Bnx2^n + Cx2^n + Dr^n + Es^n$$

where

$$r = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad s = \frac{1 - \sqrt{5}}{2}$$

<u>Example.</u> For the recursion relation

$$T_{n+1} = 3T_n - 3T_{n-1} + T_{n-2} \ ,$$

with initial values $T_1 = 5$, $T_2 = 8$, $T_3 = 17$, express T_n in terms of the roots of the auxiliary equation.

This equation is

$$x^3 - 3x^2 + 3x - 1 = 0 \ ,$$

which has a triple root of 1. Thus

$$T_{n+1} = An(n - 1) + Bn + C \ ,$$

a polynomial in n. Then

$$\begin{aligned} 5 &= \phantom{2A + {}} B + C \\ 8 &= 2A + 2B + C \\ 17 &= 6A + 3B + C \quad , \end{aligned}$$

leading to the values $A = 3$, $B = -3$, $C = 8$, so that

$$T_{n+1} = 3n^2 - 6n + 8 .$$

PROBLEMS

1. Find the recursion relation satisfied by

$$T_n = 3n^2 - 5n + 4 + 2 \times 5^n .$$

2. Given the recursion relation

$$T_{n+1} = 6T_n - 11T_{n-1} + 6T_{n-2} ,$$

and initial values

$$T_1 = 8, \qquad T_2 = 15, \qquad T_3 = 22 .$$

Express the general term T_n in terms of the roots of the auxiliary equation.

3. S_n is the Fibonacci sequence $3, 7, 10, 17, 27, \cdots$, and R_n is the geometric progression $5, 15, 45, 135, \cdots$

$$T_n = R_n + S_n .$$

Find the recursion relation for T_n.

4. If $T_n = 3n + 2 + 2(-1)^n + F_n$, find the recursion relation for T_n.

5. If $T_1 = 13$, $T_2 = 15$, $T_3 = 22$ and $T_{n+1} = 4T_n - T_{n-1} - 2T_{n-2}$ express T_n in terms of the roots of the auxiliary equation of this recursion relation.

(Solutions are on p. 302.)

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NOTICE

The two fine elementary books, The Introduction to Fibonacci Discovery and Fibonacci and Lucas Numbers, are each available for $1.50 from Brother Alfred Brousseau, St. Mary's College, California 94575.

A NOTE ON FIBONACCI NUMBERS IN HIGH SCHOOL ALGEBRA

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With the number of topics in the course of study for algebra, the teacher isn't often looking for an additional unit of work, but rather for short excursions into related material to spark student interest. This note describes such a bypath.

When teaching the multiplication and division of polynomials, excellent interest-catchers are available. In multiplication, compute $(x + 1)^6$ from its definition and then display the coefficients of $(x + 1)^n$, $n = 0, 1, \cdots, 6$ in the Pascal Triangle arrangement. Students readily find patterns in this array of numbers. One interesting pattern is the sums of ascending 45° diagonals — the Fibonacci sequence. Students can be asked to look for additional patterns and to report back to the class after their findings. In dividing polynomials when discussing the arrangement of terms in the divisor and remainders, let students compute a few terms of

$$x \div (1 - x - x^2) = x + x^2 + 2x^3 + \cdots + F_n x^n + \cdots$$

and

$$1 \div (1 - 2x) = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots .$$

The teacher can easily check how the work is progressing while walking around the room because of the pattern of the quotients.

When doing computations with radicals, let students make a table with headings

$$n,\ \alpha^n,\ \beta^n,\ \alpha^n + \beta^n,\ (\alpha^n - \beta^n)/(\alpha - \beta)$$

for

$$\alpha = (1 + \sqrt{5})/2,\ \beta = (1 - \sqrt{5})/2,\ n = 1, 2, \cdots, 10 ,$$

and then describe as many patterns observed as possible. You will be amazed at the results. Since

$$(\alpha^n - \beta^n)/(\alpha - \beta) = F_n, \quad \alpha^n + \beta^n = L_n ,$$

and

$$\alpha^n = (L_n + F_n\sqrt{5})/2, \quad \beta^n = (L_n - F_n\sqrt{5})/2 ,$$

for the Fibonacci sequence defined by

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}$$

and the Lucas sequence defined by

$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2} ,$$

the teacher can readily check the results.

If you have found interesting uses of the Fibonacci numbers in high school teaching, you are invited to send a description to the Fibonacci Quarterly.

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Continued from page 300.

SOLUTIONS TO LINEAR RECURSION RELATIONS PROBLEMS

1. $$T_{n+1} = 8T_n - 18T_{n-1} + 16T_{n-2} - 5T_{n-3}$$

2. $$T_n = -5/2 + 7 \times 2^n - (7/6)\,3^n$$

3. $$T_{n+1} = 4T_n - 2T_{n-1} - 3T_{n-2}$$

4. $$T_{n+1} = 2T_n + T_{n-1} - 3T_{n-2} + T_{n-4}$$

(5) $$T_n = 12 + \frac{1}{\sqrt{13}}\left(\frac{3 + \sqrt{13}}{2}\right)^n - \frac{1}{\sqrt{13}}\left(\frac{3 - \sqrt{13}}{2}\right)^n$$

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MULTIPLE FIBONACCI SUMS

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I. INTRODUCTION

Let us define the Fibonacci numbers by means of the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \quad \text{with} \quad F_1 = 1,\ F_2 = 1 \tag{1}$$

To derive a formula for the sum of the first m Fibonacci numbers, write (1) as $F_n = F_{n+2} - F_{n+1}$, and let $n = 1,2,3,\cdots,m$, as shown below.

$$\begin{array}{lll} F_1 & = F_3 & - F_2 \\ F_2 & = F_4 & - F_3 \\ \vdots & \vdots & \vdots \\ F_{m-1} & = F_{m+1} & - F_m \end{array} .$$

Adding, we have

$$\sum_{k=1}^{m} F_k = F_{m+2} - 1 , \tag{2}$$

a well-known and useful result. In this paper, we shall be concerned with a generalization of (2) and its subsequent derivation, as well as another possible result.

II. DERIVATION OF FORMULA

Without stating in exact form the generalization which we shall consider, let us proceed inductively. Summing both sides of (2), we obtain

$$\sum_{m=1}^{p}\sum_{k=1}^{m} F_k = \sum_{m=1}^{p}(F_{m+2} - 1) = \sum_{m=1}^{p} F_{m+2} - \sum_{m=1}^{p} 1 = F_{p+4} - F_4 - p\ ,$$

as is easily seen. Summing this again,

$$\sum_{p=1}^{q}\sum_{m=1}^{p}\sum_{k=1}^{m} F_k = \sum_{p=1}^{q}(F_{p+4} - F_4 - p) = \sum_{p=1}^{q} F_{p+4} - \sum_{p=1}^{q} F_4 - \sum_{p=1}^{q} p.$$

To evaluate this, we use the well-known formula

$$(1 + 2 + \cdots + q) = \tfrac{1}{2}q(q + 1)\ ,$$

the sum of the first q natural numbers, to give

$$\sum_{p=1}^{q}\sum_{m=1}^{p}\sum_{k=1}^{m} F_k = F_{q+6} - F_6 - qF_4 - \frac{q(q + 1)}{2}\ .$$

If we sum this result again, we have

$$\sum_{q=1}^{r}\sum_{p=1}^{q}\sum_{m=1}^{p}\sum_{k=1}^{m} F_k = \sum_{q=1}^{r}\left(F_{q+6} - F_6 - qF_4 - \frac{q(q + 1)}{2}\right)$$

$$= \sum_{q=1}^{r} F_{q+6} - \sum_{q=1}^{r} F_6 - \sum_{q=1}^{r} qF_4 - \sum_{q=1}^{r} \frac{q(q + 1)}{2}$$

To evaluate

$$\frac{1}{2}\sum_{q=1}^{r} q(q + 1)\ ,$$

we use the fact that the sum of the first r triangular numbers is the rth tetrahedral number, giving

$$\sum_{q=1}^{r}\sum_{p=1}^{q}\sum_{m=1}^{p}\sum_{k=1}^{m} F_k = F_{r+8} - F_8 - rF_6 - \frac{r(r+1)}{2}F_4 - \frac{r(r+1)(r+2)}{3!}.$$

