SUMS RELATED TO THE FIBONACCI SEQUENCE

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Abstract. We investigate sums associated with the Fibonacci sequence F_n and the golden ratio ϕ. In particular, we study the sums $G(k) = \sum_{n=1}^{\infty} n^k/F_n$ and $H(k) = \sqrt{5} \cdot \mathrm{Li}_{-k}(1/\phi) = \sum_{n=1}^{\infty} n^k \sqrt{5}/\phi^n$. These sums generalize the reciprocal Fibonacci constant $\psi = G(0)$. We prove the asymptotic equivalence $G(k) \sim H(k)$, and moreover, $G(k)/H(k) = 1 + 1/5^{k+1} + O((\log \phi/\pi)^{k+1})$ as $k \to \infty$. We express $G(k) - H(k)$ as an alternating series, allowing us to compute values of these sums to high precision, and to prove that $G(k) > H(k)$ if and only if $k \geq 2$. We also generalize the results to their Lucas sequence analogues. As a tool, we establish a widely applicable explicit bound for polylogarithms of negative integer order.

We find explicit bounds for the integer sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ defined by $H(k)/\sqrt{5} = \mathrm{Li}_{-k}(1/\phi) = A_k + B_k \phi$. We also prove several results concerning the multiplicative structure of A_k and B_k. We show that $\{A_k \pmod m\}$ and $\{B_k \pmod m\}$ are periodic for every natural number m, and that the period is a divisor of $\lambda(m)$, where λ denotes the Carmichael function.

1. Introduction

The Fibonacci sequence $\{F_n\}$ is defined recursively by the conditions $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. *Binet's formula* states that $F_n = (\phi^n - \overline{\phi}^n)/\sqrt{5}$ for all $n \geq 1$, where $\phi = (1 + \sqrt{5})/2$ is the *golden ratio* and $\overline{\phi} = (1 - \sqrt{5})/2$. It follows that the sum of the reciprocals of the Fibonacci numbers is convergent. Let $\psi = \sum_{n=1}^{\infty} 1/F_n = 3.359885666243\ldots$ denote the value of this sum, called the *reciprocal Fibonacci constant*. It is known to at least 10,000 decimal places, see for instance [6]. No closed formula is known for ψ. However, R. André-Jeannin [1] proved in 1989 that ψ is irrational.

In this paper, we study the more general sums

$$G(k) = \sum_{n=1}^{\infty} \frac{n^k}{F_n}, \qquad H(k) = \sum_{n=1}^{\infty} \frac{n^k}{\phi^n/\sqrt{5}}.$$

Note that $\psi = G(0)$. Also, $H(k) = \sqrt{5} \cdot \mathrm{Li}_{-k}(1/\phi)$, where Li_{-k} denotes the polylogarithm of order $-k$. We prove that $G(k) \sim H(k)$ as $k \to \infty$. We establish a widely applicable explicit bound for polylogarithms (Theorem 4.4): for all $1 < x < e^{2\pi}$ and $k \geq 1$, we have

$$\mathrm{Li}_{-k}\left(\frac{1}{x}\right) = \sum_{n=1}^{\infty} \frac{n^k}{x^n} = \frac{k!}{\log^{k+1} x}\left(1 + O^*\left(2\zeta(k+1)\left(\frac{\log x}{2\pi}\right)^{k+1}\right)\right),$$

where ζ is the Riemann zeta function and where $O^*(f(k))$ denotes a quantity bounded in absolute value by $f(k)$. We use this estimate to prove that $G(k)/H(k) = 1 + 1/5^{k+1} + O((\log \phi/\pi)^{k+1})$, and moreover, $G(k) > H(k)$ if and only if $k \geq 2$.

We also use this estimate to put explicit bounds on the behavior of the integer sequences $\{A_k\}$ and $\{B_k\}$ defined by $\mathrm{Li}_{-k}(1/\phi) = A_k + B_k \phi$. Finally, we prove results on the multiplicative structure of these sequences. In particular, we show that for every natural $m \geq 1$, the sequences $\{A_k \pmod m\}$ and $\{B_k \pmod m\}$ are periodic, and the period is a divisor of $\lambda(m)$, where λ is the Carmichael function.

