DIVISIBILITY PROPERTIES OF FACTORS OF THE DISCRIMINANT OF GENERALIZED FIBONACCI NUMBERS

YAO-QIANG LI

Abstract. We study some divisibility properties related to the factors of the discriminant of the characteristic polynomial of generalized Fibonacci numbers $(G_n)_{n\geq 0}$ defined by $G_0 = 0$, $G_1 = 1$, and $G_n = pG_{n-1} + qG_{n-2}$ for $n \geq 2$, where p and q are given integers. As corollaries, we give some divisibility properties of some well-known sequences.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers $1, 2, 3, \dots$ and $\mathbb{Z}$ denote the set of all integers. Given $p, q \in \mathbb{Z}$, the $< p, q >$*-Fibonacci sequence* $(G_n)_{n\geq 0}$ is defined by

$$G_0 = 0, \quad G_1 = 1, \quad \text{and} \quad G_n = pG_{n-1} + qG_{n-2} \quad \text{for all } n \geq 2.$$

For rich applications of these sequences in science and nature, see for examples [9, 10, 19].

Let $r = r(p, q) = p^2 + 4q$ denote the discriminant of the characteristic polynomial $x^2 - px - q$ of the $< p, q >$-Fibonacci sequence $(G_n)_{n\geq 0}$. For the classic Fibonacci sequence $(F_n)_{n\geq 0}$ ($p = q = 1$), it was shown by Kuipers and Shiue [11] that the only moduli for which $(F_n)_{n\geq 0}$ can possibly be uniformly distributed are the powers of the discriminant $r = 5$. Next, Niederreiter [12] proved that $(F_n)_{n\geq 0}$ is uniformly distributed mod all powers of 5. Later, the results of Niederreiter and Shiue [13, 14] on uniform distribution of linear recurring sequences in finite fields led to the observation (see [6]) that over the integers, a linear recurring sequence can be uniformly distributed mod s (and mod s^k) only if s divides the discriminant of the characteristic polynomial. These results motivate the investigation of the divisors (and their powers) of the discriminant of the characteristic polynomial of generalized Fibonacci numbers in this paper. For more overviews on uniform distribution of linear recurring sequences, we refer the reader to [6].

Throughout this paper, for $n, m \in \mathbb{Z}$, we use $n \mid m$, $n \nmid m$, and (n, m) to denote that n divides m (i.e., there exists a $k \in \mathbb{Z}$ such that $m = kn$), n does not divide m, and the greatest common divisor of n and m (if $n \neq 0$ or $m \neq 0$), respectively. Note that $0 \mid 0$. An integer sequence $(a_i)_{i\geq 0}$ with the property that

$$n \mid m \quad \text{implies} \quad a_n \mid a_m \quad \text{for all } n, m \geq 0$$

is called a *divisibility sequence*. For all $p, q \in \mathbb{Z}$, it is not difficult to prove by induction that the $< p, q >$-Fibonacci sequences are divisibility sequences (see for example [8, 2.2 Proposition]). On divisibility of the terms by subscripts, we refer the reader to [1, 7, 16, 17]. In this paper, we focus on divisibility properties related to divisors of the discriminant of the characteristic polynomial of generalized Fibonacci numbers. The following theorems and corollaries are our main results.

Theorem 1.1. *Let $p, q \in \mathbb{Z}$, $(G_n)_{n\geq 0}$ be the $< p, q >$-Fibonacci sequence, and $r = p^2 + 4q \neq 0$.*

(1) *For all $s \in \mathbb{N}$ satisfying $s \mid r$ and for all integers $k, n \geq 0$, we have*

$$s^k G_n \mid G_{s^k n}.$$

(2) *Suppose that*

$$p \text{ is odd, } (p,q)=1, \text{ and } s\in\mathbb{N} \text{ satisfying } s\mid r$$

or

$$p \text{ is even, } \left(\frac{p}{2},q\right)=1, \text{ and } s\in\mathbb{N} \text{ satisfying } s\mid \frac{r}{4}$$

or

$$(p,q)=1 \text{ and } s\geq 3 \text{ is a prime satisfying } s\mid r.$$

If $3\nmid q+1$ *or* $3\nmid s$, *then for all integers* $k,n\geq 0$,

$$s^k\mid n \quad \text{if and only if} \quad s^k\mid G_n.$$

Theorem 1.2. *Let* $p,q\in\mathbb{Z}$, $(G_n)_{n\geq 0}$ *be the* $<p,q>$*-Fibonacci sequence,* $r=p^2+4q\neq 0$, *and* $s\in\mathbb{N}$. *Suppose that*

$$p \text{ is odd, } (p,q)=1, \text{ and } s\mid r$$

or

$$p \text{ is even, } \left(\frac{p}{2},q\right)=1, \text{ and } s\mid \frac{r}{4}$$

or

$$(p,q)=1 \text{ and } s \text{ is a prime satisfying } s\mid r.$$

(1) *For all integers* $n\geq 0$,

$$s\mid n \quad \text{if and only if} \quad s\mid G_n.$$

(2) *If for all* $t\in\mathbb{N}$,

$$s\nmid t \quad \text{implies} \quad s^2\nmid G_{st},$$

then for all integers $k,n\geq 0$,

$$s^k\mid n \quad \text{if and only if} \quad s^k\mid G_n.$$

It is worth noting that the $<p,q>$-Fibonacci sequence $(G_n)_{n\geq 0}$ studied in this paper is exactly the Lucasian sequence $(U)=(U_n)_{n\geq 0}$ in [20], with the generator (characteristic polynomial) $f(x)=x^2-px-q$. If $f(x)$ is irreducible modulo a given prime, some laws of apparition of the prime in $(U_n)_{n\geq 0}$ are obtained in [20, Theorem 5.1 and 12.1]. Our results do not require $f(x)$ to be irreducible modulo a prime, but we only consider apparition of factors of the discriminant of $f(x)$. For example, let $(G_n)_{n\geq 0}$ be the $<3,4>$-Fibonacci sequence. Although the generator $f(x)=x^2-3x-4$ is not irreducible modulo 5, because 5 is a prime factor of the discriminant of $f(x)$, by applying Theorem 1.2 (1), we conclude that 5 is the unique rank of apparition (see [20] for definition) of 5 in $(G_n)_{n\geq 0}$. Besides, [20, Theorem 9.1] shows that s is a rank of apparition of any prime s in $(U_n)_{n\geq 0}$, which divides the discriminant of the generator $f(x)$. For the case that $(U_n)_{n\geq 0}$ is the $<p,q>$-Fibonacci sequence $(G_n)_{n\geq 0}$, it is straightforward to see that our Theorem 1.2 (1) (with the conditions $(p,q)=1$ and s is a prime satisfying $s\mid r$) gives [20, Theorem 9.1], noting that $(p,q)\neq 1$ will imply that there exists an integer $m\geq 2$ that divides every term of $(G_n)_{n\geq 0}$ beyond a certain point (in fact, $(p,q)\mid G_n$ for all $n\geq 2$), and this exception is stated in the postil section at the bottom of the first page in [20].

In the following, we give some corollaries according to Theorems 1.1 and 1.2.

