

NON-CLASSICAL LINEAR DIVISIBILITY SEQUENCES AND CYCLOTOMIC POLYNOMIALS

SERGIY KOSHKIN

ABSTRACT. Divisibility sequences are defined by the property that their elements divide each other whenever their indices do. The divisibility sequences that also satisfy a linear recurrence, like the Fibonacci numbers, are generated by polynomials that divide their compositions with every positive integer power. We completely characterize such polynomials in terms of their factorizations into cyclotomic polynomials using labeled Hasse diagrams, and construct new integer divisibility sequences based on them. We also show that, unlike the Fibonacci numbers, these non-classical sequences do not have the property of strong divisibility.

1. WHAT MERSENNE AND FIBONACCI HAVE IN COMMON

The Mersenne numbers are given explicitly by the formula $M_n = 2^n - 1$. The Fibonacci numbers, on the other hand, come from a recurrence relation, $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. Of course, $M_0 = 0$ and $M_1 = 1$, and M_n satisfy a similar recurrence, $M_{n+2} = 3M_{n+1} - 2M_n$. But, both sequences also have a more interesting property in common: if m divides n , then a_m divides a_n . For example, $F_7 = 13$ and $F_{14} = 377$, and we observe that $377 = 13 \cdot 29$. Such sequences are called *divisibility sequences* (the name seems to be due to Hall [5]), and *linear divisibility sequences* if they also satisfy a linear recurrence $a_{n+k} = c_1 a_{n+k-1} + \dots + c_k a_n$, like the Mersenne or Fibonacci numbers.

Lucas was first to study in depth the second order ($k = 2$) linear divisibility sequences in 1876–1880. He showed that, up to normalization, such nontrivial sequences are of the form $\frac{\alpha^n - \beta^n}{\alpha - \beta}$, where α and β are the roots of a quadratic polynomial with integer coefficients. The polynomial is then $x^2 - (\alpha + \beta)x + \alpha\beta$, and the recurrence is $a_{n+2} = (\alpha + \beta)a_{n+1} - \alpha\beta a_n$, so the sequence consists of integers, even if α and β are irrational. For the Fibonacci numbers, Lucas' form is given by the Binet formula, $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$. Based on his work with divisibility sequences, Lucas devised primality tests that helped establish the primality of M_{127} . The search for large Mersenne primes is ongoing, with the help of the internet.

What is behind the peculiar divisibility property? This question leads from numerical sequences to sequences of polynomials. It still makes sense to talk about divisibility sequences of polynomials, for example x^n is one. Slightly less obviously, so is $x^n - 1$, because $x^{dm} - 1$ has $x^m - 1$ as a factor by the difference of powers formula. The Fibonacci polynomials are a more nontrivial example [12]. Up to normalization by $\frac{1}{\alpha - \beta}$, the Lucas sequences are products of β^n and $(\frac{\alpha}{\beta})^n - 1$, which are obtained by choosing a value for x in x^n and $x^n - 1$, respectively. More generally, the product of two linear divisibility sequences, of integers or polynomials, is again a linear divisibility sequence, so it is natural to consider products $a_n = A\gamma^n \prod_i (\gamma_i^n - 1)$ that generalize Lucas sequences. One such example was studied by Lehmer [7], who refined and extended Lucas' primality tests, and more by Ward [11], in the 1930s. A general theory of such sequences was developed recently in [9], where one can also read more about their

history and applications in algebra and number theory. Other old and new results on linear divisibility sequences and their applications can be found in [4].

But x^n and $x^n - 1$ are just x and $x - 1$ composed with x^n . Classical authors thought that they generated “almost all” linear divisibility sequences, with some special exceptions [11], but they are themselves very special. One such exception, generated by composing $f(x) = (x + 1)(x^m - 1)$, m odd, with x^n is considered in [8] (but was probably already known to Ward). What other $f(x^n)$, with a polynomial f , are divisibility sequences?

Definition 1. We call $f(x)$ a *divisibility polynomial* if $f(x^n)$ is a divisibility sequence, i.e., $f(x^m)|f(x^n)$ when $m|n$. It is enough that $f(x)|f(x^n)$ for all $n \in \mathbb{N}$, since then $f(x^m)|f(x^n) = f((x^m)^d)$ for $n = md$.

The divisibility polynomials are called Lucas polynomials in [9], but that name is already in use for other purposes. In 1988, Bézivin, Pethö, and van der Poorten proved more generally [2], see also [8], [9], that non-degenerate (in a precise sense) linear divisibility sequences of integers are always of the form

$$a_n = An^k \prod_i f_i(\gamma_i^n), \quad (1)$$

where A and γ_i are complex numbers, k is a nonnegative integer, and the f_i ’s are divisibility polynomials.

Not all sequences of this form are integer-valued, of course, and the integrality conditions are still not known in general. We will fully describe the divisibility polynomials (Theorem 4), and explain where the “exceptions” come from, and why they are more of a rule than an exception. For example, if the positive integers N_i are pairwise relatively prime, then

$$f(x) = \frac{(x^{N_1} - 1) \cdots (x^{N_k} - 1)}{(x - 1)^{k-1}} \quad (2)$$

is a divisibility polynomial. When $k = 1$, we get the classical $f(x) = x - 1$, and when $k = 2$, $N_1 = 2$, and $N_2 = m$ odd, we get the “exception” $f(x) = (x + 1)(x^m - 1)$.

The divisibility polynomials have an interesting factorization theory (Sections 3, 4), and can be nicely described as certain products of cyclotomic polynomials. The cyclotomic (literally, circle dividing) polynomials are classical, and also have ancient roots. Gauss introduced them in 1796 to solve an old Greek problem of inscribing regular polygons into the circle. Our description will be visual, we will introduce Hasse diagrams with additional labels that show how to build divisibility polynomials from cyclotomic polynomials (Section 2).

