

GENERALIZED METALLIC MEANS

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ABSTRACT. The metallic means (also known as metallic ratios) may be defined as the limiting ratio of consecutive terms of sequences connected to the Fibonacci sequence via the INVERT transform. For example, the Pell sequence (INVERT transform of the Fibonacci sequence) gives the so-called silver mean, and the INVERT transform of the Pell sequence leads to the bronze mean. The limiting ratio of the Fibonacci sequence itself is known as the golden mean or ratio. We introduce a new family of k th degree metallic means obtained through INVERT transforms of the generalized k th order Fibonacci sequence. As in the case $k = 2$, each generalized metallic mean is shown to be the unique positive root of a k th degree polynomial determined by the sequence.

1. INTRODUCTION

We start by reviewing some basic information about the sequence of Fibonacci numbers, defined by the recurrence relation

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \\ F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 2. \end{aligned}$$

Their generating function $F(x) = \sum_{n=1}^{\infty} F_n x^n$ may be written as $F(x) = \frac{x}{1 - x - x^2}$, and it can be easily shown that

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varphi,$$

where φ is the unique positive root of $x^2 - x - 1 = 0$. The number $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is referred to as the Golden Mean or Golden Ratio.

More generally, for $m \in \mathbb{N}$, one can consider the sequence $(F_n^{(m)})_{n \in \mathbb{N}_0}$ defined by

$$\begin{aligned} F_0^{(m)} &= 0, \quad F_1^{(m)} = 1, \\ F_n^{(m)} &= mF_{n-1}^{(m)} + F_{n-2}^{(m)} \quad \text{for } n \geq 2, \end{aligned}$$

with generating function

$$F_m(x) = \frac{x}{1 - mx - x^2}.$$

Observe that $(F_n^{(2)})$ is the sequence of Pell numbers (see [4, A000129]), and for $m = 3$ and $m = 4$, we get sequences A006190 and A001076 in [4].

It can be shown that

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}^{(m)}}{F_n^{(m)}} = \varphi_m,$$

where φ_m is the unique positive root of $x^2 - mx - 1 = 0$. Thus,

$$\varphi_m = \frac{m + \sqrt{m^2 + 4}}{2}.$$

Note that φ_m lies in the interval $(m, m+1)$. These numbers are collectively referred to as the (quadratic) *metallic means*. For instance, $\varphi_2 = 1 + \sqrt{2} \approx 2.414$ is known as the Silver Mean, and $\varphi_3 = \frac{3+\sqrt{13}}{2} \approx 3.303$ is called the Bronze Mean.

The above family of sequences may also be generated via the INVERT transform¹. For every $m \in \mathbb{N}$, we have

$$1 + F_{m+1}(x) = \frac{1}{1 - F_m(x)}.$$

In other words, the sequence $(F_n^{(m+1)})$ is the INVERT transform of the sequence $(F_n^{(m)})$, hence it is the m th INVERT transform of the Fibonacci sequence.

In this paper, we adopt this point of view and introduce a family of generalized metallic means obtained through INVERT transforms of the generalized k th order Fibonacci sequence. In Section 2, we illustrate our approach for the cubic case ($k = 3$) that involves the Tribonacci sequence. In Section 3, we then generalize our results to arbitrary values of k . In all cases, each generalized metallic mean is shown to be the unique positive root of a polynomial determined by the sequence. In the last section, we offer some remarks about combinatorial interpretations and a possible direction for future research.

2. CUBIC METALLIC MEANS

We start by considering the sequence $(T_n)_{n \in \mathbb{N}_0}$ defined by the recurrence

$$\begin{aligned} T_0 &= 0, \quad T_1 = 1, \quad T_2 = 1, \\ T_n &= T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for } n \geq 3. \end{aligned}$$