Corollary 1.3. *Let* $p,q\in\mathbb{Z}$, $(G_n)_{n\geq 0}$ *be the* $<p,q>$*-Fibonacci sequence,* $r=p^2+4q\neq 0$, *and* $s\in\mathbb{N}$. *If*

$$p \text{ is odd, } (p,q)=1, \text{ and } s^2\mid r$$

or

$$p \text{ is even, } \left(\frac{p}{2}, q\right) = 1, \text{ and } s^2 \mid \frac{r}{4},$$

then for all integers $k, n \geq 0$*,*

$$s^k \mid n \quad \text{if and only if} \quad s^k \mid G_n.$$

Noting that the classic Fibonacci, Pell, and Jacobsthal sequences are exactly the $< 1, 1 >$, $< 2, 1 >$, and $< 1, 2 >$-Fibonacci sequences, respectively, Theorem 1.1 and Corollary 1.3 imply the following corollary.

Corollary 1.4 (Divisibility in Fibonacci, Pell, and Jacobsthal sequences)**.**
(1) *Let* $(F_n)_{n\geq 0}$ *be the Fibonacci sequence defined by*

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.$$

① *For all integers* $k, n \geq 0$*, we have*

$$5^k F_n \mid F_{5^k n}.$$

② *For all integers* $k, n \geq 0$*,*

$$5^k \mid n \quad \text{if and only if} \quad 5^k \mid F_n.$$

(2) *Let* $(P_n)_{n\geq 0}$ *be the Pell sequence defined by*

$$P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for all } n \geq 2.$$

① *For all integers* $k, n \geq 0$*, we have*

$$2^k P_n \mid P_{2^k n}.$$

② *For all integers* $k, n \geq 0$*,*

$$2^k \mid n \quad \text{if and only if} \quad 2^k \mid P_n.$$

(3) *Let* $(J_n)_{n\geq 0}$ *be the Jacobsthal sequence defined by*

$$J_0 = 0, \quad J_1 = 1, \quad \text{and} \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for all } n \geq 2.$$

① *For all integers* $k, n \geq 0$*, we have*

$$3^k J_n \mid J_{3^k n}.$$

② *For all integers* $k, n \geq 0$*,*

$$3^k \mid n \quad \text{if and only if} \quad 3^k \mid J_n.$$

The next corollary focuses on $< p, 1 >$-Fibonacci sequences, which have received much attention in recent years (see for examples [3, 4, 5, 18]).

Corollary 1.5. *Let* $p \in \mathbb{Z}$*,* $(G_n)_{n\geq 0}$ *be the* $< p, 1 >$*-Fibonacci sequence,* $r = p^2 + 4$*, and* $s \in \mathbb{N}$*. If*

$$p \text{ is odd and } s \mid r$$

or

$$p \text{ is even and } s \mid \frac{r}{4}$$

or

$$s \geq 3 \text{ is a prime satisfying } s \mid r,$$

then for all integers $k, n \geq 0$*,*

$$s^k \mid n \quad \text{if and only if} \quad s^k \mid G_n.$$

In addition, we have the following corollary.

Corollary 1.6. *Let $q \in \mathbb{Z}$ and $s \in \mathbb{N}$. Suppose that*

$$(G_n)_{n\geq 0} \text{ is the } < 1, q > \text{-Fibonacci sequence and } s \mid 4q+1$$

or

$$(G_n)_{n\geq 0} \text{ is the } < 2, q > \text{-Fibonacci sequence and } s \mid q+1.$$

(1) *For all integers $n \geq 0$,*

$$s \mid n \quad \text{if and only if} \quad s \mid G_n.$$

(2) *If $3 \nmid q+1$ or $3 \nmid s$, then for all integers $k, n \geq 0$,*

$$s^k \mid n \quad \text{if and only if} \quad s^k \mid G_n.$$

We state the last corollary as follows.

Corollary 1.7. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $(G_n)_{n\geq 0}$ be the $< p, q >$-Fibonacci sequence, and $r = p^2+4q$.*

(1) *If r is a prime, then for all integers $k, n \geq 0$,*

$$r^k \mid n \quad \text{if and only if} \quad r^k \mid G_n.$$

(2) *If $\frac{r}{4}$ is a prime and $p \neq 0$, then for all integers $k, n \geq 0$,*

$$\left(\frac{r}{4}\right)^k \mid n \quad \text{if and only if} \quad \left(\frac{r}{4}\right)^k \mid G_n.$$

Remark 1.8. *More generally, for the sequence $(G_n^*)_{n\geq 0}$ defined by*

$$G_0^* = 0, \quad G_1^* = \alpha, \quad \text{and} \quad G_n^* = pG_{n-1}^* + qG_{n-2}^* \quad \text{for all } n \geq 2,$$

where $\alpha, p, q \in \mathbb{Z}$, Theorem 1.1 (1) still holds, because $(G_n^*)_{n\geq 0} = (\alpha G_n)_{n\geq 0}$.

The paper is organized as follows. In Section 2, we give some examples to clarify that the detailed conditions in Theorems 1.1 and 1.2 and Corollaries 1.3, 1.5, 1.6, and 1.7 cannot be omitted. Then we prove the main results in Section 3, and finally present further questions in Section 4.

2. Examples

Example 2.1. *Let $p = q = 1$, $(G_n)_{n\geq 0}$ be the $< 1, 1 >$-Fibonacci sequence, $r = p^2 + 4q = 5$, and $s = 3$ ($\nmid r$). Then $G_s = 2$ and $s \nmid G_s$. This means that the condition $s \mid r$ in Theorems 1.1 and 1.2 and Corollary 1.5 and the condition $s \mid 4q + 1$ in Corollary 1.6 cannot be omitted.*

Example 2.2. *Let $p = 3$, $q = 9$, $(G_n)_{n\geq 0}$ be the $< 3, 9 >$-Fibonacci sequence, $r = p^2+4q = 45$, and $s = 3$. Then $G_2 = 3$, so we get $s \mid G_2$, but $s \nmid 2$. This means that, even if*

(1) *p is odd, $s \geq 3$ is a prime, $s \mid r$, and $3 \nmid q+1$, the condition $(p, q) = 1$ in Theorem 1.1 (2) and Theorem 1.2 cannot be omitted;*
(2) *p is odd and $s^2 \mid r$, the condition $(p, q) = 1$ in Corollary 1.3 cannot be omitted.*

Example 2.3. *Let $p = 4$, $q = 1$, $(G_n)_{n\geq 0}$ be the $< 4, 1 >$-Fibonacci sequence, and $r = p^2 + 4q = 20$.*

(1) *Let $s = 20$. By a simple calculation, we get $G_{10} = 416020$ and $s \mid G_{10}$, but $s \nmid 10$. This means that, even if*

① *$(p, q) = 1$, $s \mid r$, and $3 \nmid q+1$, the condition that p is odd in Theorem 1.1 (2), Theorem 1.2, and Corollary 1.5 cannot be omitted;*
② *p is even, $(\frac{p}{2}, q) = 1$, $s \mid r$, and $3 \nmid q + 1$, the condition $s \mid \frac{r}{4}$ in Theorem 1.1 (2), Theorem 1.2, and Corollary 1.5 cannot be omitted;*

③ $(p,q)=1$, $s\geq 3$ *satisfies* $s\mid r$, *and* $3\nmid q+1$, *the condition that* s *is a prime in Theorem 1.1 (2), Theorem 1.2, and Corollary 1.5 cannot be omitted.*

(2) *Let* $s=2$. *By* $G_2=4$, *we get* $s^2\mid G_2$, *but* $s^2\nmid 2$. *This means that, even if*

① $(p,q)=1$, s *is a prime satisfying* $s\mid r$, *and* $3\nmid q+1$, *the condition* $s\geq 3$ *in Theorem 1.1 (2) and Corollary 1.5 cannot be omitted;*

② $(p,q)=1$ *and* $s^2\mid r$, *the condition that* p *is odd in Corollary 1.3 cannot be omitted.*

Example 2.4. *Let* $p=q=4$, $(G_n)_{n\geq 0}$ *be the* $<4,4>$*-Fibonacci sequence, and* $r=p^2+4q=32$.

