INFINITE SUMS INVOLVING GIBONACCI POLYNOMIAL PRODUCTS

THOMAS KOSHY

Abstract. We explore infinite sums involving Fibonacci and Lucas polynomial products, and their Pell and Pell-Lucas implications.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \text{ and } l_n(x) = \alpha^n(x) + \beta^n(x),$$

where $\alpha(x) = \dfrac{x + \sqrt{x^2+4}}{2}$ and $\beta(x) = \dfrac{x - \sqrt{x^2+4}}{2}$. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. They also can be defined by the Binet-like formulas

$$p_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)} \text{ and } q_n(x) = \gamma^n(x) + \delta^n(x),$$

where $\gamma(x) = x + \sqrt{x^2+1}$ and $\delta(x) = x - \sqrt{x^2+1}$. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $\alpha = \alpha(1)$, $\beta = \beta(1)$, $\gamma = \gamma(1)$, $\delta = \delta(1)$, and $\Delta = \alpha(x) - \beta(x) = \sqrt{x^2+4}$, and omit a lot of basic algebra.

2. Sums of Reciprocals of Fibonacci Polynomial Products

Our discourse hinges on the *Cassini-like identity* $f_{n+k}f_{n-k} - f_n^2 = (-1)^{n+k+1}f_k^2$, *Gelin-Cesàro-like identity* $f_{n+2}f_{n+1}f_{n-1}f_{n-2} = f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2$, *addition formula* $f_{m-n} = (-1)^n(f_m f_{n-1} - f_{m-1}f_n)$, and the identities $f_{2n} = f_n l_n, l_n = f_{n+1} + f_{n-1}$, $f_{n+2} + f_{n-2} = (x^2+2)f_n$, and $xf_{n+3} = x^2 f_{n+2} + (x^2+1)f_n - f_{n-2}$ [2].

With this background, we embark on our explorations with the first infinite sum.

Theorem 2.1.

$$\sum_{n=0}^{\infty} \frac{x}{f_{2n}^2 + 1} = \alpha(x). \tag{2.1}$$

Proof. First, we will establish the formula

$$\sum_{n=0}^{m} \frac{x}{f_{2k}^2+1} = \frac{f_{2m+2}}{f_{2m+1}}, \tag{2.2}$$

using a recursive technique [2]. To this end, let A_m denote the left side of equation (2.2) and B_m its right side. Using the addition formula and the Cassini-like identity, we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{2m+2}}{f_{2m+1}} - \frac{f_{2m}}{f_{2m-1}} \\ &= \frac{f_{2m+2}f_{2m-1} - f_{2m+1}f_{2m}}{f_{2m+1}f_{2m-1}} \\ &= \frac{f_{(2m+2)-2m}}{f_{2m}^2+1} \\ &= \frac{x}{f_{2m}^2+1} \\ &= A_m - A_{m-1}. \end{aligned}$$

Thus, $A_m - A_{m-1} = B_m - B_{m-1}$; so $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_0 - B_0 = x - x = 0$. This implies $A_m = B_m$.

Because $\lim\limits_{m\to\infty} \dfrac{f_{m+1}}{f_m} = \alpha(x)$, it follows from equation (2.2) that

$$\sum_{n=0}^{\infty} \frac{x}{f_{2n}^2+1} = \alpha(x),$$

as desired. □

Equation (2.1) yields

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n}^2+1} = \frac{1+\sqrt{5}}{2},$$

as in [4, 6].

Next, we establish a corresponding result for odd-numbered Fibonacci polynomials.

Theorem 2.2.

$$\sum_{n=0}^{\infty} \frac{x^3+2x}{f_{2n+1}^2+x^2} = \alpha(x) - \beta(x). \tag{2.3}$$