Let us now generalize this procedure to the case of n summations. Thus, we consider sums of the form

$$\sum_{a_{n-1}=0}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_1=0}^{a_2}\sum_{a_0=0}^{a_1} F_{a_0},$$

where the limits in the summation are members of the sequence of arbitrary constants,

$$\left\{a_j\right\}_{j=0}^{n}.$$

Examining the specific cases we have worked out, we see that the first term of our general result will be of the form F_{a_n+2n}, the second of the form F_{2n}. The third will be $a_n F_{2n-2}$, and the fourth

$$F_{2n-4}\, a_n (a_n + 1)/2 = F_{2n-4} \sum a_{n-1}.$$

In general, we need to evaluate sums of the form

$$\sum\cdots\sum\sum a_0.$$

To do this, we have the following result [1].

$$\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_0=1}^{a_1} = f_r^{a_n} = \binom{a_n + r - 1}{r},$$

where $f_r^{a_n}$ is the rth figurate number of order a_n, and r is the number of summations plus one. Thus, we conjecture that

$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} F_{a_0} = F_{a_n+2n} - \sum_{r=0}^{n-1} F_{2(n-r)} \binom{a_n+r-1}{r} \tag{3}$$

III. PROOF OF FORMULA

Let us now prove our conjecture (3) by induction on n. By the principle of mathematical induction, we first check for $n = 1$, which is obviously formula (2), and is thus true. We then assume (3) is true for $n = s$, and show that $n = s + 1$ is also true. Thus, we have to show

$$\sum_{a_s=1}^{a_{s+1}} \left(F_{a_s+2s} - \sum_{r=0}^{s-1} F_{2(s-r)} \binom{a_s+r-1}{r} \right) = F_{a_{s+1}+2(s+1)} - \sum_{r=0}^{s} F_{2(s+1-r)} \binom{a_{s+1}+r-1}{r} . \tag{4}$$

To find the first summation on the left-hand side, we can easily derive

$$\sum_{a_s=1}^{a_{s+1}} F_{a_s+2s} = F_{a_{s+1}+2(s+1)} - F_{2+2s} . \tag{5}$$

To find the second summation, consider

$$\sum_{a_s=1}^{a_{s+1}} \sum_{r=0}^{s-1} F_{2(s-r)} \binom{a_s+r-1}{r} = \sum_{a_s=1}^{a_{s+1}} F_{2s} \binom{a_s-1}{0} + \cdots + F_2 \binom{a_s+s-1-1}{s-1}$$

$$= F_{2s} \sum_{a_s=1}^{a_{s+1}} \binom{a_s-1}{0} + F_{2(s-1)} \sum_{a_s=1}^{a_{s+1}} \binom{a_s}{1} + \cdots + F_2 \sum_{a_s=1}^{a_{s+1}} \binom{a_s+s-1-1}{s-1} . \tag{6}$$

It can easily be established by induction that for $n \geq r$,

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} . \tag{7}$$

Thus, applying (7) to (6),

$$(8)\qquad \sum_{a_{s=1}}^{a_{s+1}} \sum_{r=0}^{s-1} F_{2(s-r)} \binom{a_s + r - 1}{r} = \sum_{r=1}^{s} F_{2(s-r+1)} \binom{a_{s+1} + r - 1}{r}$$

Substituting (5) and (8) into (4), we obtain

$$F_{a_{s+1}+2(s+1)} - F_{2+2s} - \sum_{r=1}^{s} F_{2(s-r+1)} \binom{a_{s+1}+r-1}{r} = F_{a_{s+1}+2(s+1)} - \sum_{r=0}^{s} F_{2(s-r+1)} \binom{a_{s+1}+r-1}{r}$$

which proves our proposed formula.

We remark that this general formula is true for all recurrence relations of the form

$$f_{n+2} = f_{n+1} + f_n, \quad f_1 = a, \quad f_2 = b ,$$

where a and b are arbitrary integers. Thus,

$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} f_{a_0} = f_{a_n+2n} - \sum_{r=0}^{n-1} f_{2(n-r)} \binom{a_n + r - 1}{r} .$$

In particular, this result is true for the Lucas numbers defined by

$$L_{n+2} = L_{n+1} + L_n, \; L_1 = 1, \; L_2 = 3 .$$

IV. OTHER RESULTS

We shall develop a formula similar to (3), but which is derived by a different method and gives rise to a new identity. To use this method, we need a result of Hoggatt [2], namely that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad \frac{f(x)}{(1-x)^m} = \sum_{n=0}^{\infty} \left(\sum \sum \cdots \sum a_j \right) x^n ,$$

where there are m summations in the coefficient of x^n. Thus, the multiple sums are the convolutions of the a_j's with the elements of the mth column of Pascal's left-adjusted triangle. Letting

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = x(1 - x - x^2)^{-1} ,$$

then

$$\frac{f(x)}{(1-x)^m} = \frac{x}{1-x-x^2}(1-x)^{-m} = \frac{x}{1-x-x^2}\sum_{j=0}^{\infty}(-1)^j\binom{-m}{j}x^j$$

$$= \frac{1}{1-x-x^2}\sum_{j=0}^{\infty}\binom{m+j-1}{j}x^{j+1} .$$

If we carry out the indicated long division, then

$$\frac{f(x)}{(1-x)^m} = \sum_{n=0}^{\infty}\left(\sum^{n}\cdots\sum F_j\right)x^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}F_{n-j}\binom{m+j-1}{j}\right)x^n .$$

Equating coefficients of x^n, and using the notation of (3), we get

$$\sum_{a_{n-1}=0}^{a_n}\sum_{a_{n-2}=0}^{a_{n-2}}\cdots\sum_{a_0=0}^{a_1}F_{a_0} = \sum_{j=0}^{a_n}F_{a_n-j}\binom{n+j-1}{j} . \tag{9}$$

By equating (3) and (9), we derive the following identity

$$\sum_{j=0}^{a_n}F_{a_n-j}\binom{n+j-1}{j} = F_{a_n+2n} - \sum_{j=0}^{n-1}F_{2(n-j)}\binom{a_n+j-1}{j} . \tag{10}$$

We now note that this method can be used to find a general formula for

$$\sum_{a_{n-1}=0}^{a_n} \sum_{a_{n-1}=0}^{a_{n-1}} \cdots \sum_{a_0=0}^{a_1} b_{a_0} ,$$

where $\left\{b_j\right\}_0^\infty$ is a sequence of real numbers. Since

$$f(x) = \sum_{n=0}^{\infty} b_n x^n ,$$

then

$$\frac{f(x)}{(1-x)^m} = \sum_{n=0}^{\infty} b_n x^n \cdot (1-x)^{-m} = \sum_{n=0}^{\infty} b_n x^n \cdot \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} b_{n-j} \binom{m+j-1}{j} \right) x^n ,$$

by definition of the Cauchy product of two infinite series. Thus,

$$\sum_{a_{n-1}=0}^{a_n} \sum_{a_{n-2}=0}^{a_{n-1}} \cdots \sum_{a_0=0}^{a_1} b_{a_0} = \sum_{j=0}^{a_n} b_{a_n-j} \binom{n+j-1}{j} .$$

This then gives a generalization of (10) for recurrence relations of the form

$$f_{n+2} = f_{n+1} + f_n, \quad f_1 = a, \quad f_2 = b ,$$

where a and b are arbitrary integers, namely

$$\sum_{j=0}^{a_n} f_{a_n-j} \binom{n+j-1}{j} = f_{a_n+2n} - \sum_{j=0}^{n-1} f_{2(n-j)} \binom{a_n+j-1}{j} . \tag{11}$$

The author wishes to thank Dr. Verner E. Hoggatt, Jr., for all of his helpful suggestions and criticisms.

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2. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," Fibonacci Quarterly, Vol. 6, No. 4, p. 228.