The corresponding generating function $T(x)$ takes the form

$$T(x) = \frac{x}{1 - x - x^2 - x^3},$$

and its $(m-1)$ th INVERT transform is the sequence $(T_n^{(m)})_{n \in \mathbb{N}_0}$ with generating function

$$T_m(x) = \frac{x}{1 - mx - x^2 - x^3}.$$

In particular, $(T_n^{(m)})_{n \in \mathbb{N}_0}$ satisfies the recurrence relation

$$\begin{aligned} T_0^{(m)} &= 0, \quad T_1^{(m)} = 1, \quad T_2^{(m)} = m, \\ T_n^{(m)} &= mT_{n-1}^{(m)} + T_{n-2}^{(m)} + T_{n-3}^{(m)} \quad \text{for } n \geq 3. \end{aligned}$$

Observe that the corresponding characteristic polynomial $p(x) = x^3 - mx^2 - x - 1$ satisfies $p(m) < 0 < p(m+1)$, so $p(x)$ has a real root $\tau_m \in (m, m+1)$. Moreover,

$$p(x) = (x - \tau_m) \left(x^2 + \frac{\tau_m + 1}{\tau_m^2} x + \frac{1}{\tau_m} \right),$$

hence the other two roots of $p(x)$ are the complex numbers

$$\gamma_m^\pm = -\frac{1}{2\tau_m^2} \left(\tau_m + 1 \pm i\sqrt{4\tau_m^3 - (\tau_m + 1)^2} \right).$$

Since $|\gamma_m^\pm| = 1/\sqrt{\tau_m} < \tau_m$, and since

$$T_n^{(m)} = c_1 \tau_m^n + c_2 (\gamma_m^-)^n + c_3 (\gamma_m^+)^n \quad \text{for some constants } c_1, c_2, c_3,$$

¹Introduced by P. J. Cameron in [1, Section 3] as operator A .

one can easily deduce that

$$\lim_{n \rightarrow \infty} \frac{T_{n+1}^{(m)}}{T_n^{(m)}} = \tau_m.$$

This motivates the following definition.

Definition 2.1. Let $m \in \mathbb{N}$. The unique real root τ_m of the polynomial $x^3 - mx^2 - x - 1$ will be referred to as the m th cubic metallic mean. We have $m < \tau_m < m + 1$.

For example, for $m = 1, 2, 3$, we get

$$\begin{aligned} \tau_1 &= \frac{1}{3} \left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \right) \approx 1.839, \\ \tau_2 &= \frac{1}{3} \left(2 + \sqrt[3]{\frac{61+9\sqrt{29}}{2}} + \sqrt[3]{\frac{61-9\sqrt{29}}{2}} \right) \approx 2.547, \\ \tau_3 &= \frac{1}{3} \left(3 + \sqrt[3]{54 + 6\sqrt{33}} + \sqrt[3]{54 - 6\sqrt{33}} \right) \approx 3.383. \end{aligned}$$

We call τ_1 , τ_2 , and τ_3 the *cubic golden*, *silver*, and *bronze means*, respectively. The number τ_1 is also known as the *Tribonacci constant*, see for example, [6]. For more information, including geometric interpretations of τ_1 , we refer to [4, A058265] and the links therein. For example, one interpretation of τ_1 is the following. If a line segment is divided into three parts of lengths a , b , and c such that $\frac{a+b+c}{a} = \frac{a}{b} = \frac{b}{c}$, then the common ratio is precisely τ_1 (see Figure 1).

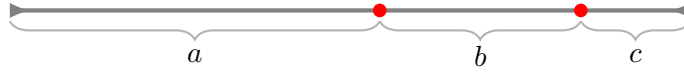


FIGURE 1. $(a + b + c) : a$ is equal to $a : b$ and equal to $b : c$.

It is worth noting that τ_1 also appears as the order of convergence of a certain iterative algorithm for solving nonlinear least squares problems, cf. [5].