(1) *Let* $s=4$. *By* $G_2=4$, *we get* $s\mid G_2$, *but* $s\nmid 2$. *This means that, even if* p *is even,* $s\mid \frac{r}{4}$, *and* $3\nmid q+1$, *the condition* $(\frac{p}{2},q)=1$ *in Theorem 1.1 (2) and Theorem 1.2 cannot be omitted.*

(2) *Let* $s=2$. *By* $G_3=20$, *we get* $s\mid G_3$, *but* $s\nmid 3$. *This means that, even if* p *is even and* $s^2\mid \frac{r}{4}$, *the condition* $(\frac{p}{2},q)=1$ *in Corollary 1.3 cannot be omitted.*

Example 2.5. *Let* $p=5$, $q=2$, $(G_n)_{n\geq 0}$ *be the* $<5,2>$*-Fibonacci sequence,* $r=p^2+4q=33$, *and* $s=3$. *Then* $G_3=27$, *so we get* $s^2\mid G_3$, *but* $s^2\nmid 3$. *This means that, even if* p *is odd,* $(p,q)=1$, $s\geq 3$ *is a prime, and* $s\mid r$, *the condition* $3\nmid q+1$ *or* $3\nmid s$ *in Theorem 1.1 (2) and the condition* $s^2\mid r$ *in Corollary 1.3 cannot be omitted.*

Example 2.6. *Let* $p=2$, $q=5$, $(G_n)_{n\geq 0}$ *be the* $<2,5>$*-Fibonacci sequence,* $r=p^2+4q=24$, *and* $s=3$. *Then* $G_3=9$, *so we get* $s^2\mid G_3$, *but* $s^2\nmid 3$. *This means that, even if*

(1) p *is even,* $(\frac{p}{2},q)=1$, *and* $s\mid\frac{r}{4}$, *the condition* $3\nmid q+1$ *or* $3\nmid s$ *in Theorem 1.1 (2) and the condition* $s^2\mid \frac{r}{4}$ *in Corollary 1.3 cannot be omitted;*

(2) $s\mid q+1$, *the condition* $3\nmid q+1$ *or* $3\nmid s$ *in Corollary 1.6 (2) cannot be omitted.*

Example 2.7. *Let* $p=q=2$, $(G_n)_{n\geq 0}$ *be the* $<2,2>$*-Fibonacci sequence, and* $r=p^2+4q=12$.

(1) *By a simple calculation, we get* $G_6=120$ *and* $r\mid G_6$, *but* $r\nmid 6$. *This means that the condition that* r *is a prime in Corollary 1.7 (1) cannot be omitted.*

(2) *Let* $s=2$. *By* $G_3=6$, *we get* $s\mid G_3$, *but* $s\nmid 3$. *This means that the condition* $s\mid q+1$ *in Corollary 1.6 cannot be omitted.*

Example 2.8. *Let* $p=4$, $q=2$, $(G_n)_{n\geq 0}$ *be the* $<4,2>$*-Fibonacci sequence, and* $r=p^2+4q=24$. *Then* $\frac{r}{4}=6$, *and* $G_3=18$, *so we get* $\frac{r}{4}\mid G_3$, *but* $\frac{r}{4}\nmid 3$. *This means that, even if* $4\mid r$ *and* $p\neq 0$, *the condition that* $\frac{r}{4}$ *is a prime in Corollary 1.7 (2) cannot be omitted.*

Example 2.9. *Let* $p=1$, $q=8$, $(G_n)_{n\geq 0}$ *be the* $<1,8>$*-Fibonacci sequence, and* $s=3$. *Then* $G_3=9$, *so we get* $s^2\mid G_3$, *but* $s^2\nmid 3$. *This means that, even if* $s\mid 4q+1$, *the condition* $3\nmid q+1$ *or* $3\nmid s$ *in Corollary 1.6 (2) cannot be omitted.*

Example 2.10. *Let* $p=0$, $q=2$, $(G_n)_{n\geq 0}$ *be the* $<0,2>$*-Fibonacci sequence, and* $r=p^2+4q=8$. *Then* $\frac{r}{4}=2$, *and* $G_3=2$, *so we get* $\frac{r}{4}\mid G_3$, *but* $\frac{r}{4}\nmid 3$. *This means that, even if* $\frac{r}{4}$ *is a prime, the condition* $p\neq 0$ *in Corollary 1.7 (2) cannot be omitted.*

Example 2.11. *Let* $p=5$, $q=-5$, $(G_n)_{n\geq 0}$ *be the* $<5,-5>$*-Fibonacci sequence, and* $r=p^2+4q=5$. *Then* $G_2=5$, *so we get* $r\mid G_2$, *but* $r\nmid 2$. *This means that, even if* $q\in\mathbb{Z}$ *and* r *is a prime, the condition* $q\in\mathbb{N}$ *in Corollary 1.7 cannot be omitted for the statement (1).*

Example 2.12. *Let* $p=4$, $q=-2$, $(G_n)_{n\geq 0}$ *be the* $<4,-2>$*-Fibonacci sequence, and* $r=p^2+4q=8$. *Then* $\frac{r}{4}=2$, *and* $G_3=14$, *so we get* $\frac{r}{4}\mid G_3$, *but* $\frac{r}{4}\nmid 3$. *This means that, even if* $q\in\mathbb{Z}$, $\frac{r}{4}$ *is a prime, and* $p\neq 0$, *the condition* $q\in\mathbb{N}$ *in Corollary 1.7 cannot be omitted for the statement (2).*

3. Proof of the Main Results

The following proposition, which says that generalized Fibonacci sequences are all divisibility sequences, follows from [8, 2.2 Proposition] (see also [2, Theorem IV]).

Proposition 3.1. *Let $p, q \in \mathbb{Z}$ and $(G_n)_{n\geq 0}$ be the $<p, q>$-Fibonacci sequence. Then for all integers $k, n \geq 0$, we have $G_n \mid G_{kn}$.*

First, we prove Theorem 1.1 (1), then Theorem 1.2, then Theorem 1.1 (2), and finally the corollaries.

