PRIMEFREE SHIFTED BINARY LINEAR RECURRENCE SEQUENCES

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Abstract. We say a sequence $\mathcal{X} = (x_n)_{n\geq 0}$ is *primefree* if $|x_n|$ is not prime for all $n \geq 0$ and, to rule out trivial situations, we require that no single prime divides all terms of $\mathcal{X}$. For $a, b, w_0, w_1 \in \mathbb{Z}$, we let $\mathcal{W}(w_0, w_1, a, b) = (w_n)_{n\geq 0}$ denote the general linear binary recurrence that is defined by

$$w_n = aw_{n-1} + bw_{n-2} \quad \text{for } n \geq 2.$$

It has been shown recently for any sequence $\mathcal{X} \in \{\mathcal{W}(0,1,a,1), \mathcal{W}(2,a,a,1)\}$, that there exist infinitely positive integers k such that both of the shifted sequences $\mathcal{X} \pm k$ are simultaneously primefree, and moreover, each term has at least two distinct prime divisors. In this article, we extend these theorems by establishing analogous results for all but finitely many sequences

$$\mathcal{X} \in \{\ \mathcal{W}(0,1,a,-1),\ \mathcal{W}(2,a,a,-1),\ \mathcal{W}(1,1,a,-1),\ \mathcal{W}(-1,1,a,-1)\ \},$$

which provides additional evidence to support a conjecture of Ismailescu and Shim.

1. Introduction

For a given sequence $\mathcal{X} = (x_n)_{n\geq 0}$, and $k \in \mathbb{Z}$, we let $\mathcal{X} + k$ denote the k-shifted sequence $(x_n + k)_{n\geq 0}$. We say that $\mathcal{X} + k$ is *primefree* if $|x_n + k|$ is not prime for all $n \geq 0$ and, to rule out trivial situations, we also require that $\mathcal{X} + k$ is not a constant sequence, and that no single prime divides all terms of $\mathcal{X} + k$. For $a, b, w_0, w_1 \in \mathbb{Z}$, we let $\mathcal{W} := \mathcal{W}(w_0, w_1, a, b) = (w_n)_{n\geq 0}$ denote the general linear binary recurrence that is defined by

$$w_n = aw_{n-1} + bw_{n-2} \quad \text{for } n \geq 2. \tag{1.1}$$

With this notation, $\mathcal{W}(0,1,a,b)$ is known in the literature as a *Lucas sequence of the first kind*, whereas $\mathcal{W}(2,a,a,b)$ is a *Lucas sequence of the second kind*. For convenience and brevity of notation, we define:

$$\begin{aligned}
\mathcal{U}(a,b) &:= \mathcal{W}(0,1,a,b)\\
\mathcal{V}(a,b) &:= \mathcal{W}(2,a,a,b)\\
\mathcal{S}(a,b) &:= \mathcal{W}\left(1,\sqrt{-b},a,b\right), \quad \text{where } b=-c^2 \text{ for some integer } c\\
\mathcal{T}(a,b) &:= \mathcal{W}\left(-1,\sqrt{-b},a,b\right), \quad \text{where } b=-c^2 \text{ for some integer } c.
\end{aligned}$$

Remark 1.1. The sequences $\mathcal{S}(a,b)$ and $\mathcal{T}(a,b)$ are defined in [10].

It has been shown recently [5, 8] for any sequence $\mathcal{X} \in \{\ \mathcal{U}(a,1),\ \mathcal{V}(a,1)\ \}$, that there exist infinitely many positive integers k such that both of the shifted sequences $\mathcal{X} \pm k$ are simultaneously primefree, and moreover, each term has at least two distinct prime divisors. In this article, we extend these previous results by establishing the following theorem:

Theorem 1.2. *Let $a \in \mathbb{Z}$, and let*

$$\mathcal{X} \in \{\ \mathcal{U}(a,-1),\ \mathcal{V}(a,-1),\ \mathcal{S}(a,-1),\ \mathcal{T}(a,-1)\ \}. \tag{1.2}$$

Then, with the exceptions $\mathcal{X} \in \{\ \mathcal{U}(\pm 2,-1),\ \mathcal{V}(2,-1),\ \mathcal{S}(\pm 2,-1),\ \mathcal{T}(2,-1)\ \}$*, there exist infinitely many integers* k*, such that both of the shifted sequences* $\mathcal{X} \pm k$ *are simultaneously primefree. Moreover, there exist infinitely many values of* k *such that every term in both of the primefree shifted sequences* $\mathcal{X} \pm k$ *has at least two distinct prime factors.*

Theorem 1.2 provides additional evidence to support the following conjecture of Ismailescu and Shim [6].

Conjecture 1.3. *Let* $\mathcal{W}$ *be an integer sequence as defined in* (1.1)*. Further assume that* $\lim_{n\to\infty} |w_n| = \infty$*. Then, there exist integers* k *that cannot be written in the form* $\pm w_n \pm p$ *for any* n *and any prime* p*.*

Maple and Magma were used to perform some of the calculations in this article.

2. Preliminaries

There are three key ingredients used in our investigations: the concept of a covering system of the integers, the notion of a primitive divisor, and the periodicity modulo a prime of the sequences $\mathcal{X}$ given in (1.2).

A *covering system*, or simply a *covering*, of the integers is a system of congruences such that every integer satisfies at least one of the congruences. This concept is originally due to Erdős [2]. All coverings in this article are finite, and it will be convenient to denote the elements in a covering $\mathcal{C}$ as ordered pairs (r, m), where $z \equiv r \pmod{m}$ is a congruence in the covering. For each sequence $\mathcal{X}$ of interest here, a covering system $\mathcal{C}^+_{\mathcal{X}}$ is constructed to find values of k such that $\mathcal{X} + k$ is primefree; and a separate covering system $\mathcal{C}^-_{\mathcal{X}}$ is also constructed to determine values of k for which the sequence $\mathcal{X} - k$ is primefree. However, we require that these two coverings be consistent so that both shifted sequences are simultaneously primefree for the exact same values of k.

We define a *primitive divisor* of the term $x_n \in \mathcal{X}$ to be a prime p such that $x_n \equiv 0 \pmod{p}$ and $x_m \not\equiv 0 \pmod{p}$ for all $0 \le m < n$ with $x_m \ne 0$. We say that $\mathcal{X}$ is n*-defective* if $x_n \in \mathcal{X}$ fails to have a primitive divisor.

It is well-known that each of the sequences $\mathcal{X}$ in (1.2) is periodic modulo a prime p [3], and that there are many beautiful symmetries in the cycles modulo p of these sequences [14]. We exploit these symmetries to allow us to use multiple residues for a single modulus in the construction of the covering systems used in this article.