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A BRAIN TEASER RELATED TO FIBONACCI NUMBERS

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A Swedish manufacturer of scientific instruments, LKB-Produkter AB, of Bromma, Sweden, has formed a tradition of sending out New Year's greetings in the form of mathematical brain teasers. A recent LKB brain teaser concerned a numerical problem encountered by the commander of a space ship and some members of his crew, which is composed of men from the Earth as well as men from Mars, Neptune and other planets.

THE PROBLEM

One day when calculating the distance which the ship had made, the Martian navigator, Lu, working with the decadic computer in the control room, obtained as a result a number the first five digits of which were 10112 and which had the property that if its last digit were moved to the first position a multiple of the number was formed.

Lu's complanetarian, Ku, tried to reconstruct Lu's number by manual calculation and was able to find a number beginning 10112 and also showing the desired property in regard of multiplication. However, Ku's number had only a little less than one-third as many digits as had Lu's.

A crew member of Neptune by name Elkeybub, who was known as a genius in mental calculation, was then called in to settle the dispute that Lu and Ku had got into because of the discrepancy between the numbers they had found. Elkeybub started the cells of his gray matter and soon came forth with his result: a number having one more digit than had Lu's, but otherwise fulfilling the same requirements as did Lu's and Ku's numbers.

The questions posed were: (1) why did not Lu, Ku, and Elkeybub get the same number, and (2) what numbers did they get?

THE GENERAL SOLUTION

It is easily shown that a number, N, having the property of being transformed into a multiple of itself when its last digit is moved to the first position has the form

(1) $$N = \frac{c(B^n - 1)}{fB - 1} ,$$

where

c = the last digit of N,

B = the base of the number system,

n = the number of digits of N

f = the multiplicity factor ($= N'/N$, where N' is the number obtained by moving c to the first position).

By simple reasoning we find that the following relationships hold between the parameters:

(2) $$2 \leq f \leq c < (B - 1)$$

(the case $f = 1$, or $N' = N$, is disregarded as being trivial).

That there exists for every set of (B, c, f) a value of n such that N is an integer can be shown by means of Fermat's theorem

(3) $$x^{\phi(m)} \equiv 1 \bmod m,$$

where x and m are integers having no common factor, and $\phi(m)$ is the number of integers less than m and prime to it.

Now, if B and $(fB - 1)$ have no common factor, Eq. (1) will give an integer value of N for

(4) $$n = \phi(fB - 1) .$$

It is immediately seen that b and $(fB - 1)$ can have no common factor, and Eq. (4) holds true.

If $(fB - 1)$ is prime, we get

(5) $$n = fB - 2 .$$

If $(fB - 1)$ is composite, that is,

(6) $$fB - 1 = d^q e^r \cdots, \quad d \neq 1, \ e \neq 0, \cdots$$

we get

$$n = \phi(fB - 1) = d^{q-1}(d - 1) \cdot e^{r-1}(e - 1)\cdots . \tag{7}$$

It should be pointed out that the values of N obtained from Eq. (1) and either of Eqs. (5) or (7) will not necessarily be the smallest values possible, a factor of $\phi(fB - 1)$ sometimes being sufficient to produce an integral value of N.

This occurs, for $(fB - 1)$ prime, with those values of $(fB - 1)$ for which B is not a primitive root. For $B = 10$ (the decadic system) we have $(fB - 1) = 19, 29, 39, \cdots, 89$. Of these, 39, 49, and 69 are composite. Of the prime values, 19, 29 and 59 have 10 as primitive root:

$$\left.\begin{array}{l} 10^{18} \equiv 1 \bmod 19 \\ 10^{28} \equiv 1 \bmod 29 \\ 10^{58} \equiv 1 \bmod 59 \end{array}\right\}, \text{ or } 10^{p-1} \equiv 1 \bmod p \quad \begin{array}{l} n_{min} = 18 \\ n_{min} = 28 \\ n_{min} = 58 \end{array} .$$

For the remaining primes, 10 is not a primitive root, and we have

$$10^{13} \equiv 1 \bmod 79, \text{ or } 10^{(p-1)/6} \equiv 1 \bmod p, \quad n_{min} = 13 ,$$

and

$$10^{44} \equiv 1 \bmod 89, \text{ or } 10^{(p-1)/2} \equiv 1 \bmod p, \quad n_{min} = 44 .$$

In the case of the composite values of $(fB - 1)$, their prime factors will decide whether $\phi(fB - 1)$ or a factor thereof will be the smallest n that satisfies Eq. (1). Here we get

$fB - 1$		$\phi(fB - 1)$	n_{min}
$39 = 3 \cdot 13$	$10^1 \equiv 1 \bmod 3$ $10^6 \equiv 1 \bmod 13$	$2 \cdot 12 = 26$	6
$49 = 7^2$	$10^6 \equiv 1 \bmod 7$	$6 \cdot 7 = 42$	42
$69 = 3 \cdot 23$	$10^{22} \equiv 1 \bmod 23$	$2 \cdot 22 = 44$	22

Note that in the case $N(10,7,5)$, we have one of the very rare cases where c is a factor of $(fB - 1)$, that is, $(fB - 1)/c = 7$, and since $10^6 \equiv 1 \bmod 7$, $N(10,7,5)$ gives $n_{min} = 6$.

An interesting property of N is related to the following reasoning:

$$N_c = N(B,c,f) = c(B^n - 1)/(fB - 1),$$
$$N_{c+1} = N(B,c+1,f) = (c+1)(B^n - 1)/(fB - 1),$$
$$N_{c+1} - N_c = (B^n - 1)/(fB - 1) = N_c/c .$$

Also, if $N_1 = c(B^n - 1)/(fB - 1)$ is an integer, then

$$N_i = c(B^{in} - 1)/(fB - 1), \quad i = 1,2,3,\cdots,$$

are integers too, which means that any N gives rise to an infinite number of such numbers, formed by cyclic repetition of N.

METHODS OF CALCULATING $N(B,c,f)$

(1) By solving n from Eqs. (5) or (7) and inserting n, B, c, and f into Eq. (1).

(2) By dividing c by $(fB - 1)$ (neglect the decimal point!) until c appears as remainder, after which the quotient will repeat periodically, the period being equal to N.

(3) By a step-by-step multiplication

$$f \cdot N(B,c,f) = N' ,$$

bearing in mind that the digit in the second position of the multiplicand (N) shall be equal to the digit in the first position of the product (N').

(4) By a step-by-step division

$$N'/f = N(B,c,f) ,$$

which is the reciprocal to the multiplication method: the digit in the $(n-1)^{th}$ position of the dividend (N') shall be equal to the digit in the nth position of the quotient, etc.

(5) By means of Fibonacci's numbers:

$$N = \sum_{i=1}^{\infty} a_i B^{n-1} - \sum_{i=1}^{\infty} a_i B^{-1} ,$$

where $a_i = a_{i-2} + (B - f)\, a_{i-1}$, $a_1 = 1$, $a_2 = c - f$.

The first term in Eq. (8) can be shown to be equal to

$$S = cB^n/(fB - 1) ,$$

and the second term,

$$S' = S/B^n = c/(fB - 1) .$$

Thus

$$N = S - S' = c(B^n - 1)/(fB - 1).$$

To illustrate method No. (5), we calculate Ku's number, N(6,5,5):

a_1	1
$a_2 = c - f$	0
$a_3 = a_1 + (B - f)a_2$	1
$a_4 = a_2 + (B - f)a_3$	1
$a_5 = a_3 + (B - f)a_4$	2
a_6 = etc.	3
a_7	5
a_8	12
a_9	21
a_{10}	3 3
a_{11}	54
a_{12}	131
a_{13}	22 5
a_{14}	4 00
a_{15}	1 025
a_{16}	142 5
a_{17}	24 54
a_{18}	4 323
a_{19}	1 122 1
a_{20}	155 44
a_{21}	31 205
...	...

$$\underbrace{101,124,045,443,151}_{N(6,5,5)},\cdots$$

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RECREATIONAL MATHEMATICS
"DIFFERENCE SERIES" RESULTING FROM SIEVING PRIMES

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Sieving techniques are notoriously simple, and yet tedious, means of listing primes. By successively eliminating integers divisible by 2, 3, 5, 7, 11, $\cdots$, $\sqrt{N}$, one has left all the primes less than N.