Proof. First, we will confirm that

$$\sum_{n=0}^{m} \frac{x^3+2x}{f_{2n+1}^2+x^2} = \frac{f_{4m+4}}{f_{2m+3}f_{2m+1}}, \tag{2.4}$$

using recursion. Again, we let A_m denote the left side of equation (2.4) and B_m its right side. Using the addition formula $f_{m-n} = (-1)^n(f_m f_{n-1} - f_{m-1} f_n)$, Cassini-like identity *Cassini-like identity* $f_{n+k}f_{n-k} - f_n^2 = (-1)^{n+k+1} f_k^2$, and the identities $f_{2m} = f_m l_m$, $l_m = f_{m+1} + f_{m-1}$, and $f_{m+3} + f_{m-1} = (x^2+2)f_{m+1}$, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{4m+4}}{f_{2m+3}f_{2m+1}} - \frac{f_{4m}}{f_{2m+1}f_{2m-1}} \\ &= \frac{f_{2m+2}(f_{2m+3}+f_{2m+1})}{f_{2m+3}f_{2m+1}} - \frac{f_{2m}(f_{2m+1}+f_{2m-1})}{f_{2m+1}f_{2m-1}} \\ &= \frac{f_{2m+3}(f_{2m+2}f_{2m-1} - f_{2m+1}f_{2m}) - f_{2m-1}(f_{2m+3}f_{2m} - f_{2m+2}f_{2m+1})}{f_{2m+3}f_{2m+1}f_{2m-1}} \\ &= \frac{f_{2m+3}f_2 + f_{2m-1}f_2}{f_{2m+3}f_{2m+1}f_{2m-1}} \\ &= \frac{x(f_{2m+3}+f_{2m-1})}{f_{2m+3}f_{2m+1}f_{2m-1}} \\ &= \frac{(x^3+2x)f_{2m+1}}{f_{2m+3}f_{2m+1}f_{2m-1}} \\ &= \frac{x^3+2x}{f_{2m+3}f_{2m-1}} \\ &= \frac{x^3+2x}{f_{2m+1}^2+x^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Consequently, $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_0 - B_0 = \frac{x^3+2x}{f_1^2+x^2} - \frac{f_4}{f_3 f_1} = 0$. So, $A_m = B_m$.

It then follows from equation (2.4) that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{x^3+2x}{f_{2n+1}^2+x^2} &= \lim_{m\to\infty} \frac{f_{2m+2}l_{2m+2}}{f_{2m+3}f_{2m+1}} \\ &= \lim_{m\to\infty} \frac{f_{2m+2}(f_{2m+3}+f_{2m+1})}{f_{2m+3}f_{2m+1}} \\ &= \alpha(x) + \frac{1}{\alpha(x)} \\ &= \alpha(x) - \beta(x), \end{aligned}$$

as expected. □

Theorem 2.2 implies that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2+1} = \frac{\sqrt{5}}{3},$$

as in [6].

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{F_n^2+1} &= \sum_{n=0}^{\infty} \frac{1}{F_{2n}^2+1} + \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2+1} \\ &= \frac{3+5\sqrt{5}}{6}, \end{aligned} \tag{2.5}$$

as in [4, 6].

The next result is interesting in its own right. It hinges on a finite sum of reciprocals of Fibonacci polynomial products studied in [3].

Theorem 2.3. *Let* $\alpha = \alpha(x)$ *and* $\beta = \beta(x)$. *Then,*

$$\sum_{n=3}^{\infty} \frac{(x^2+1)(x^3+2x)}{f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2} = \frac{2(x^4+4x^2+2)}{x^3+x} + 2\Delta^2\beta - \frac{1}{x^3+2x}. \tag{2.6}$$

Proof. With $W_n = f_n$, $k = 1$, and $l = 0$, it follows from formula (23) in [3] that

$$\sum_{n=1}^{m-1} \frac{x^3+x}{f_n f_{n+1} f_{n+2} f_{n+3}} = \frac{x^4+4x^2+2}{x^3+x} - \left(\frac{f_{m-1}}{f_m} + \frac{(x^2+2)f_m}{f_{m+1}} + \frac{f_{m+1}}{f_{m+2}}\right).$$

Consequently,

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{x^3+x}{f_{n-2}f_{n-1}f_n f_{n+1}} &= \frac{x^4+4x^2+2}{x^3+x} - \left(\frac{1}{\alpha} + \frac{x^2+2}{\alpha} + \frac{1}{\alpha}\right) \\ &= \frac{x^4+4x^2+2}{x^3+x} + \Delta^2\beta; \qquad (2.7) \\ \sum_{n=3}^{\infty} \frac{x^3+x}{f_{n-1}f_n f_{n+1} f_{n+2}} &= \frac{x^4+4x^2+2}{x^3+x} + \Delta^2\beta - \frac{1}{x^3+2x}. \qquad (2.8) \end{aligned}$$