As mentioned in Section 1, the sequences $\mathcal{U}(a, 1)$ and $\mathcal{V}(a, 1)$ have been addressed in [8] and [5], respectively. The main results for these sequences were achieved treating a as a variable and using the fact that there exists a unique generic (a polynomial in the variable a) primitive divisor of each term in the so-called generic sequences $\widehat{\mathcal{U}}(a, 1)$ and $\widehat{\mathcal{V}}(a, 1)$ [11]. In this article, we use a slightly more general technique that does not rely on the existence of unique generic primitive divisors. For example, not every term of the generic sequence $\widehat{\mathcal{U}}(a, -1)$ contains a unique generic primitive divisor. To construct the needed covering systems here, we use the concept of a generic cycle modulo a generic primitive divisor, and exploit the symmetries present in these cycles. Treating a as a variable and given *any* polynomial $f(a)$, we define the *generic cycle modulo* f for the generic version $\widehat{\mathcal{X}}$ of any sequence $\mathcal{X}$, as given in (1.2), to be

$$\Gamma_{\widehat{\mathcal{X}}}(f) := [\widehat{x}_0 \pmod{f},\quad \widehat{x}_1 \pmod{f},\quad \ldots,\quad \widehat{x}_{m-1} \pmod{f}],$$

where (assuming that $m < \infty$), $\widehat{x}_{m+i} \equiv \widehat{x}_r \pmod{f}$ for all $i \ge 0$ with $i \equiv r \pmod{m}$ and $r \in \{0, 1, \ldots, m-1\}$. We call $\left|\Gamma_{\widehat{\mathcal{X}}}(f)\right| := m$ the *generic period* of $\widehat{\mathcal{X}}$ modulo f.

Although the next lemma is essentially an observation, it will prove to be useful.

Lemma 2.1. *Let $\widehat{\mathcal{X}}$ be a generic version of any sequence $\mathcal{X}$, as given in* (1.2)*, and let $f(a)$ be a polynomial in the variable a such that $\left|\Gamma_{\widehat{\mathcal{X}}}(f)\right| < \infty$. For a particular value of a, if p is a prime with $f(a) \equiv 0 \pmod{p}$, then the period of $\mathcal{X}$ modulo p is a divisor of $\left|\Gamma_{\widehat{\mathcal{X}}}(f)\right|$.*

3. The Proof of Theorem 1.2

For $a \geq 0$, note that

$$u_n(-a) = (-1)^{n-1}u_n(a) \quad \text{and} \quad v_n(-a) = (-1)^n v_n(a)$$

for all $n \geq 0$, where $u_n(a)$, $u_n(-a)$, $v_n(a)$, and $v_n(-a)$ are, respectively, the nth terms of $\mathcal{U}(a,-1), \mathcal{U}(-a,-1)$, $\mathcal{V}(a,-1)$, and $\mathcal{V}(-a,-1)$. Thus, with the exception of the single sequence $\mathcal{V}(2,-1)$, we need only consider the cases $a \geq 0$ for $\mathcal{U}(a,-1)$ and $\mathcal{V}(a,-1)$. Since $\mathcal{V}(2,-1)$ is a constant sequence, we must address the sequence $\mathcal{V}(-2,-1)$ separately.

Also, for all $n \geq 1$, the sequence $\mathcal{V}(a,-1)$ is related to the sequence $\mathcal{U}(a,-1)$ by the relationship $v_n = u_{2n}/u_n$. Hence, the n-defective nature of $\mathcal{V}(a,-1)$ has an intimate connection to the n-defective nature of $\mathcal{U}(a,b)$. Then, using the n-defective tables given in [1], we deduce the following theorem.

Theorem 3.1. *For $a \geq 3$, the only pairs $[a,n]$ for which $\mathcal{U}(a,-1)$ is n-defective are $[a,1]$ and $[3,6]$. Consequently, for $a \geq 3$, the only pairs $[a,n]$ for which $\mathcal{V}(a,-1)$ is n-defective are $[2^c,1]$, with $c \geq 2$, and $[3,3]$.*

Throughout the remainder of this article, we let

$$\begin{array}{ll} \mathcal{U}(a,-1) = (u_n)_{n=0}^{\infty}, & \mathcal{V}(a,-1) = (v_n)_{n=0}^{\infty}, \\ \mathcal{S}(a,-1) = (s_n)_{n=0}^{\infty}, & \text{and} \quad \mathcal{T}(a,-1) = (t_n)_{n=0}^{\infty}. \end{array} \tag{3.1}$$

It turns out that the sequences $\mathcal{S}(a,-1)$ and $\mathcal{T}(a,-1)$ in (3.1) can be addressed simultaneously, and consequently, the proof of Theorem 1.2 is divided into three main parts.

3.1. The Proof for Shifting the Sequence $\mathcal{U}(a,-1)$.

Proof. We first address the cases $a \in \{0,1\}$. Observe that

$$\mathcal{U}(0,-1) = (0,1,0,-1,0,1,\ldots) \quad \text{and} \quad \mathcal{U}(1,-1) = (0,1,1,0,-1,-1,0,1,\ldots),$$

and consider the system of congruences for k:

$$\begin{array}{ll} k \equiv 0 \pmod{3} & k \equiv -1 \pmod{7} \\ k \equiv 0 \pmod{5} & k \equiv 1 \pmod{11} \\ \multicolumn{2}{c}{k \equiv 1 \pmod{2}.} \end{array} \tag{3.2}$$

Then, using the Chinese remainder theorem to solve (3.2), we get an infinite arithmetic progression of positive integers k such that all four of the sequences $\mathcal{U}(a,-1) \pm k$, where $a \in \{0,1\}$, are simultaneously (and nontrivially) primefree. The smallest positive value in this arithmetic progression is $k = 1035$. Moreover, it is easy to show that every term in any of these shifted sequences has at least two distinct prime divisors.

If $a = 2$, we get

$$\mathcal{U}(2,-1) = (0,1,2,3,4,5,\ldots) = \text{the nonnegative integers},$$

so that there does not exist an integer k for which the sequence $\mathcal{U}(2,-1) + k$ is primefree.

Now, let $a \geq 3$. Consider the list

$$N = [2,3,4,8,9,12,18,24,36]$$

of nine indices for $u_n \in \mathcal{U}(a,-1)$. By Theorem 3.1, each of the terms $u_{N[i]}$ has a primitive divisor for any $a \geq 3$. Let p_1 be a primitive divisor of $u_{N[1]} = u_2$. Since

$$\mathcal{U}(a,-1) = (0, 1, a, a^2-1, a^3-2a, a^4-3a^2+1, \ldots),$$

we see that $a \equiv 0 \pmod{p_1}$. Hence,

$$\mathcal{U}(a,-1) \pmod{p_1} = (0, 1, 0, -1, 0, 1, \ldots), \tag{3.3}$$

so that $\mathcal{U}(a,-1)$ has period 4 modulo p_1. Thus, we choose 4 as a modulus to build our coverings $\mathcal{C}_{\mathcal{U}}^{+}$ and $\mathcal{C}_{\mathcal{U}}^{-}$ for $\mathcal{U}(a,-1)+k$ and $\mathcal{U}(a,-1)-k$, respectively. Note in (3.3) that the residue 0 appears in locations $0 \pmod 4$ and $2 \pmod 4$ in the period of $\mathcal{U}(a,-1)$ modulo p_1, so that we can use (as we do) the two congruences $(0,4)$ and $(2,4)$ in each of the coverings $\mathcal{C}_{\mathcal{U}}^{+}$ and $\mathcal{C}_{\mathcal{U}}^{-}$. A similar analysis can be done for each $N[i]$. Continuing in this manner, we get the complete list

$$[4, 6, 8, 16, 18, 24, 36, 48, 72]$$

of moduli corresponding to the list of indices N. Taking advantage of the symmetries present in the cycles, we build the coverings

$$\begin{aligned}\mathcal{C}_{\mathcal{U}}^{+} = \{&(0,4),(2,4),(3,6),(5,8),(7,8),(9,16),(17,18),\\ &(17,24),(19,24),(11,36),(1,48),(59,72)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_{\mathcal{U}}^{-} = \{&(0,4),(2,4),(3,6),(1,8),(3,8),(7,16),(1,18),\\ &(5,24),(7,24),(25,36),(47,48),(13,72)\}\ .\end{aligned}$$