This paper will deal not with the list of integers which remain after each sieving procedure but with the differences between members of the remaining list of integers. The mathematical knowledge required to follow this material is strictly elementary (which may be one reason I continued working at it — if it had required more advanced mathematics, I might have drowned in a sea of mathematical notation).

If N consecutive integers are listed and all the even integers eliminated, a series of odd integers remains:

(A) $$1, 3, 5, 7, 9, 11, \cdots .$$

The difference between each successive term in (A) is 2, and the number of integers in series (A) is $N/2$.

If now all integers in series (A) which are divisible by 3 are eliminated, the following series of integers remains:

(B) $$1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, \cdots .$$

The number of integers in this series is $2N/6$ (fractions are deliberately not reduced since the numerators and denominators are important as we shall see). The differences between successive terms in series (B) is now

$$4, 2, 4, 2, 4, 2, 4, 2, 4, 2, 4, \cdots ,$$

which shows an obvious period of two terms, 4, 2. The sum of the members of this period is 6, or 2 x 3. Recall that series (B) was produced by eliminating integers divisible by 2 and 3.

Eliminating integers divisible by 5 from series (B) produces

(C) 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, $\cdots$.

The number of integers in this series is $8\underline{N}/30$, and the differences between successive terms in series (C) are

6, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 4, 6, 2, 6, $\cdots$

which has an eight-term period

6, 4, 2, 4, 2, 4, 6, 2 .

(The circled integer will be explained later.) The sum of the members of this period is 30, or $2\times3\times5$.

Bear with me for one more round and note what happens when integers from series (C) which are divisible by 7 are eliminated:

(D)
1, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59,
61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121,
127, 131, 137, 139, 143, 149, 151, 157, 163, 167, 169, 173, 179, 181,
187, 191, 193, 197, 199, 209, 211, 221, 223, 227, 229, 233, 239, 241, $\cdots$

The number of integers in this series is $48\underline{N}/210$, and the difference series derived from series (D) has 48 terms in its period:

10, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4,
2, 6, 4, 6, 8, 4, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4,
6, 2, 6, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2 .

The sum of these 48 terms is 210, or $2\times3\times5\times7$.

The difference series developed above have a number of intriguing properties. Readers undoubtedly will find more than I.

The first non-primes to appear in series (A), (B), (C), and (D) are 9, 25, 49, and 121, respectively. These are 3^2. 5^2. 7^2, and 11^2. If successive

prime divisors are $p_1, p_2, p_3, p_4, \cdots$ where $p_1 = 2$, then the first non-prime in these series is p_{n+1}^2. If integers divisible by 11 are eliminated from series (D), the first non-prime will be 13^2, or 169 ($p_5 = 11$, $p_6 = 13$).

The sum of the terms in a difference series is $p_1p_2p_3p_4\cdots p_n$. The next difference series, obtained from series (D) by eliminating integers divisible by 11, would have a sum of $2 \times 3 \times 5 \times 7 \times 11$, or 2310. The number of terms in the difference series can also be calculated. Note that the number of integers remaining in the four series (A), (B), (C), and (D) are determined as follows:

(A): $$\frac{N}{2}$$

(B): $$\frac{N}{2} - \left(\frac{N}{2}\right)\left(\frac{1}{3}\right) = \frac{2N}{6}$$

(C): $$\frac{2N}{6} - \left(\frac{2N}{6}\right)\left(\frac{1}{5}\right) = \frac{8N}{30}$$

(D): $$\frac{8N}{30} - \left(\frac{8N}{30}\right)\left(\frac{1}{7}\right) = \frac{48N}{210} \quad .$$

Going on to series (E) (left for readers with patience to develop), we have

$$\frac{48N}{210} - \left(\frac{48N}{210}\right)\left(\frac{1}{11}\right) = \frac{480N}{2310}$$

terms remaining. Generally, the number of integers remaining from the first $\underline{N}$ integers after elimination of integers divisible by successive $p_1, p_2, p_3, p_4, \cdots, p_n$, is:

$$\frac{(p_2 - 1)(p_3 - 1)(p_4 - 1)\cdots(p_n - 1)N}{p_1\, p_2\, p_3\, p_4 \cdots p_n} \quad .$$

The number of terms in the resulting difference series is

$$(p_2 - 1)(p_3 - 1)(p_4 - 1)\cdots(p_n - 1) \; .$$

Each difference series is derived from the first $(p_1p_2p_3p_4 \cdots p_n) + 1$ integers in the original set of $\underline{N}$ consecutive integers. The last term in the period of any difference series is 2. The first term in the period of any difference series is $p_{n+1} + 1$.

If $\underline{d}$ is the number of terms in a difference series $(d > 2)$ then the $\underline{d}/2^{th}$ term is of special interest. These are the circled terms in the difference series shown perviously. If we ignore the final 2, I conjecture that the difference series are symmetrical about the $\underline{d}/2^{th}$ term. I also think that this $\underline{d}/2^{th}$ term will always be 4.

Since we can calculate the number of terms in a difference series, the point of symmetry, and the last term, we can write a complete period of any difference series by sieving only half as many integers. However, as we successively eliminate integers, the work will still be rather prohibitive. Going to $p_6 = 13$, the resulting difference series has 5760 terms, at $p_7 = 17$ there will be 92,160 terms. Perhaps other relationships within or between difference series will be found by readers — and reduce the labor involved.

The sum of the terms in one period of a difference series seems to be directly related to the differences between primes which are members of arithmetical progressions of primes. Karst* lists all arithmetical progressions of primes with 12 to 16 terms, and the difference between primes in a progression is often the sum of a difference series.

Difference series have been used to quite a limited extent in computer searches for primes. Except for 2, all even numbers are eliminated by simply adding 2 to the previous odd number. This automatically saves the computer half the work of searching for primes. Only one out of every two, or fifty percent, of the integers from 1 to $\underline{N}$ need be examined for primality.

If the second difference series above (4,2) is used, only 33% of the integers from 1 to $\underline{N}$ need be examined. The third difference series (6, 4, 2, 4, 2, 4, 6, 2) would reduce this to about 27%; the next difference series (with 48 terms) would reduce this to about 23%; and the next two difference series (with 480 and 5760 terms, respectively) would reduce this to about 21% and 19%, respectively. Not much gain in search efficiency is achieved after the third difference series is exploited. Further study of these difference series is required.

*Edgar Karst, "12 to 16 Primes in Arithmetical Progression," Journal of Recreational Mathematics, Vol. 2, No. 4 (October 1969), p. .

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ON DETERMINANTS INVOLVING GENERALIZED FIBONACCI NUMBERS

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In this note we shall evaluate some determinants whose elements are the Generalized Fibonacci numbers, T_n, defined by the relations:

$$T_1 = a, \qquad T_2 = b, \qquad T_{n+2} = T_{n+1} + T_n .$$

We can express

$$T_n = C\alpha^n + D\beta^n,$$

where α,β are the roots of the equation $X^2 - X - 1 = 0$, and C and D are constants. The Fibonacci numbers, F_n, are obtained by taking $a = b = 1$, and the Lucas numbers, L_n, by taking $a = 1$, $b = 3$.

We shall make use of the following well known identities:

(i) $$F_{-n} = (-1)^{n-1}F_n ,$$

(ii) $$T_{m+n} = T_mF_{n+1} + T_{m-1}F_n ,$$

(iii) $$T^2_{n+1} - T^2_{n-1} = aT_{2n-2} + bT_{2n-1} ,$$

(iv) $$T_{m-1}T_n - T_mT_{n-1} = (-1)^{m-1}F_{n-m}D ,$$

and shall also use the formulae,

(v) $$T_{m+r}F_{n+r} + (-1)^{r+1}T_mF_n = T_{m+n+r}F_r ,$$

The truth of this formulae can be established, either by induction over r, or by substituting the values of F_n and T_n in terms of α and β.