It turns out that Lucas’ integer divisibility sequences generalize accordingly. Namely, if $f(x, y) = y^{\deg(f)} f(x/y)$ is a homogenized divisibility polynomial, and $\alpha + \beta$ and $\alpha\beta$ are integers, then $\frac{f(\alpha^n, \beta^n)}{f(\alpha, \beta)}$ is a divisibility sequence of integers (Theorem 5). When $f(x) = x - 1$, or, more generally, $x^N - 1$, we get Lucas’ sequences. But for other $f(x)$, such as in (2), the resulting sequences are *non-classical*—they are not even products of Lucas’s sequences. Nonetheless, they are closely related to them (Section 5). For the Fibonacci values of α and β , the simplest one is $\frac{1}{2}L_n F_{3n}$, where L_n are the Lucas numbers. More generally, from the template (2) for $f(x)$ we get the sequence

$$\frac{F_{N_1 n} / F_{N_1} \cdots F_{N_k n} / F_{N_k}}{(F_n)^{k-1}}.$$

We will also characterize polynomials $f(x)$ with the property of *strong divisibility* (Theorem 6), namely satisfying $\gcd(f_m, f_n) = f_{\gcd(m, n)}$, where $f_n(x) = f(x^n)$. A new connection between strong divisibility and cyclotomic polynomials was recently discovered in [3], other known

connections are discussed in [6]. This is another property that the Mersenne and Fibonacci numbers have in common, and it is distinctive to the classical sequences. Under some mild conditions on α and β , our generalized sequences are strong divisibility sequences if and only if they are classical (Theorem 7).

2. CYCLOTOMIC POLYNOMIALS AND HASSE DIAGRAMS

The divisibility polynomials are clearly special, but there are more of them than one might think. Let us call a polynomial *normal* if it has the leading coefficient 1, and a nonzero constant term. Then, any polynomial is of the form $Cx^s g(x)$ for some constant C , integer $s \geq 0$, and normal $g(x)$. In this section, we will develop diagrams that encode the structure of normal divisibility polynomials.

We start with a couple of simple observations. Let ζ be a root of a divisibility polynomial f . Since $f(x)|f(x^n)$ and $f(\zeta) = 0$, we have $f(\zeta^n) = 0$ for all positive integers n . But, a polynomial cannot have infinitely many roots, so $\zeta^k = \zeta^l$ for $k \neq l$. This means that ζ is either 0 or $\zeta^n = 1$ for some $n \geq 1$. Such ζ are called *n th roots of unity*, and they are of the form $\zeta_n^k = e^{\frac{2\pi i}{n}k}$. If $k|n$, then $\zeta_n^k = \zeta_{n/k}^1$ is also an n/k th root of unity. Those roots that are not roots for smaller n are called *primitive of order n* . By the elementary properties of cyclic groups, those are exactly the ones with $\gcd(n, k) = 1$. Bézivin, Pethö, and van der Poorten [2] found a necessary and sufficient condition on the set of roots that makes a polynomial a divisibility polynomial.

Theorem 1. *Consider a polynomial f with roots of unity as roots. Given a pair of them, let h and h' denote their orders as primitive roots of unity, and m and m' their multiplicities in f . Then, f is a divisibility polynomial if and only if for any such pair with $h|h'$ we have $m' \geq m$.*

Proof. First, suppose that the roots of $f(x)$ satisfy the conditions of the theorem. We want to show that $f(x)|f(x^n)$. If ζ is a root of $f(x)$ of multiplicity m , then $\zeta^n = \zeta'$ is also a root of multiplicity $m' \geq m$. Since $(x - \zeta^n)^{m'}$ is a factor of $f(x)$, we have $(x^n - \zeta^n)^{m'}$ is a factor of $f(x^n)$. But $(x - \zeta)|(x^n - \zeta^n)$, so $(x - \zeta)^m|(x^n - \zeta^n)^{m'}$, as $m' \geq m$. Since this is true for all roots, $f(x)|f(x^n)$.

Conversely, let f be a divisibility polynomial, and ζ and ζ' be a pair of roots from the statement of the theorem. Since $h|h'$, by the standard properties of cyclic groups, there is a $d \in \mathbb{N}$ such that $\zeta' = \zeta^d$. We have $(x - \zeta)^m$ dividing $f(x)$, and hence $f(x^d)$. But $f(x^d)$ splits into factors of the form $x^d - \xi$, where ξ is a root of $f(x)$, and $(x - \zeta)$ must divide at least one of them. This is only possible if $\xi = \zeta^d = \zeta'$, and for $(x - \zeta)^m$ to divide $(x^d - \zeta^d)^{m'}$, we must have $m' \geq m$. \square

Note that any primitive root of unity of some order is a power of any other primitive root of the same order. So if ζ is a root of $f(x)$, then so is every other primitive root of the same order, since $f(\zeta)|f(\zeta^n)$, and, by Theorem 1, they all have the same multiplicity. The product of $(x - \zeta)$ factors over all primitive roots ζ of order n is called the *n th cyclotomic polynomial* $\Phi_n(x) = \prod_{\gcd(n,k)=1} (x - \zeta_n^k)$. It follows that a normal divisibility polynomial $f(x)$ is a product of powers of cyclotomic polynomials.

From the definition of cyclotomic polynomials, Gauss derived

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \quad (3)$$

This is true because any n th root of unity must be primitive of some order that divides n . Gauss' formula allows us to find Φ_n once Φ_d for $d < n$ are already known, without any

recourse to complex numbers, by splitting off Φ_n from the product on the right. It also shows, by Gauss' lemma, that their coefficients are integers, and the leading coefficient is always 1. It is easy to find them recursively from (3): $\Phi_1(x) = x - 1$, $\Phi_2(x) = x + 1$, $\Phi_3(x) = x^2 + x + 1$, $\Phi_4(x) = x^2 + 1$, $\Phi_6(x) = x^2 - x + 1$, etc.

What kinds of cyclotomic products are the divisibility polynomials? Suppose $\Phi_h(x)$ appears as a factor in a divisibility polynomial. Theorem 1 tells us that if $d|h$, then $\Phi_d(x)$ must also be a factor. In particular, $\Phi_1(x) = x - 1$ is always a factor. We are led to the following property, known in algebra.

Definition 2. A subset $\Lambda \subset \mathbb{N}$ is called *saturated* if $d|h$ and $h \in \Lambda$ imply that $d \in \Lambda$, i.e., if with any of its elements, Λ also contains all of its divisors.