Let p_i denote a primitive divisor of $u_{N[i]}$, which is dependent on the value of a. Then, from $\mathcal{C}_{\mathcal{U}}^{+}$ and $\mathcal{C}_{\mathcal{U}}^{-}$, we get the following system of congruences for k:

$$\begin{array}{ll} k \equiv 0 \pmod{p_1} & k \equiv 1 \pmod{p_5} \\ k \equiv 0 \pmod{p_2} & k \equiv a^2 \pmod{p_6} \\ k \equiv 1 \pmod{p_3} & k \equiv -a^4+3a^2-2 \pmod{p_7} \\ k \equiv 1 \pmod{p_4} & k \equiv -1 \pmod{p_8} \\ \multicolumn{2}{c}{k \equiv a^{10}-9a^8+28a^6-35a^4+15a^2 \pmod{p_9}.} \end{array} \tag{3.4}$$

Using the Chinese remainder theorem to solve the system (3.4) produces an infinite arithmetic progression of desired values of k. This completes the proof for the sequence $\mathcal{U}(a,-1)$. $\square$

We provide an example to illustrate this case. Note that when there are two or more primitive divisors of a term, we choose the smallest one.

Example 3.2. $a = 3$.
We let

$$\mathcal{P} = [3, 2, 7, 47, 17, 23, 107, 1103, 103681]$$

be the list of chosen primitive divisors. Then, (3.4) *is*

$$\begin{array}{ll} k \equiv 0 \pmod 3 & k \equiv 1 \pmod{17} \\ k \equiv 0 \pmod 2 & k \equiv 9 \pmod{23} \\ k \equiv 1 \pmod 7 & k \equiv -56 \pmod{107} \\ k \equiv 1 \pmod{47} & k \equiv -1 \pmod{1103} \\ \multicolumn{2}{c}{k \equiv 17712 \pmod{103681},} \end{array}$$

and solving gives $k = 8435542774820400$ *as the smallest positive value.*

3.2. **The Proof for Shifting the Sequence $\mathcal{V}(a,-1)$.**

Proof. We address the cases $a \in \{0,1,\pm 2\}$ separately. Suppose first that $a=0$. Then,

$$\mathcal{V}(0,-1) = (2,0,-2,0,2,0,\ldots).$$

Consider the system of congruences for k:

$$\begin{array}{ll} k \equiv 0 \pmod{3} & k \equiv -2 \pmod{11} \\ k \equiv 0 \pmod{5} & k \equiv 2 \pmod{13} \\ k \equiv -2 \pmod{7} & k \equiv 2 \pmod{17} \\ \multicolumn{2}{c}{k \equiv 1 \pmod{2}.} \end{array} \tag{3.5}$$

It is straightforward to see, for any positive value in the solution to (3.5), that both sequences $\mathcal{V}(0,-1) \pm k$ are simultaneously primefree, and that every term in either of these shifted sequences has at least two distinct prime divisors. The smallest positive solution is $k = 64755$.

Next, suppose that $a=1$. Note that

$$\mathcal{V}(1,-1) = (2,1,-1,-2,-1,1,2,1,\ldots).$$

This case is similar to the case $a=0$. We solve the system

$$\begin{array}{ll} k \equiv -2 \pmod{3} & k \equiv 1 \pmod{11} \\ k \equiv -2 \pmod{5} & k \equiv 2 \pmod{13} \\ k \equiv -1 \pmod{7} & k \equiv 2 \pmod{17} \\ \multicolumn{2}{c}{k \equiv 1 \pmod{2}.} \end{array} \tag{3.6}$$

Then, as before, it is easy to check that every positive solution to (3.6) gives the desired result. The smallest positive solution is $k = 417913$.

Now, letting $a=2$ gives

$$\mathcal{V}(2,-1) = (2,2,2,2,\ldots).$$

It is then obvious that the shifted sequence $\mathcal{V}(2,-1)+k$ is constant for any value of k. Hence, no desired value of k exists in this case.

Since the sequence $\mathcal{V}(2,-1)$ is constant, we must address the case $a=-2$ separately. Observe that

$$\mathcal{V}(-2,-1) = (2,-2,2,-2,\ldots).$$

Then, solving the system

$$\begin{array}{ll} k \equiv -2 \pmod{3} & k \equiv 2 \pmod{7} \\ k \equiv -2 \pmod{5} & k \equiv 2 \pmod{11} \\ \multicolumn{2}{c}{k \equiv 1 \pmod{2}} \end{array} \tag{3.7}$$

produces the solution $k \equiv 1003 \pmod{2310}$, which satisfies the desired conditions. Note that we have added the congruence $k \equiv 1 \pmod{2}$ to the system (3.7) to avoid the trivial situation in which every term of the shifted sequence is divisible by 2.

Now, suppose that $a \geq 3$. We split the remaining values of a into two cases, depending on whether a is a power of 2. As in the proof for $\mathcal{U}(a,-1)$, for each particular value of a, we use primes p_i that are primitive divisors of certain terms $v_n \in \mathcal{V}(a,-1)$. However, to construct the coverings in the case of $\mathcal{V}(a,-1)$, we also require the use of primes q_i, which we refer to as *supplemental primes*, that are not primitive divisors of any term $v_n \in \mathcal{V}(a,-1)$. Associated to each of the primes p_i and q_i is a polynomial in the variable a, and a corresponding generic cycle. These polynomials are such that specialization at the particular value of a yields integers that are divisible by the corresponding prime. Most importantly, these polynomials are chosen so that their generic periods are small, since these periods are used as moduli in the construction of the coverings.