Proof of Theorem 1.1 (1). By the Binet formula (see for examples [21, Theorem 2] and [15, 2.5 Corollary]), for all integers $n \geq 0$, we have

$$G_n = \frac{(p + \sqrt{p^2 + 4q})^n - (p - \sqrt{p^2 + 4q})^n}{2^n \sqrt{p^2 + 4q}} = \frac{(p + \sqrt{r})^n - (p - \sqrt{r})^n}{2^n \sqrt{r}}, \tag{3.1}$$

where r can be negative and $\sqrt{r}$ is a complex number. For all integers $n \geq 0$, let

$$A_n = \frac{(p + \sqrt{r})^n + (p - \sqrt{r})^n}{2} \quad \text{and} \quad B_n = \frac{(p + \sqrt{r})^n - (p - \sqrt{r})^n}{2\sqrt{r}}. \tag{3.2}$$

Then A_n and B_n are both integers,

$$\begin{cases} A_n + B_n\sqrt{r} = (p + \sqrt{r})^n \\ A_n - B_n\sqrt{r} = (p - \sqrt{r})^n \end{cases} \tag{3.3}$$

and

$$G_n = \frac{B_n}{2^{n-1}}.$$

For all integers $n \geq 0$ and $s \geq 1$, by

$$A_{sn} + B_{sn}\sqrt{r} = (p + \sqrt{r})^{sn} = (A_n + B_n\sqrt{r})^s,$$

we get

$$B_{sn}\sqrt{r} = \begin{cases} \binom{s}{1}A_n^{s-1}B_n\sqrt{r} + \binom{s}{3}A_n^{s-3}(B_n\sqrt{r})^3 + \cdots + \binom{s}{s}(B_n\sqrt{r})^s, & \text{if } s \text{ is odd;} \\ \binom{s}{1}A_n^{s-1}B_n\sqrt{r} + \binom{s}{3}A_n^{s-3}(B_n\sqrt{r})^3 + \cdots + \binom{s}{s-1}A_n(B_n\sqrt{r})^{s-1}, & \text{if } s \text{ is even;} \end{cases}$$

where $\binom{s}{t} = \frac{s!}{(s-t)! \cdot t!}$ for all $t \in \{0, 1, \cdots, s\}$, and then

$$B_{sn} = \begin{cases} sA_n^{s-1}B_n + \binom{s}{3}A_n^{s-3}B_n^3 r + \cdots + \binom{s}{s}B_n^s r^{\frac{s-1}{2}}, & \text{if } s \text{ is odd;} \\ sA_n^{s-1}B_n + \binom{s}{3}A_n^{s-3}B_n^3 r + \cdots + \binom{s}{s-1}A_n B_n^{s-1} r^{\frac{s-2}{2}}, & \text{if } s \text{ is even.} \end{cases} \tag{3.4}$$

In the following, we prove that for all $s \in \mathbb{N}$ satisfying $s \mid r$, we have $s^k G_n \mid G_{s^k n}$ for all integers $k, n \geq 0$. We only need to consider $k, n \geq 1$ and $s \geq 2$. For $G_n = 0$, by Proposition 3.1, we get $G_{s^k n} = 0$, and then, $s^k G_n \mid G_{s^k n}$ follows immediately. In the following, it suffices to consider $G_n \neq 0$, which implies $B_n \neq 0$.

① Prove $sG_n \mid G_{sn}$ for all $n \in \mathbb{N}$.

i) Suppose that s is odd.
On the one hand, (3.4) implies that $\frac{B_{sn}}{B_n}$ is an integer and $s \mid \frac{B_{sn}}{B_n}$ (applying $s \mid r$). On the other hand, by Proposition 3.1, we get $G_n \mid G_{sn}$, which implies $2^{(s-1)n} \mid \frac{B_{sn}}{B_n}$. It follows from $(s, 2) = 1$ that $s2^{(s-1)n} \mid \frac{B_{sn}}{B_n}$. Thus, $sG_n \mid G_{sn}$.

ii) Suppose that s is even.
ⓐ Prove $2G_n \mid G_{2n}$, i.e., $2^{n+1}B_n \mid B_{2n}$ for all $n \in \mathbb{N}$.
Because (3.4) implies $B_{2n} = 2A_nB_n$, it suffices to prove $2^n \mid A_n$. By (3.3), we get $(A_n + B_n\sqrt{r})(A_n - B_n\sqrt{r}) = (p+\sqrt{r})^n(p-\sqrt{r})^n$ and then, $A_n^2 - B_n^2 r = (p^2-r)^n$. It follows from $r = p^2+4q$ and $B_n = 2^{n-1}G_n$ that

$$A_n^2 = 4^{n-1}rG_n^2 + (-4q)^n. \tag{3.5}$$

Because $2 \mid s$ and $s \mid r$ imply $2 \mid r$, by $r = p^2+4q$, we get $2 \mid p$ and then, $4 \mid r$. It follows from (3.5) that $4^n \mid A_n^2$, which is equivalent to $2^n \mid A_n$.
ⓑ Prove $sG_n \mid G_{sn}$ for all $n \in \mathbb{N}$.
Because s is even, there exist $a, t \in \mathbb{N}$ such that $s = 2^a t$, where t is odd. By ⓐ, we get

$$2G_{tn} \mid G_{2tn}, \quad 2G_{2tn} \mid G_{2^2tn}, \quad 2G_{2^2tn} \mid G_{2^3tn}, \quad \ldots, \quad 2G_{2^{a-1}tn} \mid G_{2^atn},$$

which imply

$$2^aG_{tn} \mid 2^{a-1}G_{2tn}, \quad 2^{a-1}G_{2tn} \mid 2^{a-2}G_{2^2tn}, \quad 2^{a-2}G_{2^2tn} \mid 2^{a-3}G_{2^3tn}, \quad \ldots, \quad 2G_{2^{a-1}tn} \mid G_{2^atn}.$$

Thus, $2^aG_{tn} \mid G_{2^atn}$. Because $t \mid r$ and t is odd, by i), we get $tG_n \mid G_{tn}$ and then, $2^atG_n \mid 2^aG_{tn}$. Therefore, $2^atG_n \mid G_{2^atn}$, i.e., $sG_n \mid G_{sn}$.

② Prove $s^kG_n \mid G_{s^kn}$ for all $n, k \in \mathbb{N}$. By ①, we get

$$sG_n \mid G_{sn}, \quad sG_{sn} \mid G_{s^2n}, \quad sG_{s^2n} \mid G_{s^3n}, \quad \ldots, \quad sG_{s^{k-1}n} \mid G_{s^kn},$$

which imply

$$s^kG_n \mid s^{k-1}G_{sn}, \quad s^{k-1}G_{sn} \mid s^{k-2}G_{s^2n}, \quad s^{k-2}G_{s^2n} \mid s^{k-3}G_{s^3n}, \quad \ldots, \quad sG_{s^{k-1}n} \mid G_{s^kn}.$$

Therefore, $s^kG_n \mid G_{s^kn}$. □

Proof of Theorem 1.2.
Case 1. Suppose that p is odd, $(p,q)=1$, and $s \mid r$.
First, we prove $(s,p)=1$. It suffices to prove $(r,p)=1$. Let $k=(r,p)$. Then, there exist $a, b \in \mathbb{Z}$ such that $r = ak$ and $p = bk$. It follows from $r = p^2+4q$ that $q = \frac{(a-b^2k)k}{4}$. Because p is odd, k must be odd. By $q \in \mathbb{N}$, we get $\frac{a-b^2k}{4} \in \mathbb{N}$. It follows from $(p,q)=1$ that $k=1$.

Because $r = p^2+4q$ is odd, we know that s is also odd. For $s=1$, the conclusions are obviously true. We only need to consider $s \geq 3$ in the following.

(1) Prove that for all integers $n \geq 0$, $s \mid n$ if and only if $s \mid G_n$.
[⇒] This follows directly from Theorem 1.1 (1).
[⇐] It suffices to consider $n \geq 1$. Suppose $s \mid G_n$. Then $s \mid B_n$. Because (3.2) implies

$$B_n = \begin{cases} np^{n-1} + \binom{n}{3}p^{n-3}r + \binom{n}{5}p^{n-5}r^2 + \cdots + \binom{n}{n}r^{\frac{n-1}{2}}, & \text{if } n \text{ is odd;} \\ np^{n-1} + \binom{n}{3}p^{n-3}r + \binom{n}{5}p^{n-5}r^2 + \cdots + \binom{n}{n-1}pr^{\frac{n-2}{2}}, & \text{if } n \text{ is even;} \end{cases} \tag{3.6}$$

it follows from $s \mid B_n$ and $s \mid r$ that $s \mid np^{n-1}$. By $(s,p)=1$, we get $s \mid n$.