Because $f_{n+2} + f_{n-2} = (x^2+2)f_n$ and

$$\begin{aligned} f_{n-2}f_{n-1}f_{n+1}f_{n+2} &= \left[f_n^2 - (-1)^n x^2\right]\left[f_n^2 + (-1)^n\right] \\ &= f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2, \end{aligned}$$

we then have

$$\begin{aligned} \frac{x^2+2}{f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2} &= \frac{x^2+2}{f_{n-2}f_{n-1}f_{n+1}f_{n+2}} \\ &= \frac{(x^2+2)f_n}{f_{n-2}f_{n-1}f_n f_{n+1}f_{n+2}} \\ &= \frac{f_{n+2}+f_{n-2}}{f_{n-2}f_{n-1}f_n f_{n+1}f_{n+2}} \\ &= \frac{1}{f_{n-2}f_{n-1}f_n f_{n+1}} + \frac{1}{f_{n-1}f_n f_{n+1}f_{n+2}}. \end{aligned}$$

Using equations (2.7) and (2.8), we then get

$$\sum_{n=3}^{\infty} \frac{(x^2+1)(x^3+2x)}{f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2} = \frac{2(x^4+4x^2+2)}{x^3+x} + 2\Delta^2\beta - \frac{1}{x^3+2x},$$

as desired. □

It follows from formula (2.6) that

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{F_n^4 - 1} &= \frac{1}{6}\left(\frac{2\cdot 7}{2} + 10\beta - \frac{1}{3}\right) \\ &= \frac{35}{18} - \frac{5\sqrt{5}}{6}, \qquad (2.9) \end{aligned}$$

as in [4, 6].

2.1. **An Interesting Byproduct.** Using equations (2.5) and (2.9), we can evaluate the sum $\sum_{n=3}^{\infty} \frac{1}{F_n^2 - 1}$:

$$\begin{aligned}
\frac{1}{F_n^2 - 1} &= \frac{2}{F_n^4 - 1} + \frac{1}{F_n^2 + 1}; \\
\sum_{n=3}^{\infty} \frac{1}{F_n^2 - 1} &= \sum_{n=3}^{\infty} \frac{2}{F_n^4 - 1} + \sum_{n=3}^{\infty} \frac{1}{F_n^2 + 1} \\
&= \frac{35 - 15\sqrt{5}}{9} + \left(\frac{-3 + 5\sqrt{5}}{6} - 1 \right) \\
&= \frac{43}{18} - \frac{5\sqrt{5}}{6},
\end{aligned}$$

as in [4, 6].

Next, we extract the Pell consequences of the above polynomial sums.

3. Pell Implications

Because $p_n(x) = f_n(2x)$, it follows from equations (2.1), (2.3), (2.7), (2.8), and (2.6) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2x}{p_{2n}^2 + 1} &= \gamma(x); \\
\sum_{n=0}^{\infty} \frac{4(2x^3 + x)}{p_{2n+1}^2 + 4x^2} &= \gamma(x) - \delta(x); \\
\sum_{n=3}^{\infty} \frac{4x^3 + x}{p_{n-2}p_{n-1}p_n p_{n+1}} &= \frac{8x^4 + 8x^2 + 1}{2(4x^3 + x)} + 2(x^2 + 1)\delta(x); \\
\sum_{n=3}^{\infty} \frac{4x^3 + x}{p_{n-1}p_n p_{n+1}p_{n+2}} &= \frac{8x^4 + 8x^2 + 1}{2(4x^3 + x)} + 2(x^2 + 1)\delta(x) - \frac{1}{8(2x^3 + x)}; \\
\sum_{n=3}^{\infty} \frac{2(2x^2 + 1)(4x^3 + x)}{p_n^4 - (-1)^n(4x^2 - 1)p_n^2 - 4x^2} &= \frac{8x^4 + 8x^2 + 1}{4x^3 + x} + 4(x^2 + 1)\delta(x) - \frac{1}{8(2x^3 + x)},
\end{aligned}$$

respectively.