Divisibility relations among positive integers in a subset can be conveniently pictured by *Hasse diagrams*. Given a subset Λ , at the lowest level of the diagram we place 1 (if it is in Λ), at the next level all prime numbers in Λ (if any), next up are their pairwise products, and so on, see Figure 1a. The edges connect the numbers to the numbers one level up that they divide, so upward paths in the diagram reflect the “increase” in divisibility. It is convenient to assign the empty diagram to the empty set \emptyset .

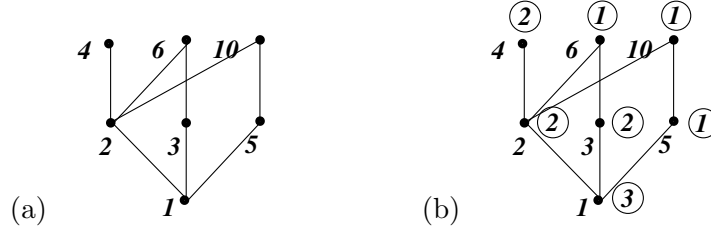


FIGURE 1. (a) Hasse diagram of $\{1, 2, 3, 4, 5, 6, 10\}$; (b) Hasse diagram of $\Phi_1^3 \Phi_2^2 \Phi_3^2 \Phi_4^2 \Phi_5 \Phi_6 \Phi_{10}$. The circled numbers are the multiplicities.

The Hasse diagrams do not reflect the multiplicities of factors Φ_h though. Those can be pictured as additional nonnegative integer labels. Theorem 1 imposes a restriction on these labels. For a product of cyclotomic polynomials $\prod_h (\Phi_h(x))^{m_h}$ to be a divisibility polynomial, “less divisible” primitive orders h must have bigger multiplicities m_h . Because divisibility “increases” along the upward paths in a Hasse diagram, the multiplicity labels must not increase when moving up along the edges, Figure 1b. To make this more precise, let us restate this in terms of maps.

Definition 3. We call maps $\lambda : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ that are nonzero at only finitely many integers *multiplicity maps*. And, we call them *order-reversing* if $h|h'$ implies $\lambda(h) \geq \lambda(h')$.

Given a product of powers of cyclotomic polynomials as above, $\lambda : h \mapsto m_h$ is such a map. And conversely, any multiplicity map λ defines a polynomial

$$\Phi_\lambda(x) = \prod_{h \in \mathbb{N}} (\Phi_h(x))^{\lambda(h)}.$$

This is true because all but finitely many factors have the exponent 0. If we assign the polynomial 1 to the zero map (and the empty diagram), then Theorem 1 can be restated as follows.

Corollary 1. There is a 1-1 correspondence between order-reversing multiplicity maps, multiplicity labeled Hasse diagrams of finite saturated sets, and the normal divisibility polynomials.

There is also a nice algebraic way to describe finite saturated sets that will come in handy later. If Λ is finite, there must be $n_1, \dots, n_k \in \Lambda$ that do not divide any of its other elements. They are maximal in terms of divisibility, and they are, obviously, uniquely determined by Λ . But for finite saturated sets the converse is also true, because upward paths in a finite Hasse diagram must terminate at a maximal element.

Corollary 2. *Finite saturated subsets of \mathbb{N} are of the form:*

$$\langle n_1, \dots, n_k \rangle = \{d \in \mathbb{N} \mid d \mid n_i \text{ for some } i\}. \quad (4)$$

In particular, $\langle n \rangle$ is the set of all (positive) divisors of n . Figures 1 and 2 contain saturated sets $\langle 4, 6, 10 \rangle$, $\langle pq \rangle$, and $\langle p, q, r \rangle$, for distinct primes p, q , and r .

Finally, we give some notational conventions. Recall that the indicator function of Λ is:

$$1_\Lambda(h) = \begin{cases} 1, & h \in \Lambda \\ 0, & h \notin \Lambda. \end{cases}$$

This is a map, and we abbreviate Φ_{1_Λ} as Φ_Λ . This is simply the product $\prod_{d \in \Lambda} \Phi_d(x)$, and it means that $\Phi_{\langle n_1, \dots, n_k \rangle}(x)$ is the product of $\Phi_d(x)$, where d runs over the set $\langle n_1, \dots, n_k \rangle$ of divisors of at least one of the positive integers n_1, \dots, n_k . They are natural generalizations of $\Phi_{\langle n \rangle}(x) = x^n - 1$, and will play a major part in what follows.

3. FACTORIZATION OF DIVISIBILITY POLYNOMIALS

With the Hasse diagrams in hand, it is easier to take stock of the divisibility polynomials. We now realize that there are many more of them than contemplated by classical authors [11]. A natural idea is to break them up into simpler pieces in some way. Note that a product of divisibility polynomials is again a divisibility polynomial. In other words, divisibility polynomials form a set that is closed under multiplication. In such sets, a natural way to decompose is to factor. Integers factor into primes, (general) polynomials factor into linear factors (over complex numbers), etc. And, those do not factor any further.

Definition 4. *A non-constant divisibility polynomial is called decomposable if it is the product of two non-constant divisibility polynomials. Otherwise, it is called indecomposable.*

We do not yet know which divisibility polynomials are indecomposable, but here again we can make use of the diagrams. The ones associated with finite saturated sets look simpler than those with additional multiplicity labels on them. The idea of decomposing them is pictured in Figure 2a. We represent multiplicities by nodes stacked over the diagram, and then slice the layers of nodes horizontally. Each slice projects to a finite saturated set that defines a factor. This idea turns out to work. Recall that Φ_λ denotes the divisibility polynomial corresponding to the multiplicity map λ .