1. THIRD-ORDER DETERMINANT

We shall show that

$$\begin{vmatrix} T_p & T_{p+m} & T_{p+m+n} \\ T_q & T_{q+m} & T_{q+m+n} \\ T_r & T_{r+m} & T_{r+m+n} \end{vmatrix} = 0 \quad , \tag{1.1}$$

for all integers p, q, r, m, and n. Using (ii), we can write

$$T_{k+m+n} = T_{k+m}F_{n+1} + T_{k+m-1}F_n, \qquad (k = p,q,r)$$

hence the determinant on the left-hand side can be written as

$$F_{n+1}\begin{vmatrix} T_p & T_{p+m} & T_{p+m} \\ T_q & T_{q+m} & T_{q+m} \\ T_r & T_{r+m} & T_{r+m} \end{vmatrix} + F_n\begin{vmatrix} T_p & T_{p+m} & T_{p+m-1} \\ T_q & T_{q+m} & T_{q+m-1} \\ T_r & T_{r+m} & T_{r+m-1} \end{vmatrix}$$

Obviously the first determinant vanishes. The second, on subtracting the elements of the 3rd column from those of the 2nd, reduces to

$$F_n\begin{vmatrix} T_p & T_{p+m-2} & T_{p+m-1} \\ T_q & T_{q+m-2} & T_{q+m-1} \\ T_r & T_{r+m-2} & T_{r+m-1} \end{vmatrix} .$$

Now on subtracting the elements of the 2nd column from the 3rd, we obtain

$$F_n\begin{vmatrix} T_p & T_{p+m-2} & T_{p+m-3} \\ T_q & T_{q+m-2} & T_{q+m-3} \\ T_r & T_{r+m-2} & T_{r+m-3} \end{vmatrix} .$$

Thus alternately subtracting the 2nd and the 3rd columns from one another, the process can be continued to reduce the suffixes. At a certain stage, if m is even, 1st and 2nd columns will become identical; and if m is odd, 1st and 3rd columns will become identical. Hence for every value of m, even or odd, the determinant vanishes.

2. EVALUATION OF THE DETERMINANT

We shall now evaluate the determinant,

$$\Delta \equiv \begin{vmatrix} T_p + k & T_{p+m} + k & T_{p+m+n} + k \\ T_q + k & T_{q+m} + k & T_{q+m+n} + k \\ T_r + k & T_{r+m} + k & T_{r+m+n} + k \end{vmatrix} ,$$

where k is an arbitrary constant, and p, q, r, m, and n are integers.

On writing the determinant as the sum of eight determinants, and using (1.1) and the property that a determinant vanishes if two columns are identical, we obtain

$$\Delta \equiv \begin{vmatrix} T_p & T_{p+m} & k \\ T_q & T_{q+m} & k \\ T_r & T_{r+m} & k \end{vmatrix} + \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} + \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} ,$$

$$= K \cdot F_m \begin{vmatrix} T_p & T_{p-1} & 1 \\ T_q & T_{q-1} & 1 \\ T_r & T_{r-1} & 1 \end{vmatrix} + \cdots + \cdots .$$

The first determinant by using (iv) can be written as

$$= D \cdot K \cdot F_m \left[(-1)^{r-1} F_{q-r} + (-1)^{p-1} F_{r-p} + (-1)^{q-1} F_{p-q} \right] .$$

Hence

$$\Delta = D \cdot K \left[(-1)^q F_{r-q} - (-1)^p F_{r-p} + (-1)^p F_{q-p} \right] \times \times \left[F_m - F_{m+n} + (-1)^m F_n \right] . \tag{2.1}$$

3. FOURTH-ORDER DETERMINANTS

We shall now evaluate the determinant,

$$\Delta \equiv \begin{vmatrix} T_{n+3} & T_{n+2} & T_{n+1} & T_n \\ T_{n+2} & T_{n+3} & T_n & T_{n+1} \\ T_{n+1} & T_n & T_{n+3} & T_{n+2} \\ T_n & T_{n+1} & T_{n+2} & T_{n+3} \end{vmatrix}$$

It can be easily shown that the determinant,

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = \left[(a+b)^2 - (c+d)^2\right] \cdot \left[(a-b)^2 - (c-d)^2\right] .$$

Hence we obtain

$$\begin{aligned} \Delta &= \left[(T_{n+3} + T_{n+2})^2 - (T_{n+1} + T_n)^2\right] \times \\ &\qquad \times \left[(T_{n+3} - T_{n+2})^2 - (T_{n+1} - T_n)^2\right] \\ &= (T_{n+4}^2 - T_{n+2}^2) \cdot (T_{n+1}^2 - T_{n-1}^2) \\ &= (aT_{2n+4} + bT_{2n+5}) \cdot (aT_{2n-2} + bT_{2n-1}) \quad , \end{aligned} \tag{3.1}$$

on using (iii).

4. EVALUATING THE CIRCULANT

We now evaluate the circulant,

$$\begin{vmatrix} T_n & T_{n+k} & \cdots & T_{n+(m-1)k} \\ T_{n+(m-1)k} & T_n & \cdots & T_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ T_{n+k} & T_{n+2k} & \cdots & T_n \end{vmatrix} .$$

Let w be any one of the m numbers

$$w_r = \cos \frac{2r\pi}{m} + i \sin \frac{2r\pi}{m} , \quad (r = 1, 2, 3, \cdots, m)$$

so that $w^m = 1$, and

$$\begin{aligned} S_1 &\equiv w_1 + w_2 + w_3 + \cdots + w_m = 0 \\ S_2 &\equiv w_1 w_2 + \cdots = 0 \\ &\cdots\cdots\cdots\cdots \\ S_m &\equiv w_1 w_2 w_3 w_4 \cdots w_m = (-1) \cdot (-1)^m = (-1)^{m+1} . \end{aligned}$$

Hence we get

$$\prod_{r=1}^{m} (y - w_r z) = y^m - z^m . \tag{4.1}$$

Therefore as discussed in [8],

$$\begin{aligned}
\Delta &= \prod_{r=1}^{m} (T_n + w_r T_{n+k} + \cdots + w_r^{m-1} T_{n+(m-1)k}) \\
&= \prod_{r=1}^{m} \left[\frac{C\alpha^n(1 - w_r^m \alpha^{mk})}{1 - w_r \alpha^k} + \frac{D\beta^n(1 - w_r^m \beta^{mk})}{1 - w_r \beta^k} \right] \\
&= \prod_{r=1}^{m} \left[\frac{(T_n - T_{n+mk}) - (-1)^k w_r (T_{n-k} - T_{n+(m-1)k})}{(1 - w_r \alpha^k)(1 - w_r \beta^k)} \right] \qquad (4.2) \\
&= \frac{(T_n - T_{n+mk})^m - (-1)^{mk}(T_{n-k} - T_{n+(m-1)k})^m}{(1 - \alpha^{mk})(1 - \beta^{mk})} \\
&= \frac{(T_n - T_{n+mk})^m - (-1)^{mk}(T_{n-k} - T_{n+(m-1)k})^m}{1 + (-1)^{mk} - L_{mk}}
\end{aligned}$$

5. EACH ELEMENT IS THE PRODUCT OF TWO NUMBERS

We shall evaluate

$$\Delta \equiv \begin{vmatrix} F_n \cdot T_{m+n} & F_{n+p} \cdot T_{m+n+p} & F_{n+p+q} \cdot T_{m+n+p+q} \\ F_{n+r} \cdot T_{m+n+r} & F_{n+r+p} \cdot T_{m+n+r+p} & F_{n+r+p+q} \cdot T_{m+n+r+p+q} \\ F_{n+s} \cdot T_{m+n+s} & F_{n+s+p} \cdot T_{m+n+s+p} & F_{n+s+p+q} \cdot T_{m+n+s+p+q} \end{vmatrix},$$

and shall show that $|\Delta|$ is independent of n.

