A NON-LINEAR RECURRENCE IDENTITY CLASS FOR TERMS OF A GENERALIZED LINEAR RECURRENCE SEQUENCE OF DEGREE THREE

PETER J. LARCOMBE AND ERIC J. FENNESSEY

Abstract. We state, and prove by a succinct matrix method, a new non-linear identity for terms generated by a general (linear) third order recurrence equation, and show it recovers a result for the well-known Horadam sequence. We then produce a fully generalized version of the identity that possesses three additional characterizing parameters and describes a new class of identities.

1. A Result and Proof

1.1. **Introduction.** Consider, with initial values $v_0 = a$, $v_1 = b$, and $v_2 = c$, the linear recursion

$$v_{n+3} = pv_{n+2} + qv_{n+1} + rv_n, \qquad n \geq 0, \tag{1.1}$$

of degree three. In this short note we state, and prove by a succinct matrix method, a non-linear recurrence identity for the sequence $\{v_n\}_{n=0}^{\infty} = \{v_n\}_0^{\infty} = \{v_n(a,b,c;p,q,r)\}_0^{\infty}$ generated therefrom; it is seen that the result recovers, as a special case, a simpler identity for the long established Horadam sequence (originating with A. F. Horadam in the 1960s) that has been discussed recently by the authors. We then derive, by the same formulation method, a fully generalized version of the identity for terms of the sequence $\{v_n(a,b,c;p,q,r)\}_0^{\infty}$ that is characterized by three extra arbitrary parameters and describes a new identity class.

1.2. **An Identity and Proof.** We state and establish an identity.

Identity. For $s, t \geq 0$,

$$v_{s+2}v_{t+1} + v_{s+1}(v_{t+2} - pv_{t+1}) + rv_s v_t = bv_{s+t+2} + (c - bp)v_{s+t+1} + arv_{s+t}.$$

Example 1. Consider, for values $s = 1$, $t = 2$ (using (1.1) as needed), the identity r.h.s. $= bv_5 + (c-bp)v_4 + arv_3 = b(pv_4 + qv_3 + rv_2) + (c-bp)v_4 + arv_3 = (bq+ar)v_3 + brv_2 + cv_4 = (v_1q + v_0r)v_3 + v_1rv_2 + v_2v_4 = (v_3 - pv_2)v_3 + rv_1v_2 + v_2v_4 = (v_3)^2 + v_2(v_4 - pv_3) + rv_1v_2 =$ l.h.s.

Example 2. Consider, for values $s = 3$, $t = 1$, the identity r.h.s. $= bv_6 + (c-bp)v_5 + arv_4 = b(pv_5 + qv_4 + rv_3) + (c-bp)v_5 + arv_4 = cv_5 + (bq+ar)v_4 + brv_3 = v_2v_5 + (v_1q + v_0r)v_4 + v_1rv_3 = v_5v_2 + v_4(v_3 - pv_2) + rv_3v_1 =$ l.h.s.

Proof. Let

$$\mathbf{R}(p,r) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -p & 0 \\ 0 & 0 & r \end{pmatrix}, \tag{I.1}$$

and define

$$\mathbf{L}_n(p,r) = \mathbf{L}_n(p,r;v_0,v_1,v_2) = (v_2,v_1,v_0)\mathbf{R}(p,r)\begin{pmatrix} v_{n+2} \\ v_{n+1} \\ v_n \end{pmatrix}, \tag{I.2}$$

so that $\mathbf{L}_n(p,r) = v_2v_{n+1} + v_1(v_{n+2} - pv_{n+1}) + v_0rv_n = cv_{n+1} + b(v_{n+2} - pv_{n+1}) + arv_n = bv_{n+2} + (c-bp)v_{n+1} + arv_n$ after a little algebra, with, in particular,

$$\mathbf{L}_{s+t}(p,r) = bv_{s+t+2} + (c-bp)v_{s+t+1} + arv_{s+t}. \tag{I.3}$$

We now introduce the matrix

$$\mathbf{J}(p,q,r) = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{I.4}$$

which captures the recurrence equation (1.1). Trivially, it yields the matrix power relation

$$\begin{pmatrix} v_{n+2} \\ v_{n+1} \\ v_n \end{pmatrix} = \mathbf{J}^n(p,q,r)\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix} \tag{I.5}$$

that holds for $n \geq 1$. Denoting matrix transposition by T, we note the important (quasi-commutativity) result

$$\mathbf{R}(p,r)\mathbf{J}(p,q,r) = [\mathbf{J}(p,q,r)]^T\mathbf{R}(p,r), \tag{I.6}$$