Theorem 2 (Slicing factorization). *A normal divisibility polynomial is indecomposable if and only if each of its cyclotomic factors has multiplicity 1. It suffices that $\Phi_1(x)$ has multiplicity 1. In other words, indecomposable normal divisibility polynomials are $\Phi_{\langle n_1, \dots, n_k \rangle}$ with $n_i \nmid n_j$ for $i \neq j$. Any normal divisibility polynomial Φ_λ factors into indecomposables:*

$$\Phi_\lambda = \prod_{j=1}^{\lambda(1)} \Phi_{\Lambda_j}, \text{ where } \Lambda_j = \{m \in \mathbb{N} \mid \lambda(m) \geq j\}. \quad (5)$$

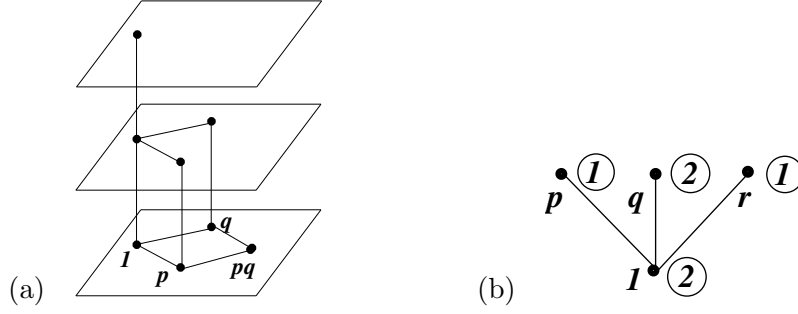


FIGURE 2. (a) Slicing decomposition of $\Phi_1^3 \Phi_p^2 \Phi_q^2 \Phi_{pq} = \Phi_{\langle 1 \rangle} \Phi_{\langle p, q \rangle} \Phi_{\langle pq \rangle}$; (b) Divisibility polynomial $\Phi_1^2 \Phi_p \Phi_q^2 \Phi_r$ with a non-unique decomposition (p, q , and r are distinct primes).

Proof. Saying that all cyclotomic factors of Φ_λ have multiplicity 1 is equivalent to saying that $\lambda = 1_\Lambda$ for a finite saturated set Λ , or that λ is an order-reversing multiplicity map with $\lambda(1) = 1$. Indeed, since $1|m$ and λ is order-reversing, $\lambda(m) \leq \lambda(1)$ for any $m \in \mathbb{N}$. If $\lambda(1) = 1$, then $\lambda(m)$ is 0 or 1. If $\lambda(1) = 0$, then λ is the zero map and $\Phi_\lambda = 1$, otherwise it is 1_Λ for some finite saturated set Λ . Moreover, if $\Phi_\lambda = \Phi_\mu \Phi_\nu$ were a nontrivial factorization, then $\lambda(1) = \mu(1) + \nu(1) \geq 2$; so none exists if $\lambda(1) = 1$.

Conversely, if $\lambda(1) \geq 2$, then we can factor Φ_λ explicitly, see Figure 2a. Let Λ_j be the projection of the j th layer of nodes above the diagram. Clearly, $\lambda = \sum_{j=1}^{\lambda(1)} 1_{\Lambda_j}$, hence $\Phi_\lambda = \prod_{j=1}^{\lambda(1)} \Phi_{\Lambda_j}$. Moreover, if $m \in \Lambda_j$ and $d|m$, then $\lambda(d) \geq \lambda(m) \geq j$. So $d \in \Lambda_j$, and Λ_j is a saturated set by definition. \square

Although the slicing factorization has much to recommend, it is not unique. Consider the multiplicity map depicted in Figure 2b. Its slicing factorization is $\Phi_{\langle q \rangle} \Phi_{\langle p, q, r \rangle}$, but $\Phi_{\langle p, q \rangle} \Phi_{\langle q, r \rangle}$ also factors into indecomposables. The situation is not unlike the simple example of multiplication of numbers of the form $3n + 1$: 1, 4, 7, 10, 13, 16, 19, 22, ..., whose set is also closed under multiplication. We have $4 \cdot 55 = 10 \cdot 22$, and 4, 22, 10, 55 do not decompose into other $3n + 1$ numbers. However, it is a simple exercise to show that if a $3n + 1$ number has a $3n + 1$ factor, then the quotient is also a $3n + 1$ number. So such a number decomposes into a product of $3n + 1$ numbers. Even this weaker property fails for normal divisibility polynomials. Every nontrivial one has $\Phi_{\langle 1 \rangle}(x) = x - 1$ as a factor, but plenty of them are indecomposable. For example, $\Phi_{\langle 2 \rangle}(x) = x^2 - 1 = (x - 1)(x + 1)$ is divisible by $\Phi_{\langle 1 \rangle}(x)$, but does not factor into divisibility polynomials. The set of divisibility polynomials shows just how far multiplicatively closed sets can be from positive integers, where the prime factorization is unique.

4. COMPRESSION OF INDECOMPOSABLE POLYNOMIALS

Factorization of divisibility polynomials is a bit of a disappointment, even aside from non-uniqueness. Although $f_m(x) = x^m - 1$ are all indecomposable, in a sense, they bring little new compared with $x - 1$. Their divisibility sequences $f_m(x^n) = x^{mn} - 1$ are rather boring subsequences of $x^n - 1$, and if we pick a numerical value for x , the resulting numerical sequences are even more alike – the chosen value is simply replaced by its m th power. We need to eliminate this redundancy.

Definition 5. A divisibility polynomial f is called compressible into a divisibility polynomial g if $f(x) = g(x^m)$, and it is called incompressible if no such g exists.

To find out which $\Phi_M(x)$ can be compressed, i.e., written as $\Phi_\Lambda(x^n)$, we need to determine how $\Phi_\Lambda(x^n)$ decompresses. We can start by decompressing their cyclotomic factors. One special case is straightforward by inspection of the roots on both sides:

$$\Phi_m(x^p) = \begin{cases} \Phi_{mp}(x), & \text{if } p \mid m \\ \Phi_m(x)\Phi_{mp}(x), & \text{if } p \nmid m, \end{cases} \quad (6)$$

where m is any positive integer, and p is prime. For divisibility polynomials we can get a better formula that does not split into cases. Recall that $\langle n \rangle = \{d \in \mathbb{N} \mid d \mid n\}$, and ST denotes the set of products of elements from sets S and T . So $\langle n \rangle \Lambda$ consists of products of divisors of n and elements of Λ .

Theorem 3 (Decompression formula). *Let $n \in \mathbb{N}$ and $\Lambda \subset \mathbb{N}$ be a finite saturated set. Then,*

$$\Phi_\Lambda(x^n) = \Phi_{\langle n \rangle \Lambda}(x) = \prod_{d \in \langle n \rangle \Lambda} \Phi_d(x). \quad (7)$$

Proof. By inspection, $\Phi_\Lambda(x^n)$ is a normal divisibility polynomial. Moreover, $x - 1$ has multiplicity 1 as a factor of $\Phi_\Lambda(x)$. No term of the form $x^n - \zeta$ is divisible by $x - 1$, except when $\zeta = 1$. Moreover, $x - 1$ has multiplicity 1 in $x^n - 1$, and, therefore, $x - 1$ has multiplicity 1 in $\Phi_\Lambda(x^n)$ as well. So, by Theorem 2, $\Phi_\Lambda(x^n)$ is indecomposable, and $\Phi_\Lambda(x^n) = \Phi_{\Lambda(n)}(x)$ for some finite saturated set. For the time being, we will denote this by $\Lambda^{(n)}$. It remains to show that $\Lambda^{(n)} = \langle n \rangle \Lambda$.