On using (v), we can write

$$F_{n+p} T_{m+n+p} + (-1)^{p+1} F_n T_{m+n} = F_p T_{m+2n+p} .$$

Hence multiplying 1st column by $(-1)^{p+1}$, $(-1)^{p+q+1}$, and adding respectively to the 2nd and 3rd columns, we obtain

$$\Delta = F_pF_{p+q}\begin{vmatrix} F_n \cdot T_{m+n} & T_{m+2n+p} & T_{m+2n+p+q} \\ F_{n+r} \cdot T_{m+n+r} & T_{m+2n+2r+p} & T_{m+2n+2r+p+q} \\ F_{n+s} \cdot T_{m+n+s} & T_{m+2n+2s+p} & T_{m+2n+2s+p+q} \end{vmatrix}$$

$$= F_pF_{p+q}F_q\begin{vmatrix} F_n\ T_{m+n} & T_{m+2n+p} & T_{m+2n+p-1} \\ F_{n+r}T_{m+n+r} & T_{m+2n+2r+p} & T_{m+2n+2r+p-1} \\ F_{n+s}T_{m+n+s} & T_{m+2n+2s+p} & T_{m+2n+2s+p-1} \end{vmatrix}$$

on using (ii).

Now alternately subtracting the 3rd and 2nd columns from one another, we can write

$$\Delta = F_pF_qF_{p+q}(-1)^{m+p}\begin{vmatrix} F_n \cdot T_{m+n} & T_0 & T_1 \\ F_{n+r} \cdot T_{m+n+r} & T_{2r} & T_{2r+1} \\ F_{n+s} \cdot T_{m+n+s} & T_{2s} & T_{2s+1} \end{vmatrix}$$

$$= F_pF_qF_{p+q}(-1)^{m+p} \cdot D\Big[F_nT_{m+n}F_{2s-2r} - F_{n+r}T_{m+n+r}F_{2s} + \\ + F_{n+s}T_{m+n+s}F_{2r}\Big]$$

on using (iv).

Now on expressing the numbers in terms of α and β, we can write

$$F_{n+s}T_{m+n+s}F_{2r} = \frac{1}{5}\Big[T_{m+2n+2s+2r} - T_{m+2n+2s-2r} + \\ + (-1)^{n+s}(T_{m-2r} - T_{m+2r})\Big]$$

Hence we have

$$\Delta = \frac{1}{5}F_pF_qF_{p+q}(-1)^{m+n+p} \cdot D\Big[(T_{m+2r-2s} - T_{m+2s-2r}) + \\ + (-1)^s(T_{m-2r} - T_{m+2r}) - (-1)^r(T_{m-2s} - T_{m+2s})\Big] \tag{5.1}$$

Also it is obvious that $|\Delta|$ is independent of n.

6. ONCE AGAIN THE FOURTH ORDER

We shall now show that

$$(6.1)\qquad \Delta \equiv \begin{vmatrix} F_pT_{p+m} & F_{p+a}T_{p+m+a} & F_{p+b}T_{p+m+b} & F_{p+c}T_{p+m+c} \\ F_qT_{q+m} & F_{q+a}T_{q+m+a} & F_{q+b}T_{q+m+b} & F_{q+c}T_{q+m+c} \\ F_rT_{r+m} & F_{r+a}T_{r+m+a} & F_{r+b}T_{r+m+b} & F_{r+c}T_{r+m+c} \\ F_sT_{s+m} & F_{s+a}T_{s+m+a} & F_{s+b}T_{s+m+b} & F_{s+c}T_{s+m+c} \end{vmatrix} = 0,$$

for all integers $p, q, r, s, m, a, b,$ and c.

Multiplying 1st column by $(-1)^{a+1}$, $(-1)^{b+1}$, $(-1)^{c+1}$ and adding to the 2nd, 3rd, and 4th columns, respectively; and using the formula (v), the determinant reduces to

$$F_a \cdot F_b \cdot F_c \cdot \begin{vmatrix} F_pT_{p+m} & T_{2p+m+a} & T_{2p+m+b} & T_{2p+m+c} \\ F_qT_{q+m} & T_{2q+m+a} & T_{2q+m+b} & T_{2q+m+c} \\ F_rT_{r+m} & T_{2r+m+a} & T_{2r+m+b} & T_{2r+m+c} \\ F_sT_{s+m} & T_{2s+m+a} & T_{2s+m+b} & T_{2s+m+c} \end{vmatrix}.$$

Expanding along the 1st column and using the result (1.1), the determinant vanishes. This can be generalized for the nth order determinants.

7. PARTICULAR CASES

A. Let us take $a = b = 1$, then $T_n \equiv F_n$ and $D = -1$.

(i) On putting $m = n$ in (1.1), we get

$$\begin{vmatrix} F_p & F_{p+n} & F_{p+2n} \\ F_q & F_{q+n} & F_{q+2n} \\ F_r & F_{r+n} & F_{r+2n} \end{vmatrix} = 0$$

— a problem suggested by Vladimir Ivanoff [4].

(ii) On taking $p = a$, $q = a + 3d$, $r = a + 6d$, $m = n = d$ in (1.1), we get

$$\begin{vmatrix} F_a & F_{a+d} & F_{a+2d} \\ F_{a+3d} & F_{a+4d} & F_{a+5d} \\ F_{a+6d} & F_{a+7d} & F_{a+8d} \end{vmatrix} = 0$$

— a problem suggested by Raphael Finkelstein [7].

(iii) On taking $p = n$, $q = n + 1$, $r = n + 2$, $m = n = 1$ in (2.1), we get

$$\begin{aligned} &\begin{vmatrix} F_n + k & F_{n+1} + k & F_{n+2} + k \\ F_{n+1} + k & F_{n+2} + k & F_{n+3} + k \\ F_{n+2} + k & F_{n+3} + k & F_{n+4} + k \end{vmatrix} \\ &= (-1)\cdot k\cdot [(-1)^{n+1} - (-1)^n + (-1)^n]\times \\ &\quad \times [F_1 - F_1 - F_2] \\ &= k\cdot(-1)^{n+1} \end{aligned} \tag{7.1}$$

— a problem suggested by Brother U. Alfred [2].

(iv) We obtain from (3.1)

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} = F_{2n+6}\cdot F_{2n}$$

— a problem suggested by George Ledin [5].

(v) We obtain from (4.1)

$$\begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+k} & F_{n+2k} & \cdots & F_n \end{vmatrix}$$

$$= \frac{(F_n - F_{n+mk})^m - (-1)^{mk}(F_{n-k} - F_{n+(m-1)k})^m}{1 - L_{mk} + (-1)^{mk}}$$

— a problem suggested by L. Carlitz [6].

(vi) On taking $m = 0$ in (5.1), we get

$$\begin{vmatrix} F_n^2 & F_{n+p}^2 & F_{n+p+q}^2 \\ F_{n+r}^2 & F_{n+r+p}^2 & F_{n+r+p+q}^2 \\ F_{n+s}^2 & F_{n+s+p}^2 & F_{n+s+p+q}^2 \end{vmatrix}$$

$$= \frac{2}{5} \cdot F_p \cdot F_q \cdot F_{p+q} (-1)^{n+p} [F_{2s-2r} + (-1)^s F_{2r} - (-1)^r F_{2s}]$$

on using result (i).

(vi)-(a) On substituting $p = q = 1$, $r = 1$, $s = 2$, we get

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix}$$

$$= \frac{2}{5} (-1)^{n+1} (F_2 + F_2 + F_4)$$
$$= 2(-1)^{n+1}$$

— a problem suggested by Brother U. Alfred [1].

(vi)-(b) On substituting $p = q = 2$, $r = 2$, $s = 4$, we get

$$\begin{vmatrix} F_n^2 & F_{n+2}^2 & F_{n+4}^2 \\ F_{n+2}^2 & F_{n+4}^2 & F_{n+6}^2 \\ F_{n+4}^2 & F_{n+6}^2 & F_{n+8}^2 \end{vmatrix}$$

$$= \frac{2}{5} \cdot 3 \cdot (-1)^n \cdot (3 + 3 - 21)$$
$$= 18 (-1)^{n+1}$$

— a problem suggested by Brother U. Alfred [3].