2. Notation and Preliminary Lemmas

Throughout the paper, $\phi = (1+\sqrt{5})/2$ is the golden ratio, $\overline{\phi} = (1-\sqrt{5})/2$ is its conjugate radical, F_n denotes the nth Fibonacci number, and $\log x$ is the natural logarithm. The use of k as a variable indicates a nonnegative integer, whereas x indicates a real number. Also, p refers to a prime number. An ellipsis $(\ldots)$ indicates that a decimal expansion is truncated. For $|x| < 1$,

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

denotes the polylogarithm of order k. We will make use of polylogarithms fairly frequently, and will rely on the following *duplication* formula:

Lemma 2.1. *We have* $\mathrm{Li}_{-k}(-x) = 2^{k+1}\mathrm{Li}_{-k}\left(x^2\right) - \mathrm{Li}_{-k}(x)$,

which can be verified using the polylogarithm definition, and the following *reciprocal* formula:

Lemma 2.2. *(See* [8, p. 151]*.) For all* $k \geq 1$*, we have* $\mathrm{Li}_{-k}(1/x) = (-1)^{k+1}\mathrm{Li}_{-k}(x)$.

These formulas apply to the analytic continuations of the polylogarithms outside of the domain of the functions with the series definition. We rely on these analytic continuations when necessary.

We define $G(k) = \sum_{n=1}^{\infty} n^k/F_n$ and $H(k) = \sqrt{5}\cdot \mathrm{Li}_{-k}(1/\phi) = \sum_{n=1}^{\infty} n^k\sqrt{5}/\phi^n$ as above. We let a_k, b_k, A_k, and B_k denote integer sequences defined by $\mathrm{Li}_{-k}(1/\phi) = a_k + b_k\sqrt{5} = A_k + B_k\phi$ for $k \geq 1$, with the exception that B_k denotes a Bernoulli number only when specified.

We let $\left\{ {n \atop k} \right\}$ denote a Stirling number of the second kind, whereas α_{nk} denotes an Eulerian number. Also, $\zeta(s)$ and $\Gamma(s)$ denote the Riemann zeta and Euler gamma functions, and $\varphi(m)$ and $\lambda(m)$ denote the Euler totient and Carmichael functions, respectively.

We use the asymptotic notation $f \sim g$ to mean that $\lim_{k\to\infty} f(k)/g(k) = 1$, as well as the Bachmann-Landau notation $f(k) = O(g(k))$ and Vinogradov notation $f(k) \ll g(k)$ to mean that there exists a constant $C > 0$ such that $|f(k)| \leq Cg(k)$ for all sufficiently large k. We write $f(k) = O^*(g(k))$ to mean $|f(k)| \leq g(k)$ for all sufficiently large k. In other words, we can take $C = 1$ above.

3. The Sums $\sum n^k/F_n$ and $\sum n^k/\phi^n$

In considering $G(k)$ and $H(k)$, we observe that, because the $\bar{\phi}^n$ term becomes insignificant relative to ϕ^n for large values of n, the two should be asymptotic. We note that Table 1 supports this idea, and justify this data later in this section.

Theorem 3.1. $G(k) \sim H(k)$.

Proof. We have that

$$\sum_{n=1}^{\infty} \frac{n^k}{F_n} = \sum_{n=1}^{\infty} \frac{n^k\sqrt{5}}{\phi^n} + \sum_{n=1}^{\infty}\left(\frac{n^k}{F_n} - \frac{n^k\sqrt{5}}{\phi^n}\right) = \sum_{n=1}^{\infty} \frac{n^k\sqrt{5}}{\phi^n} + \sum_{n=1}^{\infty} \frac{n^k\sqrt{5}}{\phi^n}\left(\frac{a^n}{1-a^n}\right),$$

where $a = \overline{\phi}/\phi$. Now, we see that $G(k)/H(k)$ equals

$$1 + \sum_{n=1}^{\infty} \frac{n^k}{\phi^n}\left(\frac{a^n}{1-a^n}\right) \Big/ \sum_{i=1}^{\infty} \frac{i^k}{\phi^i}.$$

k	$G(k)$	$H(k)$	$G(k)/H(k)$
0	3.359885666243	3.618033988749	0.928649558486
1	9.320451712281	9.472135954999	0.983986268415
2	40.15890107530	40.12461179749	1.000854569708
3	250.5643394352	250.2198067399	1.001376920155
4	2080.615034838	2079.771609537	1.000405537462
5	21611.35861433	21609.68046727	1.000077657189
6	269444.3279995	269440.8595763	1.000012872669
7	3919459.925826	3919450.752568	1.000002340444
8	65159706.65587	65159674.66202	1.000000491006
9	1218667337.130	1218667209.215	1.000000104963
10	25324964426.72	25324963888.30	1.000000021260

Table 1: Values of $G(k)$, $H(k)$, and $G(k)/H(k)$ to 12 decimal places.