First, suppose that $n = p$ is prime. If $d \in \Lambda^{(p)}$, then $\Phi_d(x)$ is a factor of $\Phi_\Lambda(x^p)$. We see from (6) that these are of the form $\Phi_m(x)$ and $\Phi_{mp}(x)$ with $m \in \Lambda$, so $d \in \langle p \rangle \Lambda$. Conversely, if $m \in \Lambda$, we need to show that $m, mp \in \Lambda^{(p)}$. That $\Phi_{mp}(x)$ is a factor of $\Phi_m(x^p)$ is immediate from (6), as it is for $\Phi_m(x)$ when $p \nmid m$. But if $p \mid m$, then $\frac{m}{p} \in \Lambda$ by the saturation property, and $\Phi_m(x)$ is always a factor of $\Phi_{\frac{m}{p}}(x^p)$. Hence, $m \in \Lambda^{(p)}$ in this case. Thus, $\Lambda^{(p)} = \langle p \rangle \Lambda$ in both cases.

To finish the proof, we need a simple observation about $\langle mn \rangle$. If $d \mid mn$, then factoring all three into primes, we can find $d_1 \mid m$ and $d_2 \mid n$ such that $d = d_1 d_2$. Conversely, if $d_1 \mid m$ and $d_2 \mid n$, then $d_1 d_2 \mid mn$. Summarizing, $\langle mn \rangle = \langle m \rangle \langle n \rangle$. The general case now follows by induction on the number of prime factors in $n = p_1 \cdots p_N$ (some p_i may repeat), since $\Phi_\Lambda(x^{(p_1 \cdots p_k) p_{k+1}}) = \Phi_{\langle p_{k+1} \rangle \Lambda}(x^{p_1 \cdots p_k})$ and $\langle p_{k+1} \rangle \langle p_1 \cdots p_k \rangle = \langle p_1 \cdots p_{k+1} \rangle$. \square

A warning: even if d can be represented as $d_1 d_2$, with $d_1 \mid n$ and $d_2 \in \Lambda$ in several different ways, $\Phi_d(x)$ still enters the product in (7) only once.

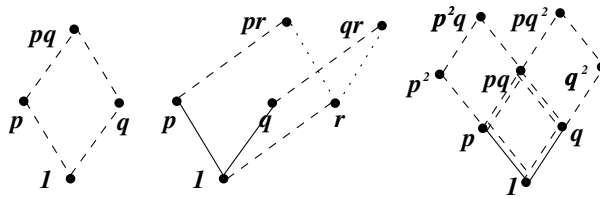


FIGURE 3. Decompression diagrams for $\Phi_{\langle 1 \rangle}(x^{pq})$, $\Phi_{\langle p,q \rangle}(x^r)$, and $\Phi_{\langle p,q \rangle}(x^{pq})$; p , q , and r are distinct primes. Adjoined (possibly twice) edges are dashed, additional edges are dotted.

When $\Lambda = \langle 1 \rangle = \{1\}$ the decompression formula reduces to Gauss' $x^n - 1 = \prod_{d \mid n} \Phi_d(x)$. Geometrically, the Hasse diagram of $\langle n \rangle \Lambda$ is obtained by adjoining a copy of $\langle n \rangle$ to every

node of Λ (or equivalently a copy of Λ to every node of $\langle n \rangle$), and possibly adding extra edges according to the diagram drawing rules, see Figure 3. This suggests that to compress a diagram we need to look for a subdiagram in the shape of Λ , to which identical diagrams in the shape of $\langle n \rangle$ are attached. This is a nontrivial search for large diagrams.

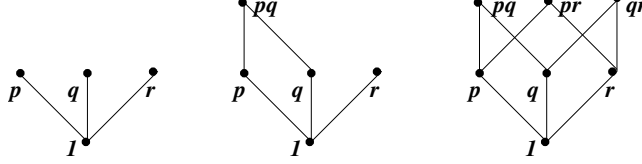


FIGURE 4. Incompressible diagrams; p , q , and r are distinct primes.

Fortunately, there is a much simpler way to detect compressibility. The reader may try to guess it by reflecting on the difference between the diagrams in Figures 3 and 4. This is one of those cases where difficult geometry is streamlined by simple algebra. Since one can easily show that $\langle d \rangle \langle n_1, \dots, n_k \rangle = \langle dn_1, \dots, dn_k \rangle$ (by the same argument as for $\langle m \rangle \langle n \rangle = \langle mn \rangle$), we can rewrite (7) as

$$\Phi_{\langle m_1, \dots, m_k \rangle}(x^d) = \Phi_{\langle dm_1, \dots, dm_k \rangle}(x).$$

So, $\Phi_{\langle n_1, \dots, n_k \rangle}$ is compressible if and only if n_i have a nontrivial common divisor.

Corollary 3. $\Phi_{\langle n_1, \dots, n_k \rangle}(x)$ is incompressible if and only if $\gcd(n_1, \dots, n_k) = 1$. Any indecomposable normal divisibility polynomial is of the form $\Phi_{\langle n_1, \dots, n_k \rangle}(x^m)$, with $m \geq 1$ and an incompressible $\Phi_{\langle n_1, \dots, n_k \rangle}(x)$.