We first address the case when $a \neq 2^c$, for any positive integer c. Consider the list of seven indices $N = [1, 2, 5, 6, 10, 15, 30]$. For each i with $1 \leq i \leq 7$, let p_i be a primitive divisor of $v_{N[i]}$. The existence of these primes is guaranteed by Theorem 3.1. Also, by definition, these primes are all distinct. Note that, since $a \neq 2^c$, we get to use the odd primitive divisor p_1 of $v_{N[1]} = v_1 = a$. In this situation, we do have the ability to choose a unique generic primitive divisor F_i for each index $N[i]$, and we use these polynomials to calculate the corresponding generic cycles which, in turn, give us the generic periods $\left|\Gamma_{\widehat{\mathcal{V}}}(F_i)\right|$ to be used as moduli. With the exception of the generic cycles, this information is provided in Table 1.

p_i	$F_i(a)$	$\left\lvert\Gamma_{\widehat{\mathcal{V}}}(F_i)\right\rvert$
p_1	a	4
p_2	$a^2 - 2$	8
p_3	$a^4 - 5a^2 + 5$	20
p_4	$a^4 - 4a^2 + 1$	24
p_5	$a^8 - 8a^6 + 19a^4 - 12a^2 + 1$	40
p_6	$a^8 - 7a^6 + 14a^4 - 8a^2 + 1$	60
p_7	$a^{16} - 16a^{14} + 105a^{12} - 364a^{10} + 714a^8$ $-784a^6 + 440a^4 - 96a^2 + 1$	120

TABLE 1. Primitive Divisors and Generic Periods for the Case $a \neq 2^c$

We also require some supplemental primes $q_i := q_i(a)$ that are odd prime divisors of $f_i(a)$ as described in Table 2.

q_i	$f_i(a)$	$\left\lvert\Gamma_{\widehat{\mathcal{V}}}(f_i)\right\rvert$
q_1	$a^2 + a - 1$	5
q_2	$a^2 - a - 1$	10
q_3	$a^4 - a^3 - 4a^2 + 4a + 1$	15
q_4	$a^4 + a^3 - 4a^2 - 4a + 1$	30

TABLE 2. Supplemental Primes and Generic Periods for the Case $a \neq 2^c$

The necessity of these supplemental primes q_i introduces complications that were not present in the case $\mathcal{U}(a, -1)$. In particular, their utilization requires the verification of the following:

(1) the existence of odd prime divisors q_i for any particular value of $a \neq 2^c$,
(2) the primes q_i are distinct and different from the primes p_i,
(3) the generic periods are as given in Table 2.

Items (1), (2), and (3) can be verified irrespective of the restriction that $a \neq 2^c$. We begin by establishing (1). Since the methods are similar for each f_i in Table 2, we only provide details for f_3. To see that f_3 always has an odd prime divisor q_3 when $a \geq 3$, we observe that the Diophantine equation

$$a^4 - a^3 - 4a^2 + 4a + 1 = 2^y$$

is impossible modulo 2 if $y \geq 1$. Hence, the only solution is $y = 0$, from which we deduce that $a(a-1)(a-2)(a+2) = 0$, so that $a \in \{0, 1, 2\}$. Therefore, we may assume the existence of an odd prime divisor q_3 of $f_3(a)$, when $a \geq 3$.

For (2), it is somewhat tedious, although straightforward, to see that the primes q_i are distinct. For example, if $q_2 = q_1$, then q_1 divides $f_1 - f_2 = 2a$. Thus, q_1 divides a since q_1

is odd, which in turn implies that q_1 divides 1, a contradiction. This same method can be employed to establish that $q_i \neq p_j$ for any i and j.

For (3), we give details only for q_2 because the other cases are similar. A simple calculation using Maple gives

$$\Gamma_{\widehat{\mathcal{V}}(a,-1)}(f_2) = [2, a, a-1, -a+1, -a, -2, -a, -a+1, a-1, a]\,, \tag{3.8}$$

so that $\left|\Gamma_{\widehat{\mathcal{V}}}(f_2)\right| = 10$. Thus, from Lemma 2.1, the actual period γ_2 of $\mathcal{V}(a,-1)$ modulo q_2 for a particular value of a, is such that $\gamma_2 \in \{1, 2, 5, 10\}$. Since $a \geq 3$ and $a^2 - a - 1$ is odd, we see from (3.8) that $\gamma_2 \in \{1, 5\}$ is impossible, whereas $\gamma_2 = 2$ only when $a = 3$ with $q_2 = 5$. Regardless, we can add 10 to the list of moduli for our covering systems, and use, for example, the pair of congruences $(3, 10)$ and $(7, 10)$ for $\mathcal{C}^+$, with the pair of congruences $(2, 10)$ and $(8, 10)$ for $\mathcal{C}^-$. Because of redundancy in the covering system $\mathcal{C}^+$ here, the congruences $(3, 10)$ and $(7, 10)$ are not needed. As always, special care must be taken to avoid the occurrence of a trivial situation.

Combining the moduli $\left|\Gamma_{\widehat{\mathcal{V}}}(F_i)\right|$ from Table 1 and the moduli $\left|\Gamma_{\widehat{\mathcal{V}}}(f_i)\right|$ from Table 2, we arrive at the complete list of moduli

$$[4, 8, 20, 24, 40, 60, 120, 5, 10, 15, 30].$$

Then, using the techniques described above, we construct the coverings

$$\begin{aligned}\mathcal{C}^+_{\mathcal{V}} = \{&(1,4),(3,4),(2,8),(6,8),(8,20),(12,20),(8,24),(16,24),\\ &(4,40),(36,40),(20,60),(40,60),(24,120),(96,120),(0,30)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}^-_{\mathcal{V}} = \{&(1,4),(3,4),(2,8),(6,8),(4,24),(20,24),\\ &(16,40),(24,40),(0,5),(2,10),(8,10),(6,15),(9,15)\}\,.\end{aligned}$$

Using the coverings $\mathcal{C}^+_{\mathcal{V}}$ and $\mathcal{C}^-_{\mathcal{V}}$, with the generic cycles $\Gamma_{\widehat{\mathcal{V}}}(F_i)$ and $\Gamma_{\widehat{\mathcal{V}}}(f_i)$, we construct a system of congruences for k. For example, since

$$\Gamma_{\widehat{\mathcal{V}}}(f_1) = [2, a, -a-1, -a-1, a],$$

and that we have used the congruence $(0, 5)$ in $\mathcal{C}^-_{\mathcal{V}}$, we solve the congruence $2 - k \equiv 0 \pmod{q_1}$ to add the congruence $k \equiv 2 \pmod{q_1}$ to the system. Continuing in this manner produces the system

$$\begin{array}{ll}
k \equiv 0 \pmod{p_1} & k \equiv 1 \pmod{p_6}\\
k \equiv 0 \pmod{p_2} & k \equiv 2 \pmod{q_1}\\
k \equiv a^2 - 2 \pmod{p_3} & k \equiv a - 1 \pmod{q_2}\\
k \equiv 1 \pmod{p_4} & k \equiv -a^3 + 3a - 1 \pmod{q_3}\\
k \equiv -a^4 + 4a^2 - 2 \pmod{p_5} & k \equiv -2 \pmod{q_4}\\
\multicolumn{2}{l}{k \equiv -a^{12} + 12a^{10} - 54a^8 + 112a^6 - 105a^4 + 36a^2 - 1 \pmod{p_7}}\\
\multicolumn{2}{c}{k \equiv 1 \pmod{2}.}
\end{array} \tag{3.9}$$

Note that the congruence $k \equiv 1 \pmod{2}$ in (3.9) is only needed to avoid a trivial situation if $a \equiv 0 \pmod{2}$.

Then, for any particular value of $a \neq 2^c$, we can use the Chinese remainder theorem to solve the system (3.9) to get an infinite arithmetic progression of desired values of k. This completes the proof for the case $\mathcal{V}(a, -1)$ when $a \neq 2^c$.