To show the sum, $G(k)/H(k) - 1$, approaches 0 as $k \to \infty$, we first write it as an alternating series, noting that $a = -0.381966\ldots < 0$. However, we cannot use the alternating series remainder theorem directly on the sum in the numerator because its summands are not immediately decreasing in absolute value. We first must establish when it transitions from increasing to decreasing. Let $g(n)$ denote the absolute value of the summand. Then, $g(n+1)/g(n) = |a|/\phi \cdot (1+1/n)^k(1-a^n)/(1-a^{n+1})$. We note that $(1+1/n)^n$ increases to a limit of e, so for $n \geq k$, $(1+1/n)^k < e$. Also $e \cdot |a|/\phi < 0.65$, so for $g(n+1)/g(n)$ to be less than 1, it is sufficient to have $(1-a^n)/(1-a^{n+1}) < 1.5$, which is true for all $n \geq 2$. Thus for all $n \geq k > 1$, $g(n)$ is decreasing. We then break up the secondary term above into two parts: $n < k$ and $n \geq k$. We then consider the ratio of these parts to $H(k)$, and show each one approaches 0. For the first part, we have that

$$\sum_{n=1}^{k-1} \frac{n^k}{\phi^n}\left(\frac{|a|^n}{1-a^n}\right) \Big/ \sum_{i=1}^{\infty} \frac{i^k}{\phi^i} \leq (k-1)\frac{(k-1)^k}{\phi^{k-1}} \Big/ \frac{(2k-2)^k}{\phi^{2k-2}} = \frac{1}{\phi}\left(\frac{\phi}{2}\right)^k (k-1),$$

with the inequality on the left resulting from $|a^n/(1-a^n)| < 1$, and $f(n) = n^k/\phi^n$ is increasing on $(0, k/\log\phi)$, so in the sum of the numerator, every term has magnitude less than $(k-1)^k/\phi^{k-1}$. This term approaches 0 as $k \to \infty$ because $0 < \phi/2 < 1$.

Now, we can use the alternating series remainder theorem on the second part. We see that

$$\sum_{n=k}^{\infty} \frac{n^k}{\phi^n}\left(\frac{|a|^n}{1-a^n}\right) \Big/ \sum_{i=1}^{\infty} \frac{i^k}{\phi^i} \leq \frac{k^k}{\phi^k}\left(\frac{|a|^k}{1-a^k}\right) \Big/ \frac{k^k}{\phi^k} = \frac{|a|^k}{1-a^k},$$

which also approaches 0 as $k \to \infty$ because $|a| < 1$. This completes the proof. □

Remark 3.2. *Although the above proof assumes that k is an integer, it can be extended to a real variable as follows: instead of splitting the secondary sum at k, we can use $\lfloor k \rfloor < k$ (assuming k is not an integer) and split the sum at $\lfloor k \rfloor$. This would yield two parts: the sum from 1 to $\lfloor k \rfloor$ and from $\lfloor k \rfloor + 1$ to ∞. Each part can then be addressed in a similar fashion to the proof above.*

The above proof provides a simple way to compute the values of $G(k)$: use $H(k)$ as an approximation, and then bound the secondary sum using the first k terms and the alternating series remainder theorem. This was the method used to provide the numerical computations at the beginning of this section. In addition to considering $G(k)/H(k)$, we note that the data in Table 1 suggests that $G(k) > H(k)$ for all $k \geq 2$. We prove this result.

Theorem 3.3. *$G(k) > H(k)$ if and only if $k \geq 2$.*

Proof. Let $b = a/\phi = 2 - \sqrt{5}$, $B = 2 + \sqrt{5}$, and $c = ba = (-11 + 5\sqrt{5})/2$. As in the proof of Theorem 3.1, we have that $(G(k) - H(k))/\sqrt{5}$ is equal to

$$\sum_{n=1}^{\infty} \frac{n^k}{\phi^n} \frac{a^n}{1-a^n} = \sum_{n=1}^{\infty} n^k b^n \left(1 + \frac{a^n}{1-a^n}\right) > \mathrm{Li}_{-k}(b) + \frac{1}{1+|a|} \mathrm{Li}_{-k}(c). \tag{3.1}$$

It suffices to show that the right side of (3.1) is positive for all $k > 10$. (We verify the theorem directly for $k \leq 10$.) Because $b < 0$, we use Lemma 2.1 to express the right side of inequality (3.1) as

$$2^{k+1} \mathrm{Li}_{-k}\left(\frac{1}{B^2}\right) - \mathrm{Li}_{-k}\left(\frac{1}{B}\right) + \frac{1}{1+|a|} \mathrm{Li}_{-k}(c).$$