(2) Suppose that for all $t \in \mathbb{N}$, $s \nmid t$ implies $s^2 \nmid G_{st}$. We prove that for all integers $k, n \geq 0$, $s^k \mid n$ if and only if $s^k \mid G_n$.
[⇒] This follows directly from Theorem 1.1 (1).
[⇐] ① First, we prove by induction that for all $t \in \mathbb{N}$ such that $s \nmid t$, we have $s^{k+1} \nmid G_{s^kt}$ for all $k \geq 0$.
i) For $k=0$, $s \nmid G_t$ follows from (1) [⇐].
ii) For $k=1$, $s^2 \nmid G_{st}$ follows from the condition that $s \nmid t$ implies $s^2 \nmid G_{st}$.

iii) Assume that, for some $k \in \mathbb{N}$, we have $s^{k+1} \nmid G_{s^k t}$ for all $t \in \mathbb{N}$ satisfying $s \nmid t$. It suffices to prove $s^{k+2} \nmid G_{s^{k+1}t}$ by contradiction. Assume $s^{k+2} \mid G_{s^{k+1}t}$ for some $t \in \mathbb{N}$ satisfying $s \nmid t$. Then $s^{k+2} \mid B_{s^{k+1}t}$. Because s is odd, by (3.4), we get

$$B_{s^{k+1}t} = B_{s(s^k t)} = sA_{s^k t}^{s-1} B_{s^k t} + \binom{s}{3} A_{s^k t}^{s-3} B_{s^k t}^3 r + \cdots + \binom{s}{s} B_{s^k t}^s r^{\frac{s-1}{2}}. \tag{3.7}$$

Noting that Theorem 1.1 (1) implies $s^k \mid G_{s^k t}$, we get $s^k \mid B_{s^k t}$, and then $s^{k+2} \mid B_{s^k t}^3$. Because $s^{k+2} \mid B_{s^{k+1}t}$ and (3.7), we get $s^{k+2} \mid sA_{s^k t}^{s-1} B_{s^k t}$, and then

$$s^{k+1} \mid A_{s^k t}^{s-1} B_{s^k t}. \tag{3.8}$$

Because (3.2) implies

$$A_{s^k t} = p^{s^k t} + c(p, r, s, k, t), \quad \text{where } r \mid c(p, r, s, k, t),$$

by $s \mid r$ and $(s, p) = 1$, we get $(s, A_{s^k t}) = 1$. It follows from (3.8) that $s^{k+1} \mid B_{s^k t}$. Because $(s, 2) = 1$ and $B_{s^k t} = 2^{s^k t - 1} G_{s^k t}$, we get $s^{k+1} \mid G_{s^k t}$, which contradicts the inductive hypothesis.

② Let $k, n \geq 0$ be integers and suppose $s^k \mid G_n$. We need to prove $s^k \mid n$. It suffices to consider $k, n \geq 1$. Let $l \geq 0$ and $t \geq 1$ be integers such that $n = s^l t$ with $s \nmid t$. By ①, we get $s^{l+1} \nmid G_n$. It follows from $s^k \mid G_n$ that $k \leq l$, which implies $s^k \mid s^l t$ $(= n)$.

Case 2. Suppose that p is even, $(\frac{p}{2}, q) = 1$, and $s \mid \frac{r}{4}$.

First, we prove $(s, \frac{p}{2}) = 1$. It suffices to prove $(\frac{r}{4}, \frac{p}{2}) = 1$. Let $k = (\frac{r}{4}, \frac{p}{2})$. Then, there exist $a, b \in \mathbb{Z}$ such that $\frac{r}{4} = ak$ and $\frac{p}{2} = bk$. It follows from $r = p^2 + 4q$ that $q = (a - b^2 k)k$. Because $(\frac{p}{2}, q) = 1$, we get $k = 1$.

(1) Prove that for all integers $n \geq 0$, $s \mid n$ if and only if $s \mid G_n$.

$\boxed{\Rightarrow}$ This follows directly from Theorem 1.1 (1).

$\boxed{\Leftarrow}$ It suffices to consider $n \geq 1$. Suppose $s \mid G_n$. Because (3.1), we get

$$G_n = \begin{cases} n(\frac{p}{2})^{n-1} + \binom{n}{3}(\frac{p}{2})^{n-3} \cdot \frac{r}{4} + \binom{n}{5}(\frac{p}{2})^{n-5}(\frac{r}{4})^2 + \cdots + \binom{n}{n}(\frac{r}{4})^{\frac{n-1}{2}}, & \text{if } n \text{ is odd;} \\ n(\frac{p}{2})^{n-1} + \binom{n}{3}(\frac{p}{2})^{n-3} \cdot \frac{r}{4} + \binom{n}{5}(\frac{p}{2})^{n-5}(\frac{r}{4})^2 + \cdots + \binom{n}{n-1}\frac{p}{2}(\frac{r}{4})^{\frac{n-2}{2}}, & \text{if } n \text{ is even.} \end{cases} \tag{3.9}$$

It follows from $s \mid G_n$ and $s \mid \frac{r}{4}$ that $s \mid n(\frac{p}{2})^{n-1}$. Because $(s, \frac{p}{2}) = 1$, we get $s \mid n$.

(2) Suppose that for all $t \in \mathbb{N}$, $s \nmid t$ implies $s^2 \nmid G_{st}$. We prove that for all $k, n \geq 0$, $s^k \mid n$ if and only if $s^k \mid G_n$.

$\boxed{\Rightarrow}$ This follows directly from Theorem 1.1 (1).

$\boxed{\Leftarrow}$ ① First, we prove by induction that for all $t \in \mathbb{N}$ such that $s \nmid t$, we have $s^{k+1} \nmid G_{s^k t}$ for all $k \geq 0$.

i) For $k = 0$, $s \nmid G_t$ follows from (1) $\boxed{\Leftarrow}$.

ii) For $k = 1$, $s^2 \nmid G_{st}$ follows from the condition that $s \nmid t$ implies $s^2 \nmid G_{st}$.

iii) Assume that for some $k \in \mathbb{N}$, we have $s^{k+1} \nmid G_{s^k t}$ for all $t \in \mathbb{N}$ satisfying $s \nmid t$. It suffices to prove $s^{k+2} \nmid G_{s^{k+1}t}$ by contradiction. Assume $s^{k+2} \mid G_{s^{k+1}t}$ for some $t \in \mathbb{N}$ satisfying $s \nmid t$. Because (3.4), we get

$$G_{s^{k+1}t} = \frac{B_{s(s^k t)}}{2^{s^{k+1}t-1}} = \begin{cases} s(\frac{A_{s^k t}}{2^{s^k t}})^{s-1} \cdot \frac{B_{s^k t}}{2^{s^k t-1}} + \binom{s}{3}(\frac{A_{s^k t}}{2^{s^k t}})^{s-3}(\frac{B_{s^k t}}{2^{s^k t-1}})^3 \cdot \frac{r}{4} \\ \qquad + \cdots + \binom{s}{s}(\frac{B_{s^k t}}{2^{s^k t-1}})^s(\frac{r}{4})^{\frac{s-1}{2}}, \quad \text{if } s \text{ is odd;} \\ s(\frac{A_{s^k t}}{2^{s^k t}})^{s-1} \cdot \frac{B_{s^k t}}{2^{s^k t-1}} + \binom{s}{3}(\frac{A_{s^k t}}{2^{s^k t}})^{s-3}(\frac{B_{s^k t}}{2^{s^k t-1}})^3 \cdot \frac{r}{4} \\ \qquad + \cdots + \binom{s}{s-1}\frac{A_{s^k t}}{2^{s^k t}}(\frac{B_{s^k t}}{2^{s^k t-1}})^{s-1}(\frac{r}{4})^{\frac{s-2}{2}}, \quad \text{if } s \text{ is even.} \end{cases} \tag{3.10}$$