(vii) On taking $m = 1$ in (5.1), we obtain

$$\begin{vmatrix} F_nF_{n+1} & F_{n+p} & F_{n+p+1} & F_{n+p+q} & F_{n+p+q+1} \\ F_{n+r}F_{n+r+1} & F_{n+r+p} & F_{n+r+p+1} & F_{n+r+p+q} & F_{n+r+p+q+1} \\ F_{n+s}F_{n+s+1} & F_{n+s+p} & F_{n+s+p+1} & F_{n+s+p+q} & F_{n+s+p+q+1} \end{vmatrix}$$

$$= \frac{1}{5} F_r \cdot F_q \cdot F_{p+q}(-1)^{n+p} [(F_{2r-2s+1} - F_{1+2s-2r}) + $$

$$+ (-1)^s(F_{1-2r} - F_{1+2r}) - (-1)^r(F_{1-2s} - F_{1+2s})]$$

$$= \frac{1}{5} F_pF_qF_{p+q} (-1)^{n+p} [-(F_{2s-2r+1} - F_{2s-2r-1}) + $$

$$+ (-1)^{s+1}(F_{2r+1} - F_{2r-1}) + (-1)^r(F_{2s+1} - F_{2s-1})]$$

$$= \frac{1}{5} (-1)^{n+p+1} F_pF_qF_{p+q}[F_{2s-2r} + (-1)^sF_{2r} - (-1)^r F_{2s}] .$$

(vii)-(a) On taking $p = q = r = 1$, and $s = 2$, we have

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} \\ F_{n+1}F_{n+2} & F_{n+2}F_{n+3} & F_{n+3}F_{n+4} \\ F_{n+2}F_{n+3} & F_{n+3}F_{n+4} & F_{n+4}F_{n+5} \end{vmatrix}$$

$$= \frac{1}{5} (-1)^n (F_2 + F_2 + F_4)$$

$$= (-1)^n \quad .$$

(viii) On taking $m = 0$ in (6.1), we get

$$\begin{vmatrix} F_p^2 & F_{p+a}^2 & F_{p+b}^2 & F_{p+c}^2 \\ F_q^2 & F_{q+a}^2 & F_{q+b}^2 & F_{q+c}^2 \\ F_r^2 & F_{r+a}^2 & F_{r+b}^2 & F_{r+c}^2 \\ F_s^2 & F_{s+a}^2 & F_{s+b}^2 & F_{s+c}^2 \end{vmatrix} = 0 \quad ,$$

for all integers $p, q, r, s, a, b,$ and c.

B. On taking $a = 1$, $b = 3$, we have $T_n \equiv L_n$ and $D = 5$.

(i) On taking $p = a$, $q = a + 3d$, $r = a + 6d$, $m = n = d$ in (1.1), we get

$$\begin{vmatrix} L_a & L_{a+d} & L_{a+2d} \\ L_{a+3d} & L_{a+4d} & L_{a+5d} \\ L_{a+6d} & L_{a+7d} & L_{a+8d} \end{vmatrix} = 0$$

— a problem suggested by Raphael Finkelstein [7].

(ii) We obtain from (2.1) that

$$\begin{vmatrix} L_p + k & L_{p+m} + k & L_{p+n} + k \\ L_q + k & L_{q+m} + k & L_{q+n} + k \\ L_r + k & L_{r+m} + k & L_{r+n} + k \end{vmatrix}$$

$$= -5 \cdot \begin{vmatrix} F_p + k & F_{p+m} + k & F_{p+n} + k \\ F_q + k & F_{q+m} + k & F_{q+n} + k \\ F_r + k & F_{r+m} + k & F_{r+n} + k \end{vmatrix}$$

for all integers p, q, r, m, and n.

(iii) We obtain from (3.1)

$$\begin{vmatrix} L_{n+3} & L_{n+2} & L_{n+1} & L_n \\ L_{n+2} & L_{n+3} & L_n & L_{n+1} \\ L_{n+1} & L_n & L_{n+3} & L_{n+2} \\ L_n & L_{n+1} & L_{n+2} & L_{n+3} \end{vmatrix}$$

$$= (L_{2n+4} + 3L_{2n+5})(L_{2n-2} + 3L_{2n-1})$$

$$= 25\, F_{2n+6} F_{2n}$$

$$= 25 \begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} .$$

(iv) We obtain from (4.1)

$$\begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \vdots & \vdots & \vdots & \vdots \\ L_{n+k} & L_{n+2k} & \cdots & L_n \end{vmatrix}$$

$$= \frac{(L_n - L_{n+mk})^m - (-1)^{mk}(L_{nk} - L_{n+(m-1)k})^m}{1 - L_{mk} + (-1)^{mk}}$$

— a problem suggested by L. Carlitz [6].

ACKNOWLEDGEMENT

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contribution are asked to enclose self-addressed stamped postcards.

B-166 Suggested by David Zeitlin's solutions to B-148, 149, and 150.

Let a and b be distinct numbers, $U_n = (a^n - b^n)/(a - b)$, and $V_n = a^n + b^n$. Establish generalizations of the formulas

(a) $$F_{(2^t n)} = F_n L_n L_{2n} \cdots L_{(2^{t-1} n)}$$

(b) $$L_{n+1} L_{n+3} + 4(-1)^{n+1} = 5F_n F_{n+4}$$

of B-148 and B-149 in which one deals with U_n and V_n instead of F_n and L_n.

B-167 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, T. P. N. G.

Let L_n be the nth Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. For which values of n is

$$nL_{n+1} > (n+1)L_n \quad ?$$

B-168 Proposed by S. H. L. King, Jacksonville University, Jacksonville, Florida.

Using each of six of the nine positive digits $1, 2, \cdots, 9$ exactly once, form an integer z such that each of z, 2z, 3z, 4z, 5z, and 6z contains the same six digits once and once only.

B-169 Proposed by C. C. Yalavigi, Government College, Mercara, India.

Prove the following identities:

(a) $$F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_nF_{n-1} - F_{n+1})^2$$

(b) $$F_n^5 + F_{n-1}^5 - F_{n+1}^5 = 5F_nF_{n-1}F_{n+1}(F_nF_{n-1} - F_{n+1}^2)$$

where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Show that these are two cases of an infinite sequence of identities.

B-170 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the binomial coefficient $\binom{m}{r}$ be zero when $m < r$ and let

$$S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} .$$

Show that $S_{n+2} - S_{n+1} + S_n = 0$ and hence $S_{n+3} = -S_n$ for $n = 0, 1, 2, \cdots$.

B-171 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $\binom{m}{r} = 0$ for $m < r$ and let

$$T_n = \sum_{j=0}^{\infty} \binom{n-2j}{2j} .$$

Obtain a fourth-order homogeneous linear recurrence formula for T_n.

SOLUTIONS

CORRECTION. In the solution to B-128 in Vol. 6, No. 4 (Oct. 1968), line 2 from the bottom of p. 295 should read:

$$S_{4n} = f_{4n+2} - f_2 = (F_{4n+1} - 1)f_2 + F_{4n}f_1 ,$$

and line 5 from the top of p. 296 should read:

$$F_{4n+1} - 1 = F_{2n}L_{2n+1} .$$

COMMENT. Mr. J. D. E. Konhauser, Macalester College, St. Paul, Minnesota, sent in the following on B-130a:

The five-disk problem is discussed in Mathematical Recreations and Essays by W. W. R. Ball, revised by H. S. M. Coxeter in 1938, on pages 97-99. Included is a reference to a 1915 paper by E. H. Neville in the Proceedings of the London Mathematical Society, second series, Vol. xiv, pp. 308-326.

ADDITIONS TO LISTS OF SOLVERS: Problem B-143 was also solved by D. V. Jaiswal (Indore, India), Amanda Neel, and A. G. Shannon (Boroko, T. P. N. G.) Problem B-143 was also solved by D. V. Jaiswal and A. G. Shannon. Problem B-146 was also solved by D. V. Jaiswal and A. G. Shannon.

TELESCOPING PRODUCT

B-148 Proposed by David Englund, Rockford College, Rockford, Illinois, and Malcolm Tallman, Brooklyn, New York.

Let F_n and L_n denote the Fibonacci and Lucas numbers and show that

$$F_{(2^t n)} = F_n L_n L_{2n} L_{4n} \cdots L_{(2^{t-1} n)} .$$

Solution by Douglas Lind, Cambridge University, Cambridge, England.