Here we rationalized numerators. We apply Theorem 4.4 (see Section 4.1 below) and then simplify, to obtain

$$2^{k+1} \mathrm{Li}_{-k}\left(\frac{1}{B^2}\right) - \mathrm{Li}_{-k}\left(\frac{1}{B}\right) = O^*\left(\frac{2k!\zeta(k+1)}{\pi^{k+1}}\left(1 + 2^{-(k+1)}\right)\right)$$

and

$$\frac{1}{1+|a|} \mathrm{Li}_{-k}(c) \geq \frac{1}{1+|a|} \frac{k!}{\log^{k+1}(1/c)} \left(1 - 2\zeta(k+1)\left(\frac{\log(1/c)}{2\pi}\right)^{k+1}\right).$$

Dividing through by $k!$, it therefore suffices to check that for all $k > 10$, we have

$$\frac{1}{1+|a|} \frac{1}{\log^{k+1}(1/c)} > 2\zeta(k+1)\left(\frac{1}{1+|a|}\left(\frac{1}{2\pi}\right)^{k+1} + \frac{1 + 2^{-(k+1)}}{\pi^{k+1}}\right).$$

We check this using that for $k > 10$, $\zeta(k+1) < 1.00025$ and $1/2^{k+1} \leq 1/4096$. $\square$

We next establish a finer estimate for $G(k)/H(k)$.

Theorem 3.4. *We have $G(k)/H(k) = 1 + 1/5^{k+1} + O\left((\log \phi/\pi)^{k+1}\right)$.*

Theorem 3.4 implies that $G(k)/H(k) - 1 \sim 1/5^{k+1}$. It also implies Theorem 3.1, as well as the assertion of Theorem 3.3 for all sufficiently large k.

Proof. We expand $(G(k) - H(k))/\sqrt{5}$ as in the proof of Theorem 3.3, obtaining

$$\sum_{n=1}^{\infty} \frac{n^k}{\phi^n} \frac{a^n}{1-a^n} = \sum_{n=1}^{\infty} n^k b^n \frac{1}{1-a^n} = \mathrm{Li}_{-k}(b) + \mathrm{Li}_{-k}(c) + \mathrm{Li}_{-k}(ca) + \sum_{n=1}^{\infty} n^k c^n \frac{a^{2n}}{1-a^n},$$

where $a = (1-\sqrt{5})/(1+\sqrt{5})$, $b = 2 - \sqrt{5}$, and $c = ab$. Dividing through by $H(k)/\sqrt{5} = \mathrm{Li}_{-k}(1/\phi)$, we have four terms to address. Noting that $\log \phi / \log(1/c) = 1/5$, the dominant term is

$$\begin{aligned} \frac{\mathrm{Li}_{-k}(c)}{\mathrm{Li}_{-k}(1/\phi)} &= \frac{k!}{\log^{k+1}(1/c)}\left(1 + O\left(\left(\frac{\log(1/c)}{2\pi}\right)^{k+1}\right)\right) \Big/ \frac{k!}{\log^{k+1}\phi}\left(1 + O\left(\left(\frac{\log \phi}{2\pi}\right)^{k+1}\right)\right) \\ &= \frac{1}{5^{k+1}} + O\left(\left(\frac{\log \phi}{2\pi}\right)^{k+1}\right). \end{aligned}$$

Here we used Theorem 4.4. We then have that $|\mathrm{Li}_{-k}(b)|/\mathrm{Li}_{-k}(1/\phi) \ll (\log\phi/\pi)^{k+1}$ as in the proof of Theorem 3.3. Also, noting that the absolute value of a sum does not exceed the sum of the absolute values of the summands, we see that $|\mathrm{Li}_{-k}(ac)|/\mathrm{Li}_{-k}(1/\phi) \ll 1/7^{k+1}$. Finally, we note that the fourth term in the expansion above can be bounded by constants times $\mathrm{Li}_{-k}(ca^2)$, and because $\mathrm{Li}_{-k}(ca^2)/\mathrm{Li}_{-k}(1/\phi) \sim 1/9^{k+1}$, this term is negligible compared with the main term by the squeeze theorem. □

The data also suggest that the limit of $G(k+1)/G(k) - G(k)/G(k-1)$ exists as $k \to \infty$ and is approximately 2.08. This is confirmed by the following, which holds by Theorem 3.1 as well as Theorem 4.4 below.