Consequently, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{P_{2n}^2 + 1} &= \frac{1 + \sqrt{2}}{2}; \\
\sum_{n=0}^{\infty} \frac{1}{P_{2n+1}^2 + 4} &= \frac{\sqrt{2}}{6}; \\
\sum_{n=3}^{\infty} \frac{1}{P_{n+1}P_n P_{n-1}P_{n-2}} &= \frac{57 - 40\sqrt{2}}{50}; \\
\sum_{n=3}^{\infty} \frac{1}{P_{n+2}P_{n+1}P_n P_{n-1}} &= \frac{679 - 480\sqrt{2}}{600};
\end{aligned}$$

$$\sum_{n=3}^{\infty} \frac{1}{P_n^4 - 3(-1)^n P_n^2 - 4} \quad = \quad \frac{1363 - 960\sqrt{2}}{3600},$$

respectively.

Next, we explore the Lucas versions of the sums in equations (2.1), (2.4), and (2.6).

4. Lucas Companions

The identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [2, p. 37] plays a pivotal role in our explorations. For example, it follows from equation (2.2) that

$$\sum_{n=0}^{m} \frac{x}{\Delta^2 f_{2n}^2 + \Delta^2} \quad = \quad \frac{f_{2m+2}}{\Delta^2 f_{2m+1}};$$

$$\sum_{n=0}^{m} \frac{x}{l_{2n}^2 - 4 + \Delta^2} \quad = \quad \frac{f_{2m+2}}{\Delta^2 f_{2m+1}};$$

$$\sum_{n=0}^{m} \frac{x}{l_{2n}^2 + x^2} \quad = \quad \frac{f_{2m+2}}{\Delta^2 f_{2m+1}}. \tag{4.1}$$

Similarly, it follows from equation (2.4) that

$$\sum_{n=0}^{m} \frac{x^3 + 2x}{l_{2n+1}^2 + (x^2+2)^2} = \frac{f_{4m+4}}{\Delta^2 f_{2m+3} f_{2m+1}}. \tag{4.2}$$

Thus,

$$\sum_{n=0}^{\infty} \frac{x}{l_{2n}^2 + x^2} \quad = \quad \frac{\alpha(x)}{x^2 + 4}; \tag{4.3}$$

$$\sum_{n=0}^{\infty} \frac{x^3 + 2x}{l_{2n+1}^2 + (x^2+2)^2} \quad = \quad \frac{1}{\sqrt{x^2+4}}. \tag{4.4}$$

Similarly, equation (2.6) yields

$$\sum_{n=3}^{\infty} \frac{(x^2+1)(x^3+2x)\Delta^4}{d(x)} \quad = \quad \frac{2(x^4+4x^2+2)}{x^3+x} + 2\Delta^2\beta(x) - \frac{1}{x^3+2x}, \tag{4.5}$$

where $d(x) = l_n^4 - (-1)^n[(x^2-1)\Delta^2 + 8]l_n^2 - x^2(x^2+2)^2$.

In particular, we get

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}^2 + 1} \quad = \quad \frac{\alpha}{5};$$

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n+1}^2 + 9} \quad = \quad \frac{\sqrt{5}}{15};$$

$$\sum_{n=3}^{\infty} \frac{1}{L_n^4 - 8(-1)^n L_n^2 - 9} \quad = \quad \frac{7}{90} - \frac{\sqrt{5}}{30}.$$

Next, we develop the Lucas counterpart of Theorem 2.3.

Theorem 4.1.

$$\sum_{n=3}^{\infty} \frac{(x^2+2)(x^4+x^2)}{l_n^4+(-1)^n(x^2-1)\Delta^2 l_n^2-\Delta^4 x^2} = \frac{(x^2+1)(x^6+6x^4+10x^2+3)}{(x^2+2)(x^2+3)(x^4+4x^2+2)} - \frac{x}{\Delta}. \tag{4.6}$$