Now, it is easy to give explicit “exceptional” examples of divisibility polynomials that are distinctive. If n_1 and n_2 are relatively prime, then $x^{n_1} - 1 = \prod_{d|n_1} \Phi_d(x)$ and $x^{n_2} - 1 = \prod_{d|n_2} \Phi_d(x)$ have only one common cyclotomic factor, $\Phi_1(x) = x - 1$. So,

$$\Phi_{\langle n_1, n_2 \rangle}(x) = \frac{(x^{n_1} - 1)(x^{n_2} - 1)}{x - 1}.$$

Similarly, if $\gcd(n_1, n_2, n_3) = 1$, then,

$$\Phi_{\langle n_1, n_2, n_3 \rangle}(x) = \frac{(x^{n_1} - 1)(x^{n_2} - 1)(x^{n_3} - 1)(x - 1)}{(x^{\gcd(n_1, n_2)} - 1)(x^{\gcd(n_2, n_3)} - 1)(x^{\gcd(n_3, n_1)} - 1)}. \quad (8)$$

In general, one can use the inclusion-exclusion formula from combinatorics to express $\Phi_{\langle n_1, \dots, n_k \rangle}(x)$ for any k . When $\gcd(n_i, n_j) = 1$ for $i \neq j$, this simplifies to

$$\Phi_{\langle n_1, \dots, n_k \rangle}(x) = \frac{(x^{n_1} - 1) \cdots (x^{n_k} - 1)}{(x - 1)^{k-1}}. \quad (9)$$

Let us summarize what we discovered about the structure of the divisibility polynomials.

Theorem 4. Any divisibility polynomial $f(x)$ is of the form $f(x) = Cx^s g(x)$, where C is a numerical constant, s is a positive integer, and $g(x)$ is a product of factors of the form $\Phi_{\langle n_1, \dots, n_k \rangle}(x^d)$, with integers $d \geq 0, n_i \geq 1, n_i \nmid n_j$ for $i \neq j$, and $\gcd(n_1, \dots, n_k) = 1$.

5. BACK TO MERSENNE AND FIBONACCI

We now have a characterization of the divisibility polynomials, and it is time to apply our results. To fully characterize linear divisibility sequences of integers, it only remains to find conditions on A and γ_i in (1), that would produce integer sequences. Unfortunately, necessary and sufficient integrality conditions are not known, even when all divisibility polynomials are x or $x - 1$ [9]. We will find sufficient integrality conditions when there is a single divisibility polynomial factor, and construct many non-classical relatives of the Fibonacci and Mersenne numbers.

It will be convenient to represent our sequences in a different form than (1), using a modification of the cyclotomic polynomials into polynomials in two variables. This modification is common in algebra, and is called *homogenization*: given a polynomial $f(x)$, we set $f(x, y) = y^{\deg(f)} f(x/y)$. It leaves x as is, but turns $x - 1$ into $x - y$, and $x^n - 1$ into $x^n - y^n$. If $f(x)$ is a product of polynomials, then its homogenization is the product of their homogenizations. In particular, $\Phi_\Lambda(x, y) = \prod_{d \in \Lambda} \Phi_d(x, y)$, where $\Phi_d(x, y)$ are the homogenized cyclotomic polynomials: $\Phi_1(x, y) = x - y$, $\Phi_2(x, y) = x + y$, $\Phi_3(x, y) = x^2 + xy + y^2$, $\Phi_4(x, y) = x^2 + y^2$, $\Phi_6(x, y) = x^2 - xy + y^2$, etc.

A key observation is that $\Phi_d(x, y)$ is symmetric in x and y for $d > 1$. The reason is because for $d > 1$, $\Phi_d(x)$ is *palindromic*, that is, the coefficients read the same forward and backward. To prove this, note that $\frac{x^n - 1}{x - 1}$ is palindromic (all coefficients are 1), and if $f(x)g(x) = h(x)$, where f and h are palindromic, then so is g . Dividing Gauss' formula (3) by $x - 1$, we have

$$\frac{x^n - 1}{x - 1} = \Phi_n(x) \prod_{d|n, 1 < d < n} \Phi_d(x). \quad (10)$$

Assuming $\Phi_d(x)$ are palindromic for $d < n$, $\Phi_n(x)$ is a quotient of palindromic polynomials, hence it is palindromic. By induction, $\Phi_n(x)$ are palindromic for all $n > 1$.

Now it is easy to see why $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ is integral, despite the irrational values $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$. By the fundamental theorem on symmetric polynomials, any such polynomial with integer coefficients is also a polynomial with integer coefficients in $x + y$ and xy , and $\alpha + \beta = 1$ and $\alpha\beta = -1$ are integers. We can generalize this integrality argument as follows.

Theorem 5. *Let α and β be complex numbers such that $\alpha + \beta$ and $\alpha\beta$ are integers. Let $f(x)$ be a normal divisibility polynomial. Then, $A_n = \frac{f(\alpha^n, \beta^n)}{f(\alpha, \beta)}$ is a linear divisibility sequence of integers.*

Proof. Since normal divisibility polynomials are products of indecomposables, and products of integer sequences are integer sequences, it suffices to consider an indecomposable $f = \Phi_\Lambda$. By the decomposition formula, $\frac{\Phi_\Lambda(x^n)}{\Phi_\Lambda(x)} = \Phi_{\langle n \rangle \Lambda}(x) / \Phi_\Lambda(x)$ is a product of cyclotomic polynomials. In particular, it is a polynomial with integer coefficients. Therefore, so is

$$\frac{\Phi_\Lambda(x^n, y^n)}{\Phi_\Lambda(x, y)} = \frac{\prod_{d \in \Lambda} \Phi_d(x^n, y^n)}{\prod_{d \in \Lambda} \Phi_d(x, y)} = \frac{x^n - y^n}{x - y} \prod_{d \in \Lambda, d > 1} \frac{\Phi_d(x^n, y^n)}{\Phi_d(x, y)}.$$

Moreover, the first factor is symmetric in x and y , because it homogenizes a palindromic $\frac{x^n - 1}{x - 1}$, and the second factor is also symmetric, because $\Phi_d(x, y)$ is symmetric for $d > 1$. By the fundamental theorem of symmetric polynomials, the product is a symmetric polynomial with integer coefficients in $x + y$ and xy , and its values are integers for $x = \alpha$ and $y = \beta$. By the same reasoning, $\frac{\Phi_\Lambda(\alpha^n, \beta^n)}{\Phi_\Lambda(\alpha^m, \beta^m)}$ is an integer when $m|n$, so $A_m | A_n$ in \mathbb{Z} . \square