It follows from (3.2) that

$$\frac{A_n}{2^n} = \begin{cases} \binom{n}{0}(\frac{p}{2})^n + \binom{n}{2}(\frac{p}{2})^{n-2} \cdot \frac{r}{4} + \cdots + \binom{n}{n}(\frac{r}{4})^{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ \binom{n}{0}(\frac{p}{2})^n + \binom{n}{2}(\frac{p}{2})^{n-2} \cdot \frac{r}{4} + \cdots + \binom{n}{n-1}\frac{p}{2}(\frac{r}{4})^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \end{cases} \tag{3.11}$$

for all $n \geq 0$, which implies that $\frac{A_{s^k t}}{2^{s^k t}}$ is an integer. Because Theorem 1.1 (1) implies $s^k \mid G_{s^k t}$, we get $s^{k+2} \mid G_{s^k t}^3$ $(= (\frac{B_{s^k t}}{2^{s^k t-1}})^3)$. It follows from $s^{k+2} \mid G_{s^{k+1}t}$ and (3.10) that $s^{k+2} \mid s(\frac{A_{s^k t}}{2^{s^k t}})^{s-1} \cdot \frac{B_{s^k t}}{2^{s^k t-1}}$ and then,

$$s^{k+1} \mid (\frac{A_{s^k t}}{2^{s^k t}})^{s-1} \cdot \frac{B_{s^k t}}{2^{s^k t-1}}. \tag{3.12}$$

Because (3.11) implies

$$\frac{A_{s^k t}}{2^{s^k t}} = \left(\frac{p}{2}\right)^{s^k t} + c(p,r,s,k,t), \quad \text{where } \frac{r}{4} \mid c(p,r,s,k,t),$$

by $s \mid \frac{r}{4}$ and $(s, \frac{p}{2}) = 1$, we get $(s, \frac{A_{s^k t}}{2^{s^k t}}) = 1$. It follows from (3.12) that $s^{k+1} \mid \frac{B_{s^k t}}{2^{s^k t-1}}$ $(= G_{s^k t})$, which contradicts the inductive hypothesis.

② In the same way as the proof of Case 1 (2) $\boxed{\Leftarrow}$ ②, we know that for all integers $k, n \geq 0$, $s^k \mid G_n$ implies $s^k \mid n$.

Case 3. Suppose that $(p,q) = 1$ and s is a prime satisfying $s \mid r$.
If p is odd, the conclusions follow immediately from Case 1. We only need to consider that p is even in the following. Because $(p,q) = 1$, we get $(\frac{p}{2}, q) = 1$. Because $2 \mid p$ implies $4 \mid r$, by $s \mid r$, we get $s \mid 2^2 \cdot \frac{r}{4}$. Noting that s is a prime, it follows that $s \mid 2$ or $s \mid \frac{r}{4}$. If $s \mid \frac{r}{4}$, the conclusions follow immediately from Case 2. In the following, we only need to consider $s \nmid \frac{r}{4}$ and $s \mid 2$, which imply $s = 2$ and $2 \nmid \frac{r}{4}$, i.e., $2 \nmid (\frac{p}{2})^2 + q$. Because $(p,q) = 1$ and $2 \mid p$ imply $2 \nmid q$, we get $2 \mid (\frac{p}{2})^2$, and then $2 \mid \frac{p}{2}$.

(1) Prove that for all $n \geq 0$, $2 \mid n$ if and only if $2 \mid G_n$.
$\boxed{\Rightarrow}$ This follows directly from Theorem 1.1 (1).
$\boxed{\Leftarrow}$ Suppose $2 \mid G_n$. Because (3.9), $2 \mid \frac{p}{2}$ and $2 \nmid \frac{r}{4}$, we know that n must be even.

(2) Suppose $2^2 \nmid G_{2t}$ for all odd $t \in \mathbb{N}$. We prove that for all integers $k, n \geq 0$, $2^k \mid n$ if and only if $2^k \mid G_n$.
$\boxed{\Rightarrow}$ This follows directly from Theorem 1.1 (1).
$\boxed{\Leftarrow}$ ① First, we prove by induction that for all odd $t \in \mathbb{N}$, we have $2^{k+1} \nmid G_{2^k t}$ for all $k \geq 0$.
i) For $k = 0$, $2 \nmid G_t$ follows from (1) $\boxed{\Leftarrow}$.
ii) For $k = 1$, $2^2 \nmid G_{2t}$ follows from the assumption for odd $t \in \mathbb{N}$.
iii) Assume that for some $k \in \mathbb{N}$, we know that $2^{k+1} \nmid G_{2^k t}$ for all odd $t \in \mathbb{N}$. It suffices to prove $2^{k+2} \nmid G_{2^{k+1}t}$ by contradiction. Assume $2^{k+2} \mid G_{2^{k+1}t}$ for some odd $t \in \mathbb{N}$. Because (3.10) implies $G_{2^{k+1}t} = 2 \cdot \frac{A_{2^k t}}{2^{2^k t}} \cdot \frac{B_{2^k t}}{2^{2^k t-1}} = 2 \cdot \frac{A_{2^k t}}{2^{2^k t}} \cdot G_{2^k t}$, we get

$$2^{k+1} \mid \frac{A_{2^k t}}{2^{2^k t}} \cdot G_{2^k t},$$

where $\frac{A_{2^k t}}{2^{2^k t}}$ is an integer. Because (3.11), $2 \mid \frac{p}{2}$ and $2 \nmid \frac{r}{4}$ imply $2 \nmid \frac{A_{2^k t}}{2^{2^k t}}$, we get $2^{k+1} \mid G_{2^k t}$, which contradicts the inductive hypothesis.

② In the same way as the proof of Case 1 (2) $\boxed{\Leftarrow}$ ②, we know that, for all integers $k, n \geq 0$, $2^k \mid G_n$ implies $2^k \mid n$.