Now suppose that $a = 2^c$, for some integer $c \geq 2$. In this case, we use the indices $N = [2, 3, 6]$ and the primes p_1, p_2, and p_3 that are odd primitive divisors of the terms

$$v_2 = a^2 - 2, \quad v_3 = a\left(a^2 - 3\right), \quad \text{and} \quad v_6 = \left(a^2 - 2\right)\left(a^4 - 4a^2 + 1\right), \tag{3.10}$$

respectively. The existence of these primitive divisors is guaranteed by Theorem 3.1. In particular, although we cannot use the index $n = 1$ since $a = 2^c$, we do get to use the index $n = 3$ since $a \neq 3$. The corresponding moduli are the generic periods of the generic primitive divisors. This information is given in Table 3.

p_i	$F_i(a)$	$\left\lvert\Gamma_{\widehat{\mathcal{V}}}(F_i)\right\rvert$
p_1	$a^2 - 2$	8
p_2	$a^2 - 3$	12
p_3	$a^4 - 4a^2 + 1$	24

Table 3. Primitive Divisors and Generic Periods for the Case $a = 2^c$

We also require some supplemental primes in this case. They, with the corresponding polynomials and their generic periods, are given in Table 4.

q_i	$f_i(a)$	$\left\lvert\Gamma_{\widehat{\mathcal{V}}}(f_i)\right\rvert$
q_1	$a^2 + a - 1$	5
q_2	$a^2 - a - 1$	10
q_3	$a^4 - a^3 - 4a^2 + 4a + 1$	15
q_4	$a^4 + a^3 - 4a^2 - 4a + 1$	30
q_5	$a + 2$	2
q_6	$a + 1$	3

Table 4. Supplemental Primes and the Generic Periods for the Case $a = 2^c$

Note that there is some overlap of the moduli in this case and the previous case $a \neq 2^c$. Also, recall that the verifications for Table 2 needed for the previous case did not actually require the restriction that $a \neq 2^c$. Consequently, here we only need to verify the following three items:

(1) the odd prime divisors q_5 and q_6 exist,
(2) $q_5 \neq q_6$, and $q_5, q_6 \notin \{p_1, p_2, p_3, q_1, q_2, q_3, q_4\}$,
(3) the generic periods $\left\lvert\Gamma_{\widehat{\mathcal{V}}}(f_5)\right\rvert$ and $\left\lvert\Gamma_{\widehat{\mathcal{V}}}(f_6)\right\rvert$ are as given in Table 4.

Using the restriction that $a = 2^c$, with $c \geq 2$, we see that the Diophantine equations $2^c + 1 = 2^y$ and $2^c + 2 = 2^y$ have no solutions, which verifies (1).

Clearly, $q_5 \neq q_6$, and as before, it is tedious and laborious, but straightforward, to verify the other conditions of (2). For example, if $q_5 = q_2$, then q_5 divides $f_2 - (a-3)f_5 = 5$. Hence, $q_5 = q_2 = 5$. Now consider the Diophantine equation

$$2^x + 1 = 5^y. \tag{3.11}$$

Reduction of (3.11) modulo 3 shows that $x \equiv 0 \pmod 2$ and $y \equiv 1 \pmod 2$. However, if $x \geq 3$, then reduction of (3.11) modulo 8 yields the contradiction that $y \equiv 0 \pmod 2$. We conclude that $x \in \{0, 2\}$, and checking these cases reveals that the only solution to (3.11) is $x = 2$ and $y = 1$. Thus, the Diophantine equation

$$a + 2 = 2^c + 2 = 2 \cdot 5^y$$

has only the solution $(c,y)=(3,1)$, which implies that

$$a^2-a-1=2^{2c}-2^c-1=5\cdot 11.$$

Therefore, we see that we can choose $q_5=5$ and $q_2=11$ in this situation. Using similar reasoning, it can be shown that if $q_6=q_3$, then $q_6=q_3=5$. But in this case, we have from the solutions to (3.11) that the Diophantine equation

$$a+1=2^c+1=5^y$$

has only the solution $(c,y)=(2,1)$, which implies that we can actually choose $q_6=5$ and $q_3=29$. These two examples represent two of the most difficult cases. In most situations, we have that $\gcd(f_i,F_j)=\gcd(f_i,f_j)=1$.

The easily-derived generic cycles for f_5 and f_6 are, respectively, $[2,-2]$ and $[2,-1,-1]$, which verifies (3).

We can now use the complete list of moduli

$$[8,12,24,5,10,15,30,2,3],$$

with the generic cycles, to construct the coverings

$$\mathcal{C}_{\mathcal{V}}^{+}=\{(0,2),(0,3),(1,8),(7,8),(5,12),(7,12),(11,24),(13,24)\}$$

and

$$\mathcal{C}_{\mathcal{V}}^{-}=\{(1,2),(1,5),(4,5),(2,10),(8,10),(5,15),(10,15),(0,30)\}\,.$$

Using the coverings $\mathcal{C}_{\mathcal{V}}^{+}$ and $\mathcal{C}_{\mathcal{V}}^{-}$, with the generic cycles $\Gamma_{\widehat{\mathcal{V}}}(F_i)$ and $\Gamma_{\widehat{\mathcal{V}}}(f_i)$, we produce the following system of congruences for k:

$$\begin{array}{ll}
k\equiv -a \pmod{p_1} & k\equiv -1 \pmod{q_3}\\
k\equiv a \pmod{p_2} & k\equiv 2 \pmod{q_4}\\
k\equiv a \pmod{p_3} & k\equiv -2 \pmod{q_5}\\
k\equiv a \pmod{q_1} & k\equiv -2 \pmod{q_6}\\
k\equiv a-1 \pmod{q_2} & k\equiv 1 \pmod{2}
\end{array}\tag{3.12}$$

Note that, since $a\equiv 0 \pmod 2$, the congruence $k\equiv 1 \pmod 2$ has been added to avoid a trivial situation. Then, for any particular value of $a=2^c$, with $c\ge 2$, we can use the Chinese remainder theorem to solve the system (3.12) to get an infinite arithmetic progression of desired values of k. This completes the proof for the case $\mathcal{V}(a,-1)$ when $a=2^c$, and hence the proof of this part of the theorem for the sequence $\mathcal{V}(a,-1)$. $\square$

Remark 3.3. The density of supplemental primes related to certain binary linear recurrence sequences has been investigated by several authors [4, 9, 12, 13].

We provide some examples to illustrate shifting the sequence $\mathcal{V}(a,-1)$. Note that, for a particular i, when there is a choice of two or more primes for p_i or q_i, we choose the smallest one.