As a first example, consider the simplest non-classical divisibility polynomial

$$f = \Phi_{\langle 2,3 \rangle} = \Phi_1 \Phi_2 \Phi_3 = (x - y)(x + y)(x^2 + xy + y^2) = (x + y)(x^3 - y^3).$$

The corresponding divisibility sequence is

$$A_n = \frac{x^n + y^n}{x + y} \cdot \frac{x^{3n} - y^{3n}}{x^3 - y^3} = B_n \cdot C_n.$$

Both factors satisfy linear recurrences of the second order, and can be generated easily:

$$B_0 = \frac{2}{x+y}, B_1 = 1, B_{n+2} = (x + y)B_{n+1} - xyB_n;$$

$$C_0 = 0, C_1 = 1, C_{n+2} = (x^3 + y^3)C_{n+1} - x^3y^3C_n.$$

For the Fibonacci values α and β , with $\alpha + \beta = 1$ and $\alpha\beta = -1$, B_n is the Lucas number L_n , and C_n is a normalized subsequence of the Fibonacci numbers: $\frac{F_{3n}}{F_3} = \frac{1}{2}F_{3n}$. So $A_n = \frac{1}{2}L_nF_{3n}$, see Table 1. This construction can be generalized by choosing any odd number N instead of 3; the resulting sequence is $A_n = L_nF_{Nn}/F_N$, a recurrence sequence of order four. Some of these sequences were entered into the On-Line Encyclopedia of Integer Sequences by Bala in 2014 [1].

n	0	1	2	3	4	5	6	7	8	9
L_n	2	1	3	4	7	11	18	29	47	76
F_{3n}/F_3	0	1	4	17	72	305	1,292	5,473	23,184	98,209
A_n	0	1	12	68	504	3,355	23,256	158,717	1,089,648	7,463,884

TABLE 1. Non-classical linear divisibility sequence constructed from the Fibonacci numbers.

A recurrent sequence of order six comes from $\Phi_{\langle 3,4 \rangle}(x) = (x + 1)(x^2 + 1)(x^3 - 1)$. For the Fibonacci values, we find $A_n = L_n \frac{L_{2n}}{L_2} \frac{F_{3n}}{F_3} = \frac{1}{6}L_nL_{2n}F_{3n}$. Note that the extra factor $\frac{L_{2n}}{L_2} = 2/3, 1, 7/3, 6, 47/3, 41, \dots$ not only is not a divisibility sequence, but is not even integer-valued.

To generalize them further, let us define

$$S_n^\Lambda(\alpha, \beta) = \frac{\Phi_\Lambda(\alpha^n, \beta^n)}{\Phi_\Lambda(\alpha, \beta)},$$

We will abbreviate $S_n^{\langle 1 \rangle}$ as S_n , so $M_n = S_n(2, 1)$ and $F_n = S_n(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, and suppress α and β from the notation when they are unimportant or understood. Any second order recurrent sequence with $S_0 = 0$ and $S_1 = 1$ is of this form, with $\Lambda = \langle 1 \rangle$ and suitable α and β . When $\Lambda = \langle N \rangle$, we get normalized subsequences of S_n , namely $S_n^{\langle N \rangle} = S_{Nn}/S_N$. More generally, we can use the expression (9) for Φ_Λ . In the homogenized form, when $\gcd(N_i, N_j) = 1$ for $i \neq j$, we have:

$$\Phi_{\langle N_1, \dots, N_k \rangle}(x, y) = \frac{(x^{N_1} - y^{N_1}) \dots (x^{N_k} - y^{N_k})}{(x - y)^{k-1}}.$$

Therefore,

$$S_n^{\langle N_1, \dots, N_k \rangle} = \frac{S_{N_1n}/S_{N_1} \dots S_{N_kn}/S_{N_k}}{(S_n)^{k-1}}. \quad (11)$$

We recover the sequences $A_n = L_nF_{Nn}/F_N$, when $S_n = F_n$, from the well-known identity $L_n = F_{2n}/F_n$, and $F_2 = 1$. We leave it to the reader to produce more general sequences based on the expressions like (8). When $\Lambda \neq \langle N \rangle$, these sequences are non-classical, and are not covered by the extension of Lucas' theory developed in [9]. Yet, all sequences from (11) are divisors of a product of classical sequences. This is not accidental; it is true for all non-degenerate linear divisibility sequences by a main result of [2].

6. STRONG DIVISIBILITY

The Fibonacci and Mersenne numbers have even more in common than being divisibility sequences. They are both *strong divisibility* sequences: $\gcd(a_n, a_m) = a_{\gcd(n, m)}$. This follows by a clever application of the Euclidean algorithm, for example. We will show that this property distinguishes them from their non-classical generalizations.

Let us start with the simplest non-classical sequence $f_n = \Phi_{\langle 2, 3 \rangle}(x^n)$. By the decomposition formula,

$$\begin{aligned} f_1 &= \Phi_1 \Phi_2 \Phi_3 \\ f_2 &= \Phi_1 \Phi_2 \Phi_3 \Phi_4 \Phi_6 \\ f_3 &= \Phi_1 \Phi_2 \Phi_3 \Phi_6 \Phi_9. \end{aligned}$$

Since Φ_d with different d have no common roots, $\gcd(f_2, f_3) = \Phi_1 \Phi_2 \Phi_3 \Phi_6 \neq \Phi_1 \Phi_2 \Phi_3 = f_1 = f_{\gcd(2, 3)}$, so this sequence is not a strong divisibility sequence. It is natural to ask when $\Phi_\Lambda(x^n)$ is a strong divisibility sequence.

Recall that $\langle n \rangle$ denotes the set of all positive divisors of n , and ST denotes the set of products of elements from sets S and T . Generalizing the above example, we see that the common cyclotomic factors of $\Phi_\Lambda(x^m) = \Phi_{\langle m \rangle \Lambda}(x)$ and $\Phi_\Lambda(x^n) = \Phi_{\langle n \rangle \Lambda}(x)$ are those with indices from $\langle m \rangle \Lambda \cap \langle n \rangle \Lambda$. Therefore, $\Phi_\Lambda(x^n)$ is a strong divisibility sequence if and only if

$$\langle m \rangle \Lambda \cap \langle n \rangle \Lambda = \langle \gcd(m, n) \rangle \Lambda. \quad (12)$$

However, we can give a much more explicit characterization.