□

Proof of Theorem 1.1 (2).
① Suppose that p is odd, $(p, q) = 1$, $s \in \mathbb{N}$ satisfying $s \mid r$, and $3 \nmid q+1$ or $3 \nmid s$. By Theorem 1.2 (2), it suffices to prove that, for all $t \in \mathbb{N}$ satisfying $s \nmid t$, we have $s^2 \nmid G_{st}$.
(By contradiction) Assume $s^2 \mid G_{st}$. Then $s^2 \mid B_{st}$. Recall from the proof of Theorem 1.2 Case 1 that $(s, p) = 1$ and s is odd. Because $s \nmid t$ implies $s \neq 1$, it follows that $s \geq 3$. Because (3.4), we get

$$B_{st} = sA_t^{s-1}B_t + \binom{s}{3}A_t^{s-3}B_t^3 r + \binom{s}{5}A_t^{s-5}B_t^5 r^2 + \cdots + \binom{s}{s}B_t^s r^{\frac{s-1}{2}}.$$

It follows from $s^2 \mid B_{st}$ and $s \mid r$ that

$$s \mid A_t^{s-1}B_t + \binom{s}{3}A_t^{s-3}B_t^3 \cdot \frac{r}{s}. \tag{3.13}$$

Because (3.2) implies

$$A_t = p^t + c_1(p, r, t), \quad \text{where } r \mid c_1(p, r, t)$$

and

$$B_t = tp^{t-1} + c_2(p, r, t), \quad \text{where } r \mid c_2(p, r, t),$$

by $s \mid r$ and (3.13), we get

$$s \mid (p^t)^{s-1}(tp^{t-1}) + \binom{s}{3}(p^t)^{s-3}(tp^{t-1})^3 \cdot \frac{r}{s},$$

$$\text{i.e.,} \quad s \mid p^{st-3}\Big(tp^2 + t^3 \cdot \frac{r(s-1)(s-2)}{6}\Big).$$

It follows from $(s, p) = 1$ that

$$s \mid tp^2 + t^3 \cdot \frac{r(s-1)(s-2)}{6}. \tag{3.14}$$

Recall the condition $3 \nmid q+1$ or $3 \nmid s$. If $3 \nmid s$, then $3 \mid s-1$ or $3 \mid s-2$. It follows from $3 \mid (s-1)(s-2)$ and $2 \mid (s-1)(s-2)$ that $\frac{(s-1)(s-2)}{6} \in \mathbb{N}$. By (3.14) and $s \mid r$, we get $s \mid tp^2$. It follows from $(s, p) = 1$ that $s \mid t$, which contradicts $s \nmid t$. Thus, we only consider $3 \mid s$. Because $(s, p) = 1$, we get $3 \nmid p$, and then, $p \equiv \pm 1 \bmod 3$. It follows that $r = p^2 + 4q \equiv 1 + q \bmod 3$. Because $3 \mid s$ implies $3 \mid r$, we get $3 \mid q+1$. This contradicts the condition $3 \nmid q+1$ or $3 \nmid s$.
② Suppose that p is even, $(\frac{p}{2}, q) = 1$, $s \in \mathbb{N}$ satisfying $s \mid \frac{r}{4}$, and $3 \nmid q+1$ or $3 \nmid s$. By Theorem 1.2 (2), it suffices to prove that for all $t \in \mathbb{N}$ satisfying $s \nmid t$, we have $s^2 \nmid G_{st}$.
(By contradiction) Assume $s^2 \mid G_{st}$ $(= \frac{B_{st}}{2^{st-1}})$. Recall from the proof of Theorem 1.2 Case 2 that $(s, \frac{p}{2}) = 1$. By (3.4), we get

$$G_{st} = \begin{cases} s(\frac{A_t}{2^t})^{s-1} \cdot \frac{B_t}{2^{t-1}} + \binom{s}{3}(\frac{A_t}{2^t})^{s-3}(\frac{B_t}{2^{t-1}})^3 \cdot \frac{r}{4} + \binom{s}{5}(\frac{A_t}{2^t})^{s-5}(\frac{B_t}{2^{t-1}})^5(\frac{r}{4})^2 \\ \qquad + \cdots + \binom{s}{s}(\frac{B_t}{2^{t-1}})^s(\frac{r}{4})^{\frac{s-1}{2}}, \quad \text{if } s \text{ is odd;} \\ s(\frac{A_t}{2^t})^{s-1} \cdot \frac{B_t}{2^{t-1}} + \binom{s}{3}(\frac{A_t}{2^t})^{s-3}(\frac{B_t}{2^{t-1}})^3 \cdot \frac{r}{4} + \binom{s}{5}(\frac{A_t}{2^t})^{s-5}(\frac{B_t}{2^{t-1}})^5(\frac{r}{4})^2 \\ \qquad + \cdots + \binom{s}{s-1}\frac{A_t}{2^t}(\frac{B_t}{2^{t-1}})^{s-1}(\frac{r}{4})^{\frac{s-2}{2}}, \quad \text{if } s \text{ is even.} \end{cases} \tag{3.15}$$

Because (3.11) implies that $\frac{A_t}{2^t}$ is an integer, it follows from (3.15), $s^2 \mid G_{st}$, and $s \mid \frac{r}{4}$ that

$$s \mid \left(\frac{A_t}{2^t}\right)^{s-1} \cdot \frac{B_t}{2^{t-1}} + \binom{s}{3}\left(\frac{A_t}{2^t}\right)^{s-3}\left(\frac{B_t}{2^{t-1}}\right)^3 \cdot \frac{r}{4s}. \tag{3.16}$$

Noting that (3.11) implies

$$\frac{A_t}{2^t} = \left(\frac{p}{2}\right)^t + c_1(p,r,t), \quad \text{where } \frac{r}{4} \mid c_1(p,r,t)$$

and (3.9) implies

$$\frac{B_t}{2^{t-1}} = t\left(\frac{p}{2}\right)^{t-1} + c_2(p,r,t), \quad \text{where } \frac{r}{4} \mid c_2(p,r,t),$$

by $s \mid \frac{r}{4}$ and (3.16), we get

$$s \mid \left(\frac{p}{2}\right)^{t(s-1)} \cdot t\left(\frac{p}{2}\right)^{t-1} + \binom{s}{3}\left(\frac{p}{2}\right)^{t(s-3)}\left(t\left(\frac{p}{2}\right)^{t-1}\right)^3 \cdot \frac{r}{4s},$$

$$\text{i.e.,} \quad s \mid \left(\frac{p}{2}\right)^{st-3}\left(t\left(\frac{p}{2}\right)^2 + t^3 \cdot \frac{r}{4} \cdot \frac{(s-1)(s-2)}{6}\right).$$

It follows from $(s, \frac{p}{2}) = 1$ that

$$s \mid t\left(\frac{p}{2}\right)^2 + t^3 \cdot \frac{r}{4} \cdot \frac{(s-1)(s-2)}{6}. \tag{3.17}$$

In a similar way as the end of ①, the contradiction follows.
③ Suppose that $(p,q) = 1$, $s \geq 3$ is a prime satisfying $s \mid r$, and $3 \nmid q+1$ or $3 \nmid s$. If p is odd, the conclusion follows immediately from ①. If p is even, in the same way as the proof of Theorem 1.2 Case 3, we get $(\frac{p}{2}, q) = 1$ and $s \mid \frac{r}{4}$, by the condition $s \geq 3$. Then, the conclusion follows immediately from ②. □

Proof of Corollary 1.3. Suppose that

$$p \text{ is odd}, \ (p,q) = 1, \ \text{and } s^2 \mid r$$

or

$$p \text{ is even}, \ \left(\frac{p}{2}, q\right) = 1, \ \text{and } s^2 \mid \frac{r}{4}.$$

By Theorem 1.2 (1), we know that for all integers $n \geq 0$,

$$s^2 \mid n \quad \text{if and only if} \quad s^2 \mid G_n. \tag{3.18}$$

To complete the proof, it suffices to check the condition of Theorem 1.2 (2). For all $t \in \mathbb{N}$ satisfying $s \nmid t$, we have $s^2 \nmid st$. It follows from (3.18) that $s^2 \nmid G_{st}$. □

Corollary 1.4 (1), (2), and (3) ① follow from Theorem 1.1, whereas (3) ② follows from Corollary 1.3.