Proposition 3.5. $\lim_{k\to\infty}(G(k+1)/G(k) - G(k)/G(k-1)) = 1/\log\phi = 2.0780869\ldots$.

Equivalently, the same relation applies to $H(k)$. We generalize Theorems 3.1, 3.3, and 3.4 to their Lucas sequence analogues, defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. Explicitly, $L_n = \phi^n + \overline{\phi}^n$. Let $L(k) = \sum_{n=1}^{\infty} n^k/L_n$.

Theorem 3.6. *We have $L(k) \sim G(k)/\sqrt{5} \sim \mathrm{Li}_{-k}(1/\phi)$ as $k \to \infty$. Moreover, $L(k)/\mathrm{Li}_{-k}(1/\phi) = 1 + O((\log\phi/\pi)^{k+1})$, and $L(k) \geq \mathrm{Li}_{-k}(1/\phi)$ for all $k \geq 0$.*

The proofs are similar to those above for the Fibonacci sequence.

4. Sequences Arising from $\sum n^k/\phi^n$

We consider the well-known formulas for the sums $\mathrm{Li}_{-k}(1/x) = \sum_{n=1}^{\infty} n^k/x^n$, which are convergent for $|x| > 1$ by the ratio test. Note that $\mathrm{Li}_{-k}(1/\phi) = H(k)/\sqrt{5}$. $\mathrm{Li}_{-k}(1/x)$ is given by the recurrence relation $\mathrm{Li}_{-(k+1)}(1/x) = -x \cdot \partial/\partial x\ \mathrm{Li}_{-k}(1/x)$. This relation can be proven directly with term-by-term differentiation, or by applying Lemma 2.2 to the result at the bottom of [8, p. 153]. Thus:

$$\mathrm{Li}_0(1/x) = 1/(x-1),$$
$$\mathrm{Li}_{-1}(1/x) = x/(x-1)^2,$$
$$\mathrm{Li}_{-2}(1/x) = (x^2+x)/(x-1)^3,$$
$$\mathrm{Li}_{-3}(1/x) = (x^3+4x^2+x)/(x-1)^4,$$

etc. This recurrence relation leads to the following well-known formula for the coefficients, which also applies to the analytic continuation of the polylogarithm.

Lemma 4.1. *For all $k \geq 1$, we have*

$$\mathrm{Li}_{-k}\left(\frac{1}{x}\right) = \frac{1}{(x-1)^{k+1}}\sum_{j=0}^{k-1}\alpha_{kj}x^{j+1},$$

where the α_{kj} are the Eulerian numbers, given recursively by

$$\alpha_{k0} = \alpha_{k(k-1)} = 1, \quad \alpha_{(k+1)j} = (j+1)\alpha_{kj} + (k+1-j)\alpha_{k(j-1)}.$$

When $x = \phi$, the golden ratio, the sums are of the form $a_k + b_k\sqrt{5}$, where a_k and b_k are positive integers for each $k \geq 1$. For instance, $\mathrm{Li}_{-1}(1/\phi) = 2+\sqrt{5}$, $\mathrm{Li}_{-2}(1/\phi) = 9+4\sqrt{5}$, and $\mathrm{Li}_{-3}(1/\phi) = 56+25\sqrt{5}$. We also define two more sequences implicitly by writing the sums in the form $A_k + B_k\phi$, where A_k and B_k are rational. Then, $\mathrm{Li}_{-k}(1/\phi) = a_k + b_k\sqrt{5} = A_k + B_k\phi$, and thus $A_k = a_k - b_k$ and $B_k = 2b_k$, so information about A_k and B_k directly gives us information about a_k and b_k, and vice versa. The first eight terms of the sequences a_k, b_k, A_k, and B_k are given in Table 2.

k	a_k	b_k	A_k	B_k
1	2	1	1	2
2	9	4	5	8
3	56	25	31	50
4	465	208	257	416
5	4832	2161	2671	4322
6	60249	26944	33305	53888
7	876416	391945	484471	783890
8	14570145	6515968	8054177	13031936

Table 2: First eight terms of the sequences $\{a_k\}$, $\{b_k\}$, $\{A_k\}$, and $\{B_k\}$.

The sequence $\{b_k\}$ has exponential generating function $f(x) = (1 - \sinh x)/(1 - 2\sinh x) - 1/2$. In other words, $b_k = f^{(k)}(0)$, where the superscript indicates the kth derivative. Thus, the terms b_k are the same as those appearing in the sequence [2] (aside from b_0). This is a consequence of Proposition 4.9.