Example 3.4. $a=3$.
We let

$$[3,7,41,23,2161,2521,241,11,5,31,61]$$

be the list of chosen primes p_i and q_i corresponding to the moduli

$$[4,8,20,24,40,60,120,5,10,15,30].$$

Then (3.9) *becomes*

$$\begin{array}{ll} k \equiv 0 \pmod{3} & k \equiv 1 \pmod{2521} \\ k \equiv 0 \pmod{7} & k \equiv 190 \pmod{241} \\ k \equiv 7 \pmod{41} & k \equiv 2 \pmod{11} \\ k \equiv 1 \pmod{23} & k \equiv 2 \pmod{5} \\ k \equiv 2114 \pmod{2161} & k \equiv 12 \pmod{31} \\ \multicolumn{2}{c}{k \equiv 59 \pmod{61},} \end{array}$$

and solving gives $k = 1258084716430844337$ *as the smallest positive value.*

Example 3.5. $a = 6$.
We let

$$[3, 17, 19, 1153, 241, 601, 1774998973441, 41, 29, 31, 5]$$

be the list of chosen primes p_i *and* q_i *corresponding to the moduli*

$$[4, 8, 20, 24, 40, 60, 120, 5, 10, 15, 30].$$

With a slight rearrangement for space considerations, (3.9) *becomes*

$$\begin{array}{ll} k \equiv 0 \pmod{3} & k \equiv 1 \pmod{601} \\ k \equiv 0 \pmod{17} & k \equiv 2 \pmod{41} \\ k \equiv 15 \pmod{19} & k \equiv 5 \pmod{29} \\ k \equiv 1 \pmod{1153} & k \equiv 18 \pmod{31} \\ k \equiv 51 \pmod{241} & k \equiv 3 \pmod{5} \\ \multicolumn{2}{l}{k \equiv 1773462176640 \pmod{1774998973441}.} \end{array} \tag{3.13}$$

Adding the congruence $k \equiv 1 \pmod{2}$ *to* (3.13) *and solving gives*

$$k = 102626069639605425993401562843$$

as the smallest positive value.

Example 3.6. $a = 1024 = 2^{10}$.
We let

$$[524287, 1048573, 3361, 1049599, 1047551, 5581, 29, 3, 5]$$

be the list of chosen primes p_i *and* q_i *corresponding to the moduli*

$$[8, 12, 24, 5, 10, 15, 30, 2, 3].$$

Then (3.12) *becomes*

$$\begin{array}{ll} k \equiv 523263 \pmod{524287} & k \equiv 5580 \pmod{5581} \\ k \equiv 1024 \pmod{1048573} & k \equiv 2 \pmod{29} \\ k \equiv 1024 \pmod{3361} & k \equiv 1 \pmod{3} \\ k \equiv 1024 \pmod{1049599} & k \equiv 3 \pmod{5} \\ k \equiv 1023 \pmod{1047551} & k \equiv 1 \pmod{2} \end{array} \tag{3.14}$$

Solving (3.14) *gives*

$$k = 1507357333171071095692796006375503$$

as the smallest positive value.

3.3. The Proof for Shifting the Sequences $\mathcal{S}(a,-1)$ and $\mathcal{T}(a,-1)$.

Proof. For any $a \in \mathbb{Z}$, note that

$$s_n(a) = (-1)^{n-1} t_n(-a)$$

for all $n \geq 0$, where $s_n(a)$ is the nth term of $\mathcal{S}(a,-1)$, and $t_n(-a)$ is the nth term of $\mathcal{T}(-a,-1)$. Thus, addressing $\mathcal{S}(a,-1)$ for any $a \in \mathbb{Z}$ will handle $\mathcal{T}(-a,-1)$ simultaneously, provided $\mathcal{S}(a,-1)$ is nonconstant. Since the sequence $\mathcal{S}(a,-1)$ is constant only for $a = 2$, the sequence $\mathcal{T}(-2,-1)$ must be addressed separately. In general, we provide fewer details here because the techniques are similar to the previous cases. The proof is divided into four main parts:

(1) $\mathcal{S}(a,-1)$ with $a \in \mathcal{E} = \{0, \pm 1, \pm 2, \pm 3, -4, 8\}$ (treated individually)
(2) $\mathcal{T}(-2,-1)$
(3) $\mathcal{S}(a,-1)$ with $|a| \geq 5$, $|a| \neq 2^c$
(4) $\mathcal{S}(a,-1)$ with $|a| = 2^c$, $a \notin \mathcal{E}$.

As for the previous sequence $\mathcal{V}(a,-1)$, here we also require the use of supplemental primes q_i for (3) and (4). Consequently, certain conditions, such as that the primes q_i can be chosen to be distinct, must be verified. Additionally, special care must be taken to avoid trivial situations. For example, if $a \equiv 2 \pmod 3$, then $s_n \equiv 1 \pmod 3$ for every term $s_n \in \mathcal{S}(a,-1)$. Hence, we must avoid the use of congruences in our coverings that produce the congruence $k \equiv 2 \pmod 3$, since then $s_n + k \equiv 0 \pmod 3$ for all $n \geq 0$. As previously mentioned, in most situations, we leave the details of these verifications to the reader.

3.3.1. *Proof of Part* (1). Since the cases $a \in \{0, \pm 1\}$ are straightforward and similar to previous situations, we simply provide the arithmetic progressions for k:

$$a = \begin{cases} 0 & k \equiv 11 \pmod{30} \\ 1 & k \equiv 21 \pmod{2310} \\ -1 & k \equiv 37 \pmod{51870}. \end{cases}$$

For $a = -2$, we claim that no value of k exists such that $|s_n \pm k|$ is composite for all $n \geq 0$. Observe that

$$\mathcal{S}(-2,-1) = \left((-1)^{n+1}(2n-1)\right)_{n=0}^{\infty}.$$

Thus, if such a value of k exists, we must have $k \equiv 0 \pmod 2$ to avoid the trivial situation in which all terms $s_n + k$ are divisible by 2. But, note for any $k \equiv 0 \pmod 2$ and prime $p > k$, we see that there exists n such that either $p - k = s_n$ or $k - p = s_n$. In either case, we have violated the desired composite condition, and the claim is established. From this analysis, we can also conclude that no such value of k exists such that every term of $\mathcal{T}(2,-1) + k$ is composite.

For $a = 2$, we have that $\mathcal{S}(2,-1) = (1, 1, 1, \ldots)$, and it is clear that no such desired value of k exists in this case.

For the other values of $a \in \mathcal{E}$, it will be convenient to indicate the congruences in the coverings as ordered triples (r, m, p), where $x \equiv r \pmod m$ is an actual congruence in the covering, and p is the corresponding prime such that m is the period of $\mathcal{S}(a,-1)$ modulo p.

For $a = -3$, we construct the coverings:

$$\mathcal{C}_{\mathcal{S}}^{+} = \{(0,3,2), (1,3,2), (2,9,17), (8,9,17), (5,18,19), (14,18,19)\},$$

$$\mathcal{C}_{\mathcal{S}}^{-} = \{(0,3,2), (1,3,2), (2,4,3), (3,4,3), (4,8,7), (5,8,7), (8,24,23), (17,24,23)\}.$$

From the actual residues in each cycle, these coverings produce the system

$$\begin{array}{ll} k \equiv 1 \pmod{2} & k \equiv 2 \pmod{3} \\ k \equiv 4 \pmod{17} & k \equiv 6 \pmod{7} \\ k \equiv 0 \pmod{19} & k \equiv 16 \pmod{23}, \end{array}$$

which has $k = 163799$ as its least positive integer solution.