and, appealing to this and (I.5) as needed, we write directly from (I.2),

$$\begin{aligned}
\mathbf{L}_{s+t}(p,r) &= (v_2,v_1,v_0)\mathbf{R}(p,r)\begin{pmatrix} v_{s+t+2} \\ v_{s+t+1} \\ v_{s+t} \end{pmatrix} \\
&= (v_2,v_1,v_0)\mathbf{R}(p,r)\mathbf{J}^{s+t}(p,q,r)\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix} \\
&= (v_2,v_1,v_0)\mathbf{R}(p,r)\mathbf{J}^{s}(p,q,r)\mathbf{J}^{t}(p,q,r)\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix} \\
&= (v_2,v_1,v_0)[\mathbf{J}^{s}(p,q,r)]^T\mathbf{R}(p,r)\mathbf{J}^{t}(p,q,r)\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix} \\
&= \left[\mathbf{J}^{s}(p,q,r)\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix}\right]^T\mathbf{R}(p,r)\mathbf{J}^{t}(p,q,r)\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix} \\
&= \begin{pmatrix} v_{s+2} \\ v_{s+1} \\ v_s \end{pmatrix}^T\begin{pmatrix} 0 & 1 & 0 \\ 1 & -p & 0 \\ 0 & 0 & r \end{pmatrix}\begin{pmatrix} v_{t+2} \\ v_{t+1} \\ v_t \end{pmatrix} \\
&= (v_{s+2},v_{s+1},v_s)\begin{pmatrix} v_{t+1} \\ v_{t+2} - pv_{t+1} \\ rv_t \end{pmatrix} \\
&= v_{s+2}v_{t+1} + v_{s+1}(v_{t+2} - pv_{t+1}) + rv_sv_t;
\end{aligned} \tag{I.7}$$

the proof is complete upon equating $\mathbf{L}_{s+t}(p,r)$ as found in (I.7) and (I.3). □

1.3. **Recovery of a Horadam Sequence Identity.** The well-known second order recursion

$$w_n = pw_{n-1} - qw_{n-2}; \qquad w_0 = a,\ w_1 = b, \tag{1.2}$$

characterized by the four parameters a, b, p, q, defines the so called Horadam sequence written $\{w_n(a, b; p, q)\}_0^\infty$ (the notation for which, and form of (1.2), originates with A. F. Horadam in the mid 1960s when the sequence was first introduced as a fully generalized one on which analysis began in earnest). Thus, noting that $w_2 = pw_1 - qw_0 = pb - qa$, then clearly

$$\{w_n(a, b; p, q)\}_0^\infty = \{v_n(a, b, pb - qa; p, -q, 0)\}_0^\infty \tag{1.3}$$

on which basis our identity reads, in terms of Horadam elements (setting $c = pb - qa$, $r = 0$),

$$w_{t+1}(w_{s+2} - pw_{s+1}) + w_{s+1}w_{t+2} = bw_{s+t+2} + (-qa)w_{s+t+1}, \tag{1.4}$$

that is, deploying (1.2),

$$w_{t+1}(-qw_s) + w_{s+1}w_{t+2} = bw_{s+t+2} - qaw_{s+t+1}. \tag{1.5}$$

In other words (for $s, t \geq 0$),

$$w_{s+1}w_{t+1} - qw_s w_t = bw_{s+t+1} - qaw_{s+t}, \tag{1.6}$$

as formulated in [1]. Note that an equivalent version of (1.6) appeared in each of Horadam's two oft cited seminal papers of 1965, being found at the time directly from the Horadam sequence ordinary generating function (see [1] for details).

2. An Identity Class

The matrix $\mathbf{R}(p, r)$ (I.1) is but one of an infinite number of matrices that exhibit quasi-commutativity with $\mathbf{J}(p, q, r)$ (I.4). Writing

$$\mathbf{S} = \mathbf{S}(s_1, s_2, \ldots, s_9) = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{pmatrix}, \tag{2.1}$$

then the condition $\mathbf{S}(s_1, \ldots, s_9)\mathbf{J}(p, q, r) = [\mathbf{J}(p, q, r)]^T\mathbf{S}(s_1, \ldots, s_9)$ of quasi-commutativity for $\mathbf{S}$ results in nine equations in its elements, the solution of which (a straightforward exercise, see the Appendix) allows a general form of $\mathbf{S}$ to be determined. It is found that, for arbitrary β, γ, δ,

$$\mathbf{S}^{[\beta,\gamma,\delta]}(p, q, r) = \begin{pmatrix} (\gamma p + \delta)/r & \beta & \gamma \\ \beta & [\gamma(pq + r) + \delta q - \beta pr]/r & \delta \\ \gamma & \delta & \beta r - \gamma q \end{pmatrix}. \tag{2.2}$$

Omitting the details, then if the same proof procedure is followed as before (with $\mathbf{S}^{[\beta,\gamma,\delta]}(p, q, r)$ replacing $\mathbf{R}(p, r)$) a previously unseen result is yielded of which the earlier identity is the $\beta = 1$, $\gamma = \delta = 0$ instance (with $\mathbf{S}^{[1,0,0]}(p, q, r) = \mathbf{R}(p, r)$); since β, γ, and δ are arbitrary, we have a *complete class of infinite cases* available to us.