We note by Lemma 4.1 that

$$\mathrm{Li}_{-k}\left(\frac{1}{\phi}\right) = \frac{1}{(\phi-1)^{k+1}}\sum_{j=0}^{k-1}\alpha_{kj}\phi^{j+1} = \phi^{k+1}\sum_{j=0}^{k-1}\alpha_{kj}\phi^{j+1} = \sum_{j=0}^{k-1}\alpha_{kj}\phi^{k+2+j}.$$

Using the identity $\phi^k = F_k\phi + F_{k-1}$, this yields a formula for A_k and B_k in terms of Fibonacci and Eulerian numbers. In particular, A_k and B_k are natural numbers.

Proposition 4.2. *For all $k \geq 1$, we have*

$$A_k = \sum_{j=0}^{k-1}\alpha_{kj}F_{k+j+1}, \quad B_k = \sum_{j=0}^{k-1}\alpha_{kj}F_{k+j+2}.$$

4.1. **Estimates for Polylogarithms and the Sequences $\mathbf{A_k}$ and $\mathbf{B_k}$.** To obtain estimates for A_k and B_k, we use the following expansion, see [8, p. 149].

Lemma 4.3. *For $s \notin \mathbb{N}$ and $|\log z| < 2\pi$, we have*

$$\mathrm{Li}_s(z) = \Gamma(1-s)\log^{s-1}\left(\frac{1}{z}\right) + \sum_{n=0}^{\infty}\zeta(s-n)\frac{\log^n z}{n!}.$$

We now prove a general explicit estimate for the polylogarithm.

Theorem 4.4. *For all $1 < x < e^{2\pi}$ and $k \geq 1$, we have*

$$\mathrm{Li}_{-k}\left(\frac{1}{x}\right) = \frac{k!}{\log^{k+1} x}\left(1 + O^*\left(2\zeta(k+1)\left(\frac{\log x}{2\pi}\right)^{k+1}\right)\right).$$

Proof. Taking $s = -k$ and $z = 1/x$ in Lemma 4.3, we have

$$\mathrm{Li}_{-k}\left(\frac{1}{x}\right) = \frac{k!}{\log^{k+1} x}\left(1 + \sum_{n=0}^{\infty}\zeta(-(n+k))\frac{(-1)^n\log^{n+k+1} x}{n!k!}\right)$$

from Lemma 4.3. Assume first that $k \geq 2$ is even. We thus have $\zeta(-(n+k)) = 0$ if n is even, due to the trivial zeros of zeta at the negative even integers. Changing the index of summation to represent only odd terms $2n+1$, $n \geq 0$, we have

$$\begin{aligned}\mathrm{Li}_{-k}\left(\frac{1}{x}\right) &= \frac{k!}{\log^{k+1} x}\left(1 + \sum_{n=0}^{\infty} \zeta(-(2n+k+1)) \frac{-\log^{2n+k+2} x}{(2n+1)!k!}\right) \\ &= \frac{k!}{\log^{k+1} x}\left(1 + \sum_{n=0}^{\infty} \frac{B_{2n+k+2} \log^{2n+k+2} x}{(2n+k+2)(2n+1)!k!}\right).\end{aligned}$$

Here we used the well-known formula $\zeta(-m) = -B_{m+1}/(m+1)$, where B_{m+1} denotes a Bernoulli number, and where $m = 2n+k+1$. We next use the expansion for the Bernoulli numbers, $B_{2N} = 2(-1)^{N+1}(2N)!\zeta(2N)/(2\pi)^{2N}$, with $2N = m+1 = 2n+k+2$. Thus,

$$\mathrm{Li}_{-k}\left(\frac{1}{x}\right) = \frac{k!}{\log^{k+1} x}\left(1 + 2(-1)^{\frac{k}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \zeta(2n+k+2)}{\left(\frac{2\pi}{\log x}\right)^{2n+k+2}} \binom{2n+k+1}{k}\right).$$

Let $y = y(x) = \log x/(2\pi)$. We show that the series above is $O^*(\zeta(k+1)y^{k+1})$. Using the definition of the Riemann zeta function and changing the order of summation, this series is equal to

$$\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n \left(\frac{y}{j}\right)^{2n+k+2} \binom{2n+k+1}{k}.$$