Proof. With $W_n = l_n$, $k = 1$, and $l = 0$, it follows from formula (23) in [3] that

$$\begin{aligned}
\sum_{n=1}^{m-1} \frac{x^2(x^2+1)\Delta^2}{l_n l_{n+1} l_{n+2} l_{n+3}} &= \left(\frac{f_{m-1}}{l_m} + \frac{l_2 f_m}{l_{m+1}} + \frac{f_{m+1}}{l_{m+2}}\right) - \frac{\Delta^2 f_4}{l_2 l_3};\\
\sum_{n=1}^{m-1} \frac{x^4+x^2}{l_n l_{n+1} l_{n+2} l_{n+3}} &= \frac{1}{\Delta^2}\left[\frac{f_{m-1}}{l_m} + \frac{(x^2+2) f_m}{l_{m+1}} + \frac{f_{m+1}}{l_{m+2}}\right] - \frac{x^3+2x}{(x^2+2)(x^3+3x)}\\
&= \frac{1}{\Delta^2}\left[\frac{f_{m-1}}{l_m} + \frac{(x^2+2) f_m}{l_{m+1}} + \frac{f_{m+1}}{l_{m+2}}\right] - \frac{1}{x^2+3}.
\end{aligned}$$

Because $\lim_{m\to\infty} \frac{f_m}{l_{m+1}} = \frac{1}{\Delta\alpha} = -\frac{\beta}{\Delta}$, this yields

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{x^4+x^2}{l_{n-2} l_{n-1} l_n l_{n+1}} &= \frac{1}{\Delta^2}\left(\frac{1}{\Delta\alpha} + \frac{x^2+2}{\Delta\alpha} + \frac{1}{\Delta\alpha}\right) - \frac{1}{x^2+3}\\
&= -\frac{\beta}{\Delta} - \frac{1}{x^2+3}\\
&= -\frac{\Delta\, x - \Delta^2}{2\Delta^2} - \frac{1}{x^2+3}\\
&= \frac{(x^2+1)\Delta^2 - (x^3+3x)\Delta}{2(x^2+3)\Delta^2}.
\end{aligned} \tag{4.7}$$

Consequently,

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{x^4+x^2}{l_{n-1} l_n l_{n+1} l_{n+2}} &= \frac{(x^2+1)\Delta^2 - (x^3+3x)\Delta}{2(x^2+3)\Delta^2} - \frac{x^4+x^2}{l_1 l_2 l_3 l_4}\\
&= \frac{(x^2+1)\Delta^2 - (x^3+3x)\Delta}{2(x^2+3)\Delta^2} - \frac{x^2+1}{(x^2+2)(x^2+3)(\Delta^2 x^2+2)}.
\end{aligned} \tag{4.8}$$

Using the identities $l_{n+2} + l_{n-2} = (x^2+2)l_n$ [2, p. 57] and $l_{n+k} l_{n-k} - l_n^2 = (-1)^{n+k}\Delta^2 f_k^2$ [2, p. 58], we have

$$\begin{aligned}
\frac{1}{l_{n-2} l_{n-1} l_n l_{n+1}} + \frac{1}{l_{n-1} l_n l_{n+1} l_{n+2}} &= \frac{l_{n+2} + l_{n-2}}{l_{n-2} l_{n-1} l_n l_{n+1} l_{n+2}}\\
&= \frac{x^2+2}{l_{n-2} l_{n-1} l_{n+1} l_{n+2}}\\
&= \frac{x^2+2}{l_n^4 + (-1)^n (x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2}.
\end{aligned}$$

Thus, by equations (4.7) and (4.8), we get

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{(x^2+2)(x^4+x^2)}{l_n^4+(-1)^n(x^2-1)\Delta^2 l_n^2-\Delta^4 x^2} &= \frac{(x^2+1)\Delta^2-(x^3+3x)\Delta}{(x^2+3)\Delta^2} \\ &\quad -\frac{x^2+1}{(x^2+2)(x^2+3)(\Delta^2x^2+2)} \\ &= \frac{x^2+1}{x^2+3}-\frac{x}{\Delta}-\frac{x^2+1}{(x^2+2)(x^2+3)(\Delta^2x^2+2)} \\ &= \frac{(x^2+1)(x^6+6x^4+10x^2+3)}{(x^2+2)(x^2+3)(x^4+4x^2+2)}-\frac{x}{\Delta},\end{aligned}$$

as expected. □

It follows from equation (4.6) that

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{1}{L_n^4-25} &= \frac{1}{6}\left(\frac{2\cdot 20}{3\cdot 4\cdot 7}-\frac{1}{\sqrt{5}}\right) \\ &= \frac{5}{63}-\frac{\sqrt{5}}{30},\end{aligned}$$

as in [5, 7].