Remark. As an aside, we note that the product matrices $\mathbf{SJ}$ and $\mathbf{J}^T\mathbf{S}$ are, being equal, each symmetric, for we see that $(\mathbf{J}^T\mathbf{S})^T = \mathbf{S}^T(\mathbf{J}^T)^T = \mathbf{S}^T\mathbf{J} = \mathbf{SJ}$ ($\mathbf{S}$ is symmetric) $= \mathbf{J}^T\mathbf{S}$ (by quasi-commutativity). Alternatively, $(\mathbf{SJ})^T = \mathbf{J}^T\mathbf{S}^T = \mathbf{J}^T\mathbf{S}$ ($\mathbf{S}$ is symmetric) $= \mathbf{SJ}$ (by quasi-commutativity).

Identity (Generalized). For $s, t \geq 0$, and arbitrary constants β, γ, δ combining to form parameters $s_1, \ldots, s_9$,

$$\begin{aligned} v_{s+2}(s_1v_{t+2} + s_2v_{t+1} + s_3v_t) + v_{s+1}(s_4v_{t+2} + s_5v_{t+1} + s_6v_t) + v_s(s_7v_{t+2} + s_8v_{t+1} + s_9v_t) \\ = (cs_1 + bs_4 + as_7)v_{s+t+2} + (cs_2 + bs_5 + as_8)v_{s+t+1} + (cs_3 + bs_6 + as_9)v_{s+t}. \end{aligned}$$

We have validated the identity by using a suite of s, t value choices, with wide variety, while keeping a, b, c, p, q, r, β, γ, δ symbolic.

Summary

We have formulated a complex and highly non-linear recurrence identity class for terms of a linear recursion of degree three using a pleasing matrix method. We believe this result to be a new one in the literature. It is clear that the methodology applied here (and in [1]) lends itself to fully general recurrence equations of higher order, which we leave as an open area for study.

Appendix

The quasi-commutativity condition $\mathbf{S}(s_1, \ldots, s_9)\mathbf{J}(p, q, r) = [\mathbf{J}(p, q, r)]^T\mathbf{S}(s_1, \ldots, s_9)$ of Section 2 generates nine equations for the entries $s_i = s_i(p, q, r; \beta, \gamma, \delta)$ of $\mathbf{S}$ $(i = 1, \ldots, 9)$; matching like terms across the equation, we write down

$$\begin{aligned} ps_1 + s_2 &= ps_1 + s_4, \\ qs_1 + s_3 &= ps_2 + s_5, \\ rs_1 &= ps_3 + s_6, \\ ps_4 + s_5 &= qs_1 + s_7, \\ qs_4 + s_6 &= qs_2 + s_8, \\ rs_4 &= qs_3 + s_9, \\ ps_7 + s_8 &= rs_1, \\ qs_7 + s_9 &= rs_2, \\ rs_7 &= rs_3, \end{aligned} \tag{A.1}$$

whose (infinite) solution set (containing three arbitrary constants β, γ, δ) is, assuming $r \neq 0$, seen in (2.2) as $s_1 = (\gamma p + \delta)/r$, $s_2 = \beta = s_4$, $s_3 = \gamma = s_7$, $s_5 = [\gamma(pq + r) + \delta q - \beta pr]/r$, $s_6 = \delta = s_8$, $s_9 = \beta r - \gamma q$.

References

[1] P. J. Larcombe and E. J. Fennessey, *A new non-linear recurrence identity class for Horadam sequence terms*, Palestine J. Mathematics, **7** (2018), 406–409.

MSC2010: 11K31, 15A24

Department of Computing and Mathematics, College of Engineering and Technology, University of Derby, Kedleston Road, Derby DE22 1GB, England, U.K.
E-mail address: p.j.larcombe@derby.ac.uk

BAE Systems Integrated System Technologies, Broad Oak, The Airport, Portsmouth PO3 5PQ, England, U.K.
E-mail address: Eric.Fennessey@baesystems.com