Changing the order of summation is justified because the original double series is absolutely convergent by the ratio test. We proceed by writing this expression as

$$\sum_{j=1}^{\infty}\left(\frac{y}{j}\right)^{k+1} \sum_{n=0}^{\infty}(-1)^n \left(\frac{y}{j}\right)^{2n+1} \binom{2n+k+1}{k}$$

which is equal to

$$\sum_{j=1}^{\infty}\left(\frac{y}{j}\right)^{k+1} \cdot \frac{\sin\left((k+1)\tan^{-1}\left(\frac{y}{j}\right)\right)}{\left(\left(\frac{y}{j}\right)^2 + 1\right)^{\frac{k+1}{2}}} = O^*\left(\sum_{j=1}^{\infty}\left(\frac{y}{j}\right)^{k+1}\right).$$

(This can be proved by induction on k with the sum formulas for the sine and cosine.) We complete the proof of the case where k is even by noting that the expression in the O^* symbol above is $y^{k+1}\zeta(k+1)$. The odd case follows by a similar argument. $\square$

In a sense, we can view B_k as a polylog transform of the Fibonacci sequence, as shown in the following explicit formulas for A_k and B_k, see [4] and [5].

Proposition 4.5. *For all $k \geq 1$, we have*

$$B_k = \frac{(-1)^{k+1}}{\sqrt{5}}\left(\mathrm{Li}_{-k}(\phi) - \mathrm{Li}_{-k}(\overline{\phi})\right), \quad A_k = \frac{(-1)^{k+1}}{\sqrt{5}}\left(\phi \mathrm{Li}_{-k}(\overline{\phi}) + \frac{1}{\phi}\mathrm{Li}_{-k}(\phi)\right). \tag{4.1}$$

It follows that if $k \geq 1$ is even (respectively odd), we have

$$B_k = \frac{2^{k+1}}{\sqrt{5}}\mathrm{Li}_{-k}\left(\frac{1}{\phi^2}\right), \quad B_k = \frac{2}{\sqrt{5}}\left(\mathrm{Li}_{-k}\left(\frac{1}{\phi}\right) - 2^k \mathrm{Li}_{-k}\left(\frac{1}{\phi^2}\right)\right), \tag{4.2}$$

and if $k \geq 1$ is even (respectively odd), we have

$$A_k = \mathrm{Li}_{-k}\left(\frac{1}{\phi}\right) - \frac{2^{k+1}\phi}{\sqrt{5}}\mathrm{Li}_{-k}\left(\frac{1}{\phi^2}\right), \quad A_k = \frac{-1}{\sqrt{5}}\left(\mathrm{Li}_{-k}\left(\frac{1}{\phi}\right) - 2^{k+1}\phi\mathrm{Li}_{-k}\left(\frac{1}{\phi^2}\right)\right). \tag{4.3}$$

Proof. We establish (4.1) using algebraic manipulations with Lemma 4.1 and Proposition 4.2. Equations (4.2) and (4.3) follow from equation (4.1) by Lemmas 2.1 and 2.2. ☐

The following theorem is a consequence of Theorem 4.4 and Proposition 4.5.

Theorem 4.6. *For all $k \geq 1$, we have*

$$A_k = \frac{2}{5+\sqrt{5}}\frac{k!}{\log^{k+1}\phi}\left(1 + O^*\left(2\phi\zeta(k+1)(\sqrt{5}\cdot 2^{-(k+1)} + \phi)\left(\frac{\log\phi}{\pi}\right)^{k+1}\right)\right),$$

$$B_k = \frac{1}{\sqrt{5}}\frac{k!}{\log^{k+1}\phi}\left(1 + O^*\left(2\zeta(k+1)(2^{-k}+1)\left(\frac{\log\phi}{\pi}\right)^{k+1}\right)\right).$$

Using Theorem 4.6, we establish bounds for the counting functions of A_k and B_k. Let $A(x) = |\{k : A_k \leq x\}|$ and $B(x) = |\{k : B_k \leq x\}|$. The following theorem shows that $A(x)$ and $B(x)$ belong to the same class of functions as the counting function of the factorial numbers.

Theorem 4.7. *We have*

$$A(x) = \frac{\log x}{\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)$$

and the same estimate holds for $B(x)$. In particular, $A(x) \sim B(x) \sim \log x/\log\log x$.