By the well-known formula $F_{2n} = F_n L_n$, we have

$$F_{2^t n} = F_{2^{t-1} n} L_{2^{t-1} n} = F_{2^{t-2} n} L_{2^{t-2} n} L_{2^{t-1} n} = \cdots = F_n L_n L_{2n} \cdots L_{2^{t-1} n} .$$

Also solved by Christine Anderson, Serge Hamelin (Canada), Bruce W. King, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, Michael Yoder, David Zeitlin, and the proposer.

A QUADRATIC IDENTITY

B-149 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

$$L_{n+1}L_{n+3} + 4(-1)^{n+1} = 5F_nF_{n+4}.$$

Solution by Carol A. Vespe, Student, University of New Mexico, Albuquerque, New Mexico.

Let $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Since both sides of the equation are of the form

$$c_1(a^2)^n + c_2(ab)^n + c_3(b^2)^n,$$

with constant c_i, it suffices to note that the identity holds for $n = 0, 1$, and 2.

Also solved by Clyde A. Bridger, Juliette Davenport, Herta T. Freitag, Serge Hamelin (Canada), Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), C. C. Yalavigi (Mercara, India), Michael Yoder, David Zeitlin, and the Proposer.

ANOTHER QUADRATIC IDENTITY

B-150 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

$$L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}.$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let U_n and V_n be solutions of $W_{n+2} = aW_{n+1} + bW_n$, where $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = a$. Noting that

$$V_n = 2U_{n+1} - aU_n \equiv U_{n+1} + bU_{n-1},$$

we obtain

$$V_n^2 - a^2U_n^2 = 4bU_{n-1}U_{n+1}. \tag{1}$$

The desired result is obtained from (1) with $a = b = 1$, $V_n \equiv L_n$, and $U_n \equiv F_n$.

Also solved by Clyde A. Bridger, Juliette Davenport, David Englund, Herta T. Freitag, Serge Hamelin (Canada), John E. Homer, Jr., Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind (England), John W. Milsom, C. B. A. Peck, Gerald Satlow, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, Michael Yoder, and the Proposer.

MISSING TERMS

B-151 Proposed by Hal Leonard, San Jose State College, San Jose, California.

Let $m = L_1 + L_2 + \cdots + L_n$ be the sum of the first n Lucas numbers. Let

$$P_n(x) = \prod_{i=1}^{n} (1 + x^{L_i}) = a_0 + a_1 x + \cdots + a_m x^m .$$

Let q_n be the number of integers k such that both $0 < k < m$ and $a_k = 0$. Find a recurrence relation for the q_n.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Note that

$$m = m_n = L_1 + L_2 + \cdots + L_n = L_{n+2} - 3$$

and that q_n is the number of integers in $\{1, 2, 3, \cdots, m-1\}$ that are not expressible in the form

$$c_1 L_1 + c_2 L_2 + \cdots + c_n L_n \; ; \quad c_i \in \{0,1\} \quad \text{for} \quad 1 \le i \le n .$$

A generalization of this problem is dealt with by David A. Klarner in "Representations of N as a Sum of Distinct Elements from Special Sequences," Fibonacci Quarterly, Vol. 4, No. 4 (Dec. 1966), pp. 289-306.

Using

$$m_{2k} = L_1 + L_2 + L_3 + \cdots + L_{2k} = L_3 + L_5 + L_7 + \cdots + L_{2k+1}$$
$$m_{2k-1} = L_1 + L_2 + \cdots + L_{2k-1} = L_1 + (L_4 + L_6 + \cdots + L_{2k}) ,$$

and formula (43) on page 303 of Klarner's paper, one has

$$q_{2k} = m_{2k} - (F_4 + F_6 + \cdots + F_{2k+2}) = L_{2k+2} - 3 - (F_{2k+3} - F_3)$$

$$q_{2k-1} = m_{2k-1} - F_2 - (F_5 + F_7 + \cdots + F_{2k+1}) = L_{2k+1} - 3 - 1 - (F_{2k+2} - F_4).$$

Now $L_n = F_{n+1} + F_{n-1}$ leads to $q_n = F_{n+1} - 1$ for all n. Hence

$$(q_{n+2} + 1) = (q_{n+1} + 1) + (q_n + 1) \quad \text{or} \quad q_{n+2} = q_{n+1} + q_n + 1$$

Also solved by Serge Hamelin (Quebec, Canada), C. B. A. Peck, and Carol A. Vespe. Hamelin gave the homogeneous recursion formula $q_{n+3} = 2q_{n+2} - q_n$.

FIBONACCI ADDITION FORMULA

B-152 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that

$$F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}.$$

Solution by John E. Homer, Jr., Lisle, Illinois.

From the well-known formulas

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n$$

$$F_{m+n-1} = F_m F_n + F_{m-1}F_{n-1},$$

we have

$$F_{m+1}F_{n+1} - F_{m-1}F_{n-1} = F_{m+n+1} - F_{m+n-1} = F_{m+n}.$$

Also solved by Clyde A. Bridger, Juliette Davenport, David Englund, Herta T. Freitag, Serge Hamelin (Canada), Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind (England), John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, C. C. Yalavigi (Mercara, India), Michael Yoder, and the Proposer.

[Continued in p. 276.]

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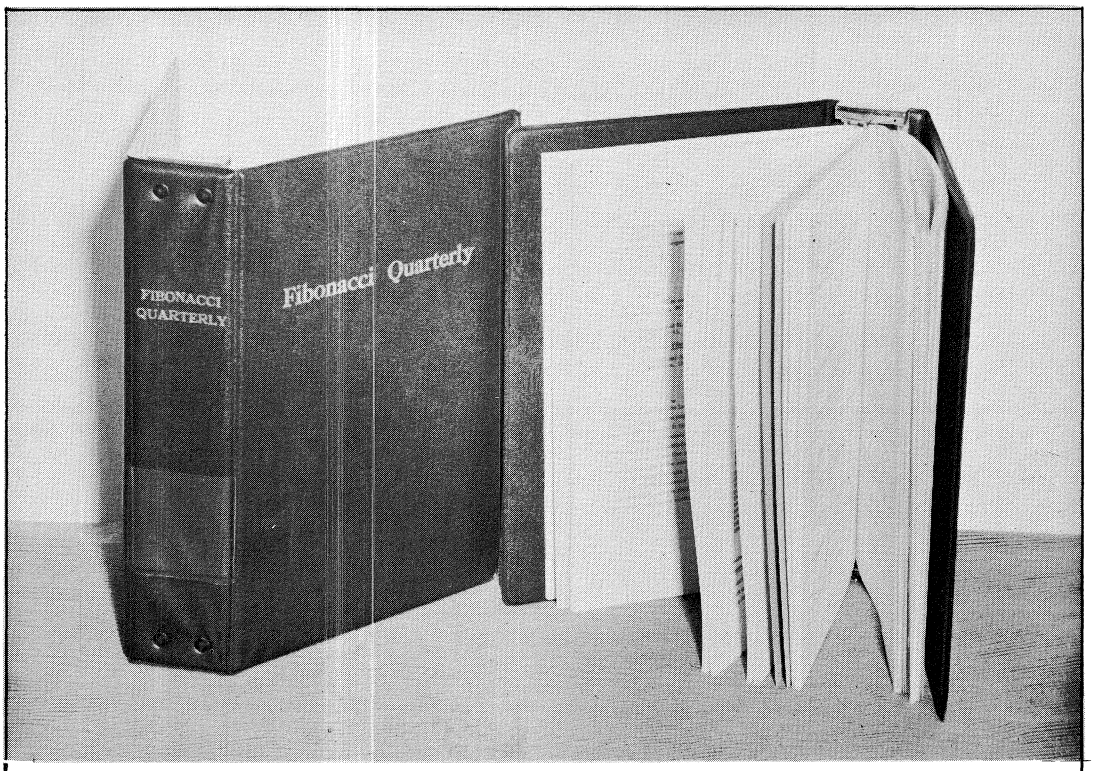

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