3. GENERALIZED METALLIC MEANS

Motivated by the quadratic and cubic metallic means defined via the INVERT transform of the Fibonacci and Tribonacci sequences, respectively, we now consider the generalized k th order Fibonacci sequence ($k \geq 2$) with generating function

$$G(x) = \frac{x}{1 - x - x^2 - \dots - x^k}.$$

If we write $G(x) = \sum_{n=1}^{\infty} g_n x^n$, then $(g_n)_{n \in \mathbb{N}_0}$ satisfies the recurrence relation

$$g_n = g_{n-1} + g_{n-2} + \dots + g_{n-k} \quad \text{for } n \geq k,$$

with initial values $g_0 = 0$, $g_1 = 1$, and if $k \geq 3$,

$$g_j = 2^{j-2} \quad \text{for } j \in \{2, \dots, k-1\}.$$

We apply the INVERT transform $m - 1$ times and arrive at the sequence $(g_n^{(m)})_{n \in \mathbb{N}_0}$ with generating function

$$G_m(x) = \frac{x}{1 - mx - x^2 - \dots - x^k}, \quad (3.1)$$

thus we have the recurrence relation

$$g_n^{(m)} = mg_{n-1}^{(m)} + g_{n-2}^{(m)} + \cdots + g_{n-k}^{(m)} \text{ for } n \geq k.$$

As a consequence, if $\gamma_1, \dots, \gamma_k$ are the distinct² roots of the polynomial

$$p_m(x) = x^k - mx^{k-1} - x^{k-2} - \cdots - x - 1, \quad (3.2)$$

then we have

$$g_n^{(m)} = c_1\gamma_1^n + \cdots + c_k\gamma_k^n,$$

for some constants c_1, \dots, c_k .

Theorem 3.1. *For every $m \in \mathbb{N}$, the polynomial $p_m(x)$ has a unique positive root ϱ_m with $m < \varrho_m < m + 1$. Moreover, every other root γ_j of $p_m(x)$ satisfies $|\gamma_j| < \varrho_m$.*

Proof. Since $(x - 1)p_m(x) = x^{k+1} - (m + 1)x^k + (m - 1)x^{k-1} + 1$, we have

$$p_m(x) = \frac{x^{k+1} - (m + 1)x^k + (m - 1)x^{k-1} + 1}{x - 1} \text{ for } x \neq 1. \quad (3.3)$$

Hence,

$$p_m(m) = \begin{cases} 1 - k & \text{if } m = 1, \\ \frac{1 - m^{k-1}}{m - 1} & \text{if } m > 1, \end{cases} \quad \text{and} \quad p_m(m + 1) = \frac{(m - 1)(m + 1)^{k-1} + 1}{m}.$$

Clearly, $p_m(m) < 0 < p_m(m + 1)$ for every m , which implies that $p_m(x)$ must have a real root between m and $m + 1$. We call this root ϱ_m . By Descartes' rule of signs, ϱ_m is the only positive real root of $p_m(x)$.

As a consequence, $p_m(x) < 0$ for every $x \in (0, \varrho_m)$ and $p_m(x) > 0$ for every $x > \varrho_m$. Let $\gamma \neq \varrho_m$ be such that $p_m(\gamma) = 0$. Then, $\gamma^k = m\gamma^{k-1} + \gamma^{k-2} + \cdots + \gamma + 1$ and so

$$|\gamma|^k \leq m|\gamma|^{k-1} + |\gamma|^{k-2} + \cdots + |\gamma| + 1.$$

This implies $p_m(|\gamma|) \leq 0$, hence $|\gamma| \leq \varrho_m$. We will show that this inequality is strict.

Note that, by (3.3), $p(x) = 0$ if and only if $(m + 1)x^k = x^{k+1} + (m - 1)x^{k-1} + 1$.

Assume that $\gamma \neq \varrho_m$ is a root with $|\gamma| = \varrho_m$. Thus $p_m(\gamma) = 0 = p_m(|\gamma|)$, hence

$$(m + 1)\gamma^k = \gamma^{k+1} + (m - 1)\gamma^{k-1} + 1, \quad (3.4)$$

$$(m + 1)|\gamma|^k = |\gamma|^{k+1} + (m - 1)|\gamma|^{k-1} + 1. \quad (3.5)$$

Equation (3.4) together with Lemma 3.2 gives

$$|(m + 1)\gamma^k| + \underbrace{\left(3 - \left|\frac{\gamma^{k+1}}{|\gamma|^{k+1}} + \frac{\gamma^{k-1}}{|\gamma|^{k-1}} + 1\right|\right)}_{\geq 0} \leq |\gamma^{k+1}| + |(m - 1)\gamma^{k-1}| + 1,$$