Theorem 6. *Let $\Lambda \subset \mathbb{N}$ be a finite saturated set. Then, $\Phi_\Lambda(x^n)$ is a strong divisibility sequence of polynomials if and only if $\Lambda = \langle N \rangle$ for some $N \geq 1$.*

Proof. First, suppose $\Lambda = \langle N \rangle$. Then,

$$\langle m \rangle \langle N \rangle \cap \langle n \rangle \langle N \rangle = \langle mN \rangle \cap \langle nN \rangle = \langle \gcd(mN, nN) \rangle,$$

because every common divisor of two numbers is a divisor of their gcd, and vice versa. Multiplying every line of the Euclidean algorithm for m and n by N , we may conclude that $\gcd(mN, nN) = \gcd(m, n)N$. Therefore, $\langle \gcd(mN, nN) \rangle = \langle \gcd(m, n) \rangle \langle N \rangle$, and (12) holds, so $\Phi_\Lambda(x^n)$ is a strong divisibility sequence.

To prove the converse, we will first show that if (12) holds, and $m, n \in \Lambda$, then $\text{lcm}(m, n) \in \Lambda$. Clearly, $mn \in \langle m \rangle \Lambda \cap \langle n \rangle \Lambda = \langle \gcd(m, n) \rangle \Lambda$. Therefore, there is a $d \mid \gcd(m, n)$ and $h \in \Lambda$ such that $mn = dh$. But then,

$$h = \frac{mn}{\gcd(m, n)} \cdot \frac{\gcd(m, n)}{d} = \text{lcm}(m, n) \cdot \frac{\gcd(m, n)}{d},$$

so $\text{lcm}(m, n) \mid h$. Since $h \in \Lambda$, and Λ is saturated, this implies that $\text{lcm}(m, n) \in \Lambda$. Because this is true for any $m, n \in \Lambda$, we have $\Lambda = \langle N \rangle$, where $N = \text{lcm}\{h \mid h \in \Lambda\}$. \square

Corollary 4. *Every indecomposable divisibility polynomial that generates a strong divisibility sequence is compressible into $x - 1$, i.e., it is of the form $x^N - 1$.*

In other words, strong divisibility sequences of polynomials are the classical ones. Does this extend to the corresponding integer sequences? The question is subtle because we may get ± 1 when substituting values into $\Phi_d(x, y)$, even though the polynomial itself is nontrivial. Barring such degeneracy, we can characterize which linear divisibility sequences defined in Section 5 are strong divisibility sequences.

Theorem 7. *Let α and β be complex numbers such that $\alpha + \beta$ and $\alpha\beta$ are integers, α/β is not a root of unity, and $|\Phi_d(\alpha, \beta)| \neq 1$ for all large enough d . Then, $S_n^\Lambda(\alpha, \beta)$ is a strong divisibility sequence of integers if and only if $\Lambda = \langle N \rangle$ for some $N \geq 1$.*

Proof. If $\Lambda = \langle N \rangle$, strong divisibility follows directly from Theorem 6, even for the polynomials $\Phi_\Lambda(x^n, y^n)$. If $\Lambda \neq \langle N \rangle$, by the same argument as in the proof of the theorem, there are $m, n \in \Lambda$ such that for any positive integer k . So we have $kmn \in \langle km \rangle \Lambda \cap \langle kn \rangle \Lambda$, but $kmn \notin \langle \gcd(km, kn) \rangle \Lambda$. Therefore, $\gcd(S_{km}^\Lambda, S_{kn}^\Lambda)$ multiplies $S_{\gcd(km, kn)}^\Lambda$ by at least $\Phi_{kmn}(\alpha, \beta)$. Note that $S_n^\Lambda(\alpha, \beta)$ can only be 0 for $n > 0$ if α/β is a root of unity, and, by assumption, $\Phi_{kmn}(\alpha, \beta) \neq \pm 1$ for large enough k . Thus, S_n^Λ is not a strong divisibility sequence. \square

For the sequences with real α and β , like the Mersenne and Fibonacci numbers, we can rule out the numerical degeneracy easily.

Corollary 5. *Let α and β be real numbers such that $\alpha + \beta$ and $\alpha\beta$ are integers, and $\alpha\beta \neq 0$ and $\alpha \neq \pm\beta$. Then, $|\Phi_d(\alpha, \beta)| > 1$ for $d > 2$, and the only strong divisibility sequences among S_n^Λ are the classical ones S_{Nn}/S_N .*

Proof. From the definition, $\Phi_d(x, y)$ is the product of $x - \zeta y$, where ζ runs over all primitive roots of unity of order d . Consider two circles centered at the origin with radii $|\alpha|$ and $|\beta|$. Then $\zeta\beta$ is on the second circle, since $|\zeta| = 1$, and its distance $|\alpha - \zeta\beta|$ to α is no less than the distance between the circles, $|\alpha + \beta|$ or $|\alpha - \beta|$, depending on the signs of α and β . Moreover, since α and β are real, the distance can equal $|\alpha \pm \beta|$ only if $\zeta = \pm 1$, i.e., the order of ζ is 1 or 2. But $\alpha + \beta$ is an integer, so $|\alpha + \beta| \geq 1$, unless $\alpha = -\beta$. Similarly, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ is an integer, so $|\alpha - \beta| \geq 1$, unless $\alpha = \beta$. Thus, $|\alpha - \zeta\beta| > 1$ for $\zeta \neq \pm 1$, and $|\Phi_d(\alpha, \beta)| > 1$ for $d > 2$. The second conclusion follows from Theorem 7. \square

This implies that of the divisibility sequences we constructed in Section 5, the only strong divisibility sequences are the classical ones, S_n and S_{Nn}/S_N . One can show that the conditions of Theorem 7 also hold for complex conjugate α and β , when $\alpha\beta \neq 0$ and α/β is not a root of unity. But, the proof relies on a non-elementary estimate of Stewart [10] for $|\Phi_d(\alpha, \beta)|$, and we omit it.

ACKNOWLEDGEMENT

The author is grateful to Brian Pasko and the anonymous referees for their comments on a previous version of this paper.

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MSC2010: 11B39, 11B83, 37P05, 11B37

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HOUSTON-DOWNTOWN, HOUSTON, TEXAS, 77002, USA

E-mail address: `koshkins@uhd.edu`