Corollary 1.5 follows immediately from taking $q = 1$ in Theorem 1.1 (2).

Proof of Corollary 1.6. Let $q \in \mathbb{Z}$ and $s \in \mathbb{N}$. If $(G_n)_{n\geq 0}$ is the $< 1, q >$-Fibonacci sequence and $s \mid 4q+1$, or $(G_n)_{n\geq 0}$ is the $< 2, q >$-Fibonacci sequence and $s \mid q+1$ with $q \neq -1$, then the conclusions follow directly from Theorem 1.2 (1) and Theorem 1.1 (2). If $(G_n)_{n\geq 0}$ is the $< 2, -1 >$-Fibonacci sequence, it is straightforward to get $G_n = n$ for all $n \geq 0$, and then, the conclusions follow. □

Proof of Corollary 1.7. Let $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $r = p^2 + 4q \geq 4$.
(1) Suppose that r is a prime. Then we have the following results.

① p is odd, because if p is even, then $4 \mid r$ will contradict that r is a prime;
② $(p,q) = 1$, because $(p,q) \mid r$, r is a prime, and $(p,q) \leq q < 4q \leq r$.
③ $3 \nmid r$, because r is a prime and $r \geq 4$.

By taking $s = r$ in Theorem 1.1 (2), the conclusion follows.
(2) Suppose that $4 \mid r$, $\frac{r}{4} = (\frac{p}{2})^2 + q$ is a prime, and $p \neq 0$.

① p is even, because $4 \mid r$.

② $(\frac{p}{2}, q) = 1$, because $(\frac{p}{2}, q) \mid \frac{r}{4}$, $\frac{r}{4}$ is a prime, and $(\frac{p}{2}, q) \leq q < \frac{r}{4}$.

Let $s = \frac{r}{4}$. If $3 \nmid s$, the conclusion follows immediately from ①, ②, and Theorem 1.1 (2). We only need to consider $3 \mid s$ in the following argument. Because s is a prime, we get $s = 3$. It follows from $s = (\frac{p}{2})^2 + q$, $p \in \mathbb{Z} \setminus \{0\}$, and $q \in \mathbb{N}$ that $(\frac{p}{2})^2 = 1$ and $q = 2$. To complete the proof, by Theorem 1.2 (2), it suffices to check that for all $t \in \mathbb{N}$ satisfying $3 \nmid t$, we have $3^2 \nmid G_{3t}$.
(By contradiction) Assume $3^2 \mid G_{3t}$. In the same way as the proof of Theorem 1.1 (2) ②, we get (3.17). That is, $3 \mid t + t^3$. By $3 \nmid t$, there exists an integer $m \geq 0$ such that $t = 3m + 1$ or $3m + 2$. Thus, $3 \mid (3m + 1) + (3m + 1)^3$ or $(3m + 2) + (3m + 2)^3$, which implies $3 \mid 1 + 1$ or $2 + 8$. This is impossible. □

4. Further Questions

In Theorems 1.1, 1.2 and Corollaries 1.3, 1.5, 1.6, we give sufficient conditions (on p, q, r, s) for the equivalences of

$$s \mid n \quad \text{and} \quad s \mid G_n \quad \text{for all integers } n \geq 0,$$

and

$$s^k \mid n \quad \text{and} \quad s^k \mid G_n \quad \text{for all integers } k, n \geq 0.$$

What are the necessary and sufficient conditions for these equivalences?

Acknowledgment

The author thanks Prof. Jean-Paul Allouche for his advice, Dr. Shuo Li for discussions, the referee for useful suggestions, and the Oversea Study Program of Guangzhou Elite Project (GEP) for financial support (JY201815).

References

[1] R. André-Jeannin, *Divisibility of generalized Fibonacci and Lucas numbers by their subscripts*, The Fibonacci Quarterly, **29.4** (1991), 364–366.
[2] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. of Math., (2) **15** (1913/14), 30–48.
[3] S. Falcon, *On the powers of the k-Fibonacci numbers*, Ars Combin., **127** (2016), 329–338.
[4] S. Falcon and A. Plaza, *On k-Fibonacci numbers of arithmetic indexes*, Appl. Math. Comput., **208.1** (2009), 180–185.
[5] S. Falcón and Á. Plaza, *On the Fibonacci k-numbers*, Chaos Solitons Fractals, **32.5** (2007), 1615–1624.
[6] T. Herendi, *Uniform distribution of linear recurring sequences modulo prime powers*, Finite Fields Appl., **10.1** (2004), 1–23.
[7] V. E. Hoggatt, Jr. and G. E. Bergum, *Divisibility and congruence relations*, The Fibonacci Quarterly, **12.2** (1974), 189–195.
[8] P. Horak and L. Skula, *A characterization of the second-order strong divisibility sequences*, The Fibonacci Quarterly, **23.2** (1985), 126–132.
[9] E. Karaduman, *An application of Fibonacci numbers in matrices*, Appl. Math. Comput., **147.3** (2004), 903–908.
[10] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Vol. 2, Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2019.
[11] L. Kuipers and J.-S. Shiue, *A distribution property of the sequence of Fibonacci numbers*, The Fibonacci Quarterly, **10.4**, (1972), 375–376.
[12] H. Niederreiter, *Distribution of Fibonacci numbers mod* 5^k, The Fibonacci Quarterly, **10.4** (1972), 373–374.

[13] H. Niederreiter and J.-S. Shiue, *Equidistribution of linear recurring sequences in finite fields*, Indag. Math., **39** (1977), 397–405.

[14] H. Niederreiter and J.-S. Shiue, *Equidistribution of linear recurring sequences in finite fields, II*, Acta Arith., **38** (1980), 197–207.

[15] Z. Şiar and R. Keskin, *Some new identities concerning generalized Fibonacci and Lucas numbers*, Hacet. J. Math. Stat., **42.3** (2013), 211–222.

[16] C. Smyth, *The terms in Lucas sequences divisible by their indices*, J. Integer Seq., **13.2** (2010), Article 10.2.4, 18 pp.

[17] L. Somer, *Divisibility of terms in Lucas sequences by their subscripts*, Applications of Fibonacci Numbers, Vol. 5 (St. Andrews, 1992), Kluwer Acad. Publ., Dordrecht, 1993, 515–525.

[18] P. Trojovský, *On the sum of powers of two k-Fibonacci numbers which belongs to the sequence of k-Lucas numbers*, Tatra Mt. Math. Publ., **67** (2016), 41–46.

[19] S. Vajda, *Fibonacci & Lucas numbers, and the golden section. Theory and applications*, With chapter XII by B. W. Conolly. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1989.

[20] M. Ward, *The law of apparition of primes in a Lucasian sequence*, Trans. Amer. Math. Soc., **44.1** (1938), 68–86.

[21] Y. Yazlik and N. Taskara, *A note on generalized k-Horadam sequence*, Comput. Math. Appl., **63.1** (2012), 36–41.

MSC2010: 11B39

Institut de Mathématiques de Jussieu - Paris Rive Gauche, Sorbonne Université - Campus Pierre et Marie Curie, Paris, 75005, France

Email address: `yaoqiang.li@imj-prg.fr` `yaoqiang.li@etu.upmc.fr`

School of Mathematics, South China University of Technology, Guangzhou, 510641, P. R. China

Email address: `scutyaoqiangli@gmail.com` `scutyaoqiangli@qq.com`