which by (3.5) implies $\left|\frac{\gamma^{k+1}}{|\gamma|^{k+1}} + \frac{\gamma^{k-1}}{|\gamma|^{k-1}} + 1\right| = 3$. If $\gamma = \varrho_m e^{i\theta}$, this can be written as

$$\left|e^{i\theta(k+1)} + e^{i\theta(k-1)} + 1\right| = 3,$$

which is only possible if $\theta = 0$ or if $\theta = \pi$ (for odd k). However, $\theta = 0$ would contradict $\gamma \neq \varrho_m$, and if $\theta = \pi$ and k is odd, the left side of (3.4) would be negative, whereas the right side would be positive, a contradiction.

In conclusion, if $\gamma \neq \varrho_m$ and $p_m(\gamma) = 0$, then $|\gamma| < \varrho_m$. □

The following lemma was motivated by a theorem of Maligranda [3].

²This fact will be discussed in the proof of Theorem 3.3.

Lemma 3.2. *Let $z_1, z_2, z_3 \in \mathbb{C}$ be such that $0 < |z_1| \leq |z_2| \leq |z_3|$. Then,*

$$|z_1 + z_2 + z_3| + \left(3 - \left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} + \frac{z_3}{|z_3|}\right|\right) |z_1| \leq |z_1| + |z_2| + |z_3|.$$

Proof. By the triangle inequality, and since $|z_1| \leq |z_2| \leq |z_3|$, we have

$$\begin{aligned} |z_1 + z_2 + z_3| &= \left| \frac{|z_1|}{|z_1|} z_1 + \frac{|z_1|}{|z_2|} z_2 + \frac{|z_1|}{|z_3|} z_3 + \left(1 - \frac{|z_1|}{|z_2|}\right) z_2 + \left(1 - \frac{|z_1|}{|z_3|}\right) z_3 \right| \\ &\leq \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} + \frac{z_3}{|z_3|} \right| |z_1| + \left(1 - \frac{|z_1|}{|z_2|}\right) |z_2| + \left(1 - \frac{|z_1|}{|z_3|}\right) |z_3| \\ &= \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} + \frac{z_3}{|z_3|} \right| |z_1| + |z_2| - |z_1| + |z_3| - |z_1|. \end{aligned}$$

Adding and subtracting $|z_1|$, we arrive at

$$|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3| + \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} + \frac{z_3}{|z_3|} \right| |z_1| - 3|z_1|,$$

which is equivalent to the claimed inequality. \square

Theorem 3.3. *Fix $k, m \in \mathbb{N}$, $k \geq 2$, and let $(g_n^{(m)})_{n \in \mathbb{N}}$ be the sequence defined by the generating function $G_m(x)$ from (3.1). Then,*

$$\lim_{n \rightarrow \infty} \frac{g_{n+1}^{(m)}}{g_n^{(m)}} = \varrho_m,$$

where ϱ_m is the unique positive root of the polynomial (3.2). Consistent with the quadratic and cubic cases, we call ϱ_m the m th metallic mean of degree k .

Proof. We already showed that $p_m(x)$ has a unique positive real root $\varrho_m \in (m, m+1)$, and we claim that every root of $p_m(x)$ is simple. To see this, consider the polynomial

$$q(x) = (x-1)p_m(x) = x^{k+1} - (m+1)x^k + (m-1)x^{k-1} + 1.$$

Clearly, $p_m(x)$ and $q(x)$ share all of their roots (including multiplicity) except for $x = 1$. Now, since

$$q'(x) = x^{k-2} ((k+1)x^2 - k(m+1)x + (k-1)(m-1)),$$

and since the quadratic polynomial $(k+1)x^2 - k(m+1)x + (k-1)(m-1)$ has two distinct positive real roots (the discriminant is $k^2(m-1)^2 + 4(k^2 + m-1)$), we conclude that the roots of $q(x)$, and therefore the roots of $p_m(x)$, are all simple.