Next, we extract the Pell-Lucas implications of identities (4.1) through (4.6).

5. Pell-Lucas Implications

Because $q_n(x) = l_n(2x)$ and $2Q_n = l_n(2)$, equations (4.1) through (4.6) yield the following results:

$$\begin{aligned}\sum_{n=0}^{m} \frac{2x}{q_{2n}^2+4x^2} &= \frac{p_{2m+2}}{4(x^2+1)p_{2m+1}}; \\ \sum_{n=0}^{\infty} \frac{2x}{q_{2n}^2+4x^2} &= \frac{\gamma(x)}{4(x^2+1)}; \\ \sum_{n=0}^{m} \frac{4(2x^3+x)}{q_{2n+1}^2+4(2x^2+1)^2} &= \frac{p_{4m+4}}{4(x^2+1)p_{2m+3}p_{2m+1}}; \\ \sum_{n=0}^{\infty} \frac{4(2x^3+x)}{q_{2n+1}^2+4(2x^2+1)^2} &= \frac{1}{2\sqrt{x^2+1}}; \\ \sum_{n=3}^{\infty} \frac{4(2x^3+x)}{e(x)} &= \frac{1}{16(4x^2+1)f^2(x)}\left[\frac{2(8x^4+8x^2+1)}{4x^3+x}-\frac{8f(x)}{\gamma(x)}-\frac{1}{4(2x^3+x)}\right]; \\ \sum_{n=3}^{\infty} \frac{32(2x^2+1)(4x^4+x^2)}{h(x)} &= \frac{(x^2+1)(4x^2+1)g(2x)-(4x^3+3x)[g(2x)+1]\sqrt{x^2+1}}{(x^2+1)(2x^2+1)(4x^2+3)(8x^4+8x^2+1)},\end{aligned}$$

where $e(x) = q_n^4 - 4(-1)^n(4x^4+3x^2+1)q_n^2 - 16(2x^3+x)^2$, $f(x) = x^2+1, h(x) = q_n^4 + 4(-1)^n(x^2+1)(4x^2-1)q_n^2 - 64(x^3+x)^2$, and $g(x) = x^6+6x^4+10x^2+3$.

They imply the following results:

$$\begin{aligned}
\sum_{n=0}^{m} \frac{1}{Q_{2n}^2+1} &= \frac{P_{2m+2}}{4P_{2m+1}};\\
\sum_{n=0}^{\infty} \frac{1}{Q_{2n}^2+1} &= \frac{1+\sqrt{2}}{4};\\
\sum_{n=0}^{m} \frac{3}{Q_{2n+1}^2+9} &= \frac{P_{4m+4}}{8P_{2m+3}P_{2m+1}};\\
\sum_{n=0}^{\infty} \frac{3}{Q_{2n+1}^2+9} &= \frac{1}{2\sqrt{2}};\\
\sum_{n=3}^{\infty} \frac{12}{Q_n^4-8(-1)^nQ_n^2-9} &= \frac{1367}{1200}-\frac{4\sqrt{2}}{5};\\
\sum_{n=3}^{\infty} \frac{1}{Q_n^4+6(-1)^nQ_n^2-16} &= \frac{29}{306}-\frac{\sqrt{2}}{15},
\end{aligned}$$

respectively.

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References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[3] R. S. Melham, *Finite sums that involve reciprocal of products of generalized Fibonacci numbers*, Integers, **13.4** (2013), A40.
[4] H. Ohtsuka, *Problem H-783*, The Fibonacci Quarterly, **54.1** (2016), 87.
[5] Á. Plaza, *Problem H-810*, The Fibonacci Quarterly, **55.3** (2017), 282.
[6] Á. Plaza, *Solution to Problem H-783*, The Fibonacci Quarterly, **56.1** (2018), 90–91.
[7] Á. Plaza, *Solution to Problem H-810*, The Fibonacci Quarterly, **57.3** (2019), 281.

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Department of Mathematics, Framingham State University, Framingham, MA 01701, USA
Email address: tkoshy@emeriti.framingham.edu