For $a = 3$, we construct the coverings:

$$\mathcal{C}_{\mathcal{S}}^{+} = \{(0,3,2),(1,3,2),(0,5,11),(1,5,11),(3,10,5),(8,10,5),\\ (2,15,31),(14,15,31)\},$$

$$\mathcal{C}_{\mathcal{S}}^{-} = \{(0,3,2),(1,3,2),(2,4,3),(3,4,3),(0,8,7),(1,8,7),\\ (5,24,23),(20,24,23)\}.$$

Using these coverings, we construct the system

$$\begin{array}{ll} k \equiv 1 \pmod{2} & k \equiv 29 \pmod{31} \\ k \equiv 10 \pmod{11} & k \equiv 2 \pmod{3} \\ k \equiv 0 \pmod{5} & k \equiv 1 \pmod{7}, \\ \multicolumn{2}{c}{k \equiv 11 \pmod{23},} \end{array}$$

which has $k = 212255$ as its least positive integer solution.

Remark 3.7. Since the sequence $\mathcal{S}(3,-1)$ is equal to the sequence $\{F_{2n-1}\}_{n=0}^{\infty}$, where F_m is the mth Fibonacci number, the values of k found in either [6] or [7] for the Fibonacci sequence could have been used in this case. However, the coverings constructed here are much smaller, and so we present them for completeness.

For $a = -4$, we construct the coverings:

$$\mathcal{C}_{\mathcal{S}}^{+} = \{(0,5,11),(1,5,11),(2,6,5),(5,6,5),(0,8,7),(1,8,7),(3,10,19),\\ (8,10,19),(0,12,13),(1,12,13),(2,20,181),(19,20,181),(10,24,193),\\ (15,24,193),(7,30,29),(24,30,29),(4,40,37441),(37,40,37441),\\ (9,60,61),(52,60,61),(27,120,1201),(94,120,1201)\},$$

$$\mathcal{C}_{\mathcal{S}}^{-} = \{(2,6,5),(5,6,5),(4,8,7),(5,8,7),(0,9,17),(1,9,17),(3,10,19),\\ (8,10,19),(6,12,13),(7,12,13),(0,15,241),(1,15,241),(7,18,53),\\ (12,18,53),(9,20,181),(12,20,181),(3,24,193),(22,24,193),\\ (9,30,29),(22,30,29),(3,36,37),(34,36,37),(17,40,37441),\\ (24,40,37441),(6,45,89),(40,45,89),(21,60,661),(40,60,661),\\ (34,120,1201),(87,120,1201)\}.$$

Using these coverings, we construct the system

$$\begin{array}{ll} k \equiv 0 \pmod{5} & k \equiv 34 \pmod{53} \\ k \equiv 6 \pmod{7} & k \equiv 14 \pmod{61} \\ k \equiv 10 \pmod{11} & k \equiv 79 \pmod{89}, \\ k \equiv 12 \pmod{13} & k \equiv 5 \pmod{181} \\ k \equiv 1 \pmod{17} & k \equiv 19 \pmod{193} \\ k \equiv 0 \pmod{19} & k \equiv 1 \pmod{241} \\ k \equiv 21 \pmod{29} & k \equiv 171 \pmod{661} \\ k \equiv 19 \pmod{37} & k \equiv 58 \pmod{1201} \\ \multicolumn{2}{c}{k \equiv 71 \pmod{37441},} \end{array}$$

which has

$$k = 598785063474098287870556862489 0$$

as its least positive integer solution.

For $a = 8$, we construct the coverings:

$$\begin{aligned} \mathcal{C}_{\mathcal{S}}^{+} = \{&(0,5,71),(1,5,71),(3,6,7),(4,6,7),(0,8,31),(1,8,31),(3,10,5),\\ &(8,10,5),(4,10,11),(7,10,11),(2,20,19),(19,20,19),(5,24,23),\\ &(20,24,23),(12,30,29),(19,30,29)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{\mathcal{S}}^{-} = \{&(0,6,7),(1,6,7),(4,8,31),(5,8,31),(3,10,5),(8,10,5),(2,10,11),\\ &(9,10,11),(3,12,61),(10,12,61),(5,15,3361),(11,15,3361),\\ &(9,24,167),(16,24,167),(14,30,149),(17,30,149)\}. \end{aligned}$$

From these coverings, we derive the system

$$\begin{array}{ll} k \equiv 70 \pmod{71} & k \equiv 18 \pmod{23} \\ k \equiv 1 \pmod{7} & k \equiv 27 \pmod{29} \\ k \equiv 30 \pmod{31} & k \equiv 55 \pmod{61}, \\ k \equiv 0 \pmod{5} & k \equiv 48 \pmod{3361} \\ k \equiv 7 \pmod{11} & k \equiv 68 \pmod{167} \\ k \equiv 12 \pmod{19} & k \equiv 142 \pmod{149} \end{array}$$

which has

$$k = 16179664513290603970$$

as its least positive integer solution.

3.3.2. *Proof of Part* (2). Observe that

$$\mathcal{T}(-2,-1) = (-1,1,-1,1,-1,1,\ldots),$$

consider the system

$$\begin{array}{ll} k \equiv 0 \pmod{2} & k \equiv 1 \pmod{5} \\ k \equiv 1 \pmod{3} & k \equiv -1 \pmod{7} \\ \multicolumn{2}{c}{k \equiv -1 \pmod{11}.} \end{array}$$

The smallest positive solution is $k = 76$. For any positive solution k, it is straightforward to see that each term of $\mathcal{T}(-2,-1) \pm k$ has at least two distinct prime divisors.

p_i or q_i	$h_i(a)$	$\left\lvert\Gamma_{\widehat{\mathcal{S}}}(h_i)\right\rvert$
q_1	a	4
q_2	a^2+a-1	5
p_1	$a-1$	6
q_3	a^2-2	8
p_2	a^2-a-1	10
q_4	a^2-3	12
q_5	$a^4-a^3-4a^2+4a+1$	15
q_6	a^4-5a^2+5	20
q_7	a^4-4a^2+1	24

TABLE 5. Primitive Divisors p_i and Supplemental Primes q_i, with the Generic Periods of $\widehat{\mathcal{S}}$ modulo h_i for the Case $|a| \neq 2^c$ with $|a| \geq 5$

3.3.3. *Proof of Part* (3). Now suppose that $|a| \geq 5$ and $|a| \neq 2^c$ for any positive integer c. We have consolidated the primitive divisors p_i and supplemental prime divisors q_i into Table 5.

We use h_i to indicate a generic primitive divisor (associated with some p_i) or some other irreducible polynomial (associated with some q_i). As mentioned earlier, many of the verification details, as described in Section 3.2 (such as proving that all primes p_i and q_i can be chosen so that they are distinct), are routine, and we leave most of them to the reader. As an example for the purpose of illustration, we show that q_7 can be chosen so that $q_7 \geq 3$ to ensure that $q_7 \neq p_1 = 2$ in the event that $a = 2^c + 1$. To see this, consider the Diophantine equation $a^4 - 4a^2 + 1 = 2^y$, which can be written as

$$\left(a^2-2\right)^2 = 2^y + 3. \tag{3.15}$$

If $y \geq 2$, then (3.15) is impossible modulo 4. Then, checking $y = 0$ and $y = 1$ in (3.15) reveals that the only solutions are $a = 0$ and $a = 2$. Hence, q_7 can be chosen to be odd.