Let $\gamma_1, \dots, \gamma_{k-1}, \varrho_m$ be the k distinct roots of the polynomial $p_m(x)$ associated with the sequence $(g_n^{(m)})_{n \in \mathbb{N}}$. Then, there are constants c_1, \dots, c_k such that

$$g_n^{(m)} = c_1 \gamma_1^n + \dots + c_{k-1} \gamma_{k-1}^n + c_k \varrho_m^n.$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{g_{n+1}^{(m)}}{g_n^{(m)}} &= \lim_{n \rightarrow \infty} \frac{c_1 \gamma_1^{n+1} + \dots + c_{k-1} \gamma_{k-1}^{n+1} + c_k \varrho_m^{n+1}}{c_1 \gamma_1^n + \dots + c_{k-1} \gamma_{k-1}^n + c_k \varrho_m^n} \\ &= \lim_{n \rightarrow \infty} \frac{c_1 \gamma_1 \left(\frac{\gamma_1}{\varrho_m}\right)^n + \dots + c_{k-1} \gamma_{k-1} \left(\frac{\gamma_{k-1}}{\varrho_m}\right)^n + c_k \varrho_m}{c_1 \left(\frac{\gamma_1}{\varrho_m}\right)^n + \dots + c_{k-1} \left(\frac{\gamma_{k-1}}{\varrho_m}\right)^n + c_k} \\ &= \varrho_m. \end{aligned}$$

The last step follows since, for $j \in \{1, \dots, k-1\}$, we have $|\gamma_j| < \varrho_m$, which implies $\lim_{n \rightarrow \infty} \left(\frac{\gamma_j}{\varrho_m}\right)^n = 0$. \square

4. CONCLUDING REMARKS

Motivated by the (quadratic) metallic means, which may be defined through sequences that are related to the Fibonacci sequence via the INVERT transform, in this paper we have introduced a family of generalized metallic means of arbitrary degree $k > 2$.

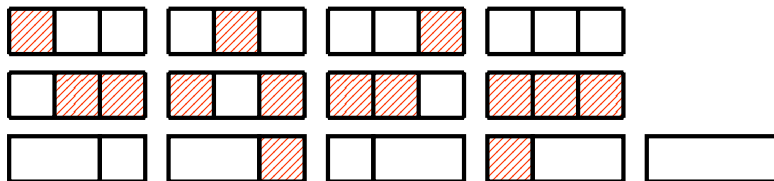
Observe that our definition is consistent with the quadratic case ($k = 2$), and the generalized metallic means of degree k as well as the corresponding sequences $(g_n^{(m)})_{n \in \mathbb{N}_0}$, all satisfy similar properties as their quadratic counterparts.

Combinatorially, it is known and easy to prove that, for $n \geq 2$, the sequence F_n gives the number of tilings of an $(n-1) \times 1$ rectangular board by 1×1 and 2×1 tiles. In that context, $F_n^{(m)}$ gives the number of such tilings, where the 1×1 tiles come in m colors. In general, for $k, n \geq 2$, $g_n^{(m)}$ gives the number of tilings of an $(n-1) \times 1$ rectangular board by tiles of sizes $1 \times 1, 2 \times 1, \dots, k \times 1$, where the 1×1 tiles come in m colors.

For example, if $k = 3$ and $m = 2$, we get the sequence $(T_n^{(2)})_{n \in \mathbb{N}}$ with terms

$$1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, \dots$$

The 13 such tilings of a 3×1 rectangular board are:



Other combinatorial interpretations are certainly possible.

We finish our exposition by mentioning that, for the generalized golden mean ($m = 1$) of arbitrary degree, Hare, Prodinger, and Shallit [2] gave series representations for ϱ_1 , $1/\varrho_1$, and $1/(2 - \varrho_1)$. It would be interesting to find corresponding series representations for the m th generalized metallic mean ϱ_m .

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