We construct the coverings:

$$\mathcal{C}_{\mathcal{S}}^{+} = \{(2,4),(3,4),(2,6),(5,6),(0,8),(1,8),(0,12),(1,12),(4,24),(21,24)\},$$

$$\begin{aligned}\mathcal{C}_{\mathcal{S}}^{-} = \{&(0,4),(1,4),(0,5),(1,5),(2,6),(5,6),(2,10),(9,10),(3,15),(13,15),\\ &(7,20),(14,20)\}.\end{aligned}$$

These coverings, with the generic cycles corresponding to the information given in Table 5, yield the system

$$\begin{array}{ll} k \equiv 1 \pmod{q_1} & k \equiv a-1 \pmod{p_2} \\ k \equiv 1 \pmod{q_2} & k \equiv -1 \pmod{q_4} \\ k \equiv 0 \pmod{p_1} & k \equiv a^2-a-1 \pmod{q_5}, \\ k \equiv -1 \pmod{q_3} & k \equiv -a^3+a^2+2a-1 \pmod{q_6} \\ \multicolumn{2}{c}{k \equiv -a^3+a^2+2a-1 \pmod{q_7}.} \end{array} \tag{3.16}$$

3.3.4. *Proof of Part* (4). Now suppose that $|a| = 2^c$ for some positive integer c and $a \notin \mathcal{E}$. Following the format of Section (3.3.3), we have consolidated the primitive divisors p_i and supplemental prime divisors q_i into Table 6. As before, the details of the usual necessary verifications are omitted.

p_i or q_i	$h_i(a)$	$\lvert\Gamma_{\widehat{\mathcal{S}}}(h_i)\rvert$
q_1	$a+1$	3
q_2	a^2+a-1	5
p_1	$a-1$	6
p_2	a^2-a-1	10
q_3	a^2-3	12
q_4	$a^4-a^3-4a^2+4a+1$	15
q_5	a^4-5a^2+5	20
p_3	$a^4+a^3-4a^2-4a+1$	30

TABLE 6. Primitive Divisors p_i and Supplemental Primes q_i, with the Generic Periods of $\widehat{\mathcal{S}}$ modulo h_i for the Case $|a| = 2^c$ with $a \notin \mathcal{E}$

The coverings we construct here are:

$$\mathcal{C}_{\mathcal{S}}^{+} = \{(0,3),(1,3),(2,6),(5,6)\}\ ,$$

$$\begin{aligned}\mathcal{C}_{\mathcal{S}}^{-} = \{&(0,5),(1,5),(2,6),(5,6),(2,10),(9,10),(3,12),(10,12),(3,15)\\ &(13,15),(4,20),(17,20),(7,30),(24,30)\}\ .\end{aligned}$$

Combining the information here, we arrive at the following system:

$$\begin{array}{ll} k \equiv -1 \pmod{q_1} & k \equiv -a+2 \pmod{q_3} \\ k \equiv 1 \pmod{q_2} & k \equiv a^2-a-1 \pmod{q_4} \\ k \equiv 0 \pmod{p_1} & k \equiv a^3-a^2-2a+1 \pmod{q_5}, \\ k \equiv a-1 \pmod{p_2} & k \equiv -a^3+a^2+3a-2 \pmod{p_3} \\ \multicolumn{2}{c}{k \equiv -a^3+a^2+2a-1 \pmod{q_7}.} \end{array} \tag{3.17}$$

Finally, given that $\mathcal{X}$ is a sequence in (1.2) for which there exist infinitely many positive integers k such that every term of each sequence $\mathcal{X} \pm k$ is composite, the same argument used in [8] and [5], to prove the existence of infinitely many positive integers k such that every term of each sequence $\mathcal{X} \pm k$ has at least two distinct prime divisors, can be used here to establish the analogous result. Since the details are identical, we refer the reader to either of those references. □

We provide some examples for shifting the sequence $\mathcal{S}(a,-1)$.

Example 3.8. $a = 5$.
Following Section 3.3.3, we choose the list

$$[5, 29, 2, 23, 19, 11, 421, 101, 263]$$

of primes p_i and q_i corresponding to the moduli

$$[4, 5, 6, 8, 10, 12, 15, 20, 24].$$

Then (3.16) *becomes*

$$\begin{array}{ll} k \equiv 1 \pmod{5} & k \equiv 4 \pmod{19} \\ k \equiv 1 \pmod{29} & k \equiv -1 \pmod{11} \\ k \equiv 0 \pmod{2} & k \equiv 19 \pmod{421} \\ k \equiv -1 \pmod{23} & k \equiv -91 \pmod{101} \\ \multicolumn{2}{c}{k \equiv -91 \pmod{263},} \end{array}$$

and solving gives $k = 3922571199216$ *as the smallest positive value.*

Example 3.9. $a = -5$.
Following Section 3.3.3, we choose the list

$$[5, 19, 3, 23, 29, 11, 631, 101, 263]$$

of primes p_i *and* q_i *corresponding to the moduli*

$$[4, 5, 6, 8, 10, 12, 15, 20, 24].$$

Then (3.16) *becomes*

$$\begin{array}{ll} k \equiv 1 \pmod{5} & k \equiv -6 \pmod{29} \\ k \equiv 1 \pmod{19} & k \equiv -1 \pmod{11} \\ k \equiv 0 \pmod{3} & k \equiv 29 \pmod{631} \\ k \equiv -1 \pmod{23} & k \equiv 139 \pmod{101} \\ \multicolumn{2}{c}{k \equiv 139 \pmod{263},} \end{array}$$

and solving gives $k = 2766221917656$ *as the smallest positive value.*

Example 3.10. $a = 4$.
Following Section 3.3.4, we choose the list

$$[5, 19, 3, 11, 13, 29, 181, 241]$$

of primes p_i *and* q_i *corresponding to the moduli*

$$[3, 5, 6, 10, 12, 15, 20, 30].$$

Then (3.17) *becomes*

$$\begin{array}{ll} k \equiv -1 \pmod{5} & k \equiv -2 \pmod{13} \\ k \equiv 1 \pmod{19} & k \equiv 11 \pmod{29} \\ k \equiv 0 \pmod{3} & k \equiv 41 \pmod{181} \\ k \equiv 3 \pmod{11} & k \equiv -38 \pmod{241} \end{array}$$

and solving gives $k = 36033026274$ *as the smallest positive value.*

Example 3.11. $a = -16$.
Following Section 3.3.4, we choose the list

$$[5, 239, 17, 271, 23, 13709, 359, 1951]$$

of primes p_i *and* q_i *corresponding to the moduli*

$$[3, 5, 6, 10, 12, 15, 20, 30].$$

Then (3.17) *becomes*

$$\begin{array}{ll} k \equiv -1 \pmod{5} & k \equiv 18 \pmod{23} \\ k \equiv 1 \pmod{239} & k \equiv 271 \pmod{13709} \\ k \equiv 0 \pmod{17} & k \equiv -4319 \pmod{359} \\ k \equiv -17 \pmod{271} & k \equiv 4302 \pmod{1951} \end{array}$$

and solving gives $k = 861108588991441709$ *as the smallest positive value.*

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