Proof. We prove the theorem for $B(x)$, noting that the same argument applies to $A(x)$. By definition, $B(x)$ is the number k such that $B_k \leq x < B_{k+1}$. Thus,

$$\frac{\log B_k}{k\log k} \leq \frac{\log x}{k\log k} < \frac{\log B_{k+1}}{k\log k}. \tag{4.4}$$

By Theorem 4.6, we have $B_k = ck!/\log^{k+1}\phi\cdot(1+O(y^{k+1}))$, where $c = 1/\sqrt{5}$ and $y = \log\phi/\pi$. We apply Stirling's formula in the form $k! = k^k/e^k\cdot\sqrt{2\pi k}(1+O(1/k))$. Thus $B_k = ck^k/e^k\cdot\sqrt{2\pi k}/\log^{k+1}\phi\cdot(1+O(1/k))$, so that

$$\log B_k = \log c + k\log k - k + \frac{1}{2}\log 2\pi + \frac{1}{2}\log k - (k+1)\log\log\phi + O(1/k).$$

Simplifying, we have $\log B_k = k\log k + O(k)$. Similarly, $\log B_{k+1} = (k+1)\log(k+1) + O(k+1) = (k+1)(\log k + \log(1+1/k)) + O(k) = k\log k + O(k)$. Therefore,

$$\frac{\log B_k}{k\log k} = 1 + O\left(\frac{1}{\log k}\right), \quad \frac{\log B_{k+1}}{k\log k} = 1 + O\left(\frac{1}{\log k}\right). \tag{4.5}$$

Combining (4.4) with (4.5) and recalling that $k = B(x)$, we have

$$\frac{\log x}{B(x)\log B(x)} = 1 + O\left(\frac{1}{\log B(x)}\right). \tag{4.6}$$

Note that $B(x) \to \infty$ as $x \to \infty$, so that $1/\log B(x) \to 0$ as $x \to \infty$. Thus in particular, we have $B(x) \sim \log x/\log B(x)$. Rearranging (4.6), we have

$$B(x) = \frac{\log x}{\log B(x)}\left(1 + O\left(\frac{1}{\log B(x)}\right)\right). \tag{4.7}$$

Taking logarithms in (4.7), we have

$$\log B(x) = \log\log x - \log\log B(x) + O\left(\frac{1}{\log B(x)}\right).$$

Dividing by $\log B(x)$, we obtain

$$1 + \frac{\log\log B(x)}{\log B(x)} - \frac{\log\log x}{\log B(x)} = O\left(\frac{1}{\log^2 B(x)}\right). \tag{4.8}$$

Because $B(x) \to \infty$ as $x \to \infty$, the right side and the middle term on the left side both have limit 0 as $x \to \infty$. This implies that $\lim_{x\to\infty} \log\log x/\log B(x) = 1$, or equivalently, $\log\log x \sim \log B(x)$. Combining this with the estimate $B(x) \sim \log x/\log B(x)$ (see the line below (4.6)), we have $B(x) \sim \log x/\log\log x$. It follows that $\log\log B(x) \sim \log\log\log x$. Thus from (4.8), we obtain

$$\frac{\log\log x}{\log B(x)} = 1 + O\left(\frac{\log\log\log x}{\log\log x}\right).$$

Combining this with (4.7), completes the proof of Theorem 4.7. □

4.2. **Asymptotic Relationships of $\mathbf{a_k}$, $\mathbf{b_k}$, $\mathbf{A_k}$, and $\mathbf{B_k}$.** We have the following two propositions as immediate consequences of Theorem 4.6.

Proposition 4.8. *For any $\varepsilon > 0$, there exists an integer N_ε such that for all $k \geq N_\varepsilon$ we have*

$$\frac{B_k}{A_k} = \phi\left(1 + O^*\left((C+\varepsilon)\left(\frac{\log\phi}{\pi}\right)^{k+1}\right)\right),$$

where $C = 2(1+\phi^2) = 5+\sqrt{5}$. In particular, $\lim_{k\to\infty} B_k/A_k = \phi$.

Proof. By Theorem 4.6, we have

$$B_k = \frac{k!}{\sqrt{5}\log^{k+1}\phi}\left(1+f(k)\right), \qquad A_k = \frac{2k!}{(5+\sqrt{5})\log^{k+1}\phi}\left(1+g(k)\right),$$

where
$f(k) = O^*\left(2\zeta(k+1)\left(2^{-k}+1\right)y^{k+1}\right)$ and $g(k) = O^*\left(2\zeta(k+1)\phi\left(\sqrt{5}\cdot 2^{-k-1}+\phi\right)y^{k+1}\right)$ for $y = \log\phi/\pi$. Then,

$$\frac{B_k}{A_k} = \phi\frac{1+f(k)}{1+g(k)} = \phi\left(1 + \frac{f(k)-g(k)}{1+g(k)}\right).$$

Then for any $\varepsilon > 0$ and sufficiently large N_ε, for all $k \geq N_\varepsilon$, we have
$B_k/A_k = \phi\left(1 + O^*\left((C+\varepsilon)\,y^{k+1}\right)\right)$. □

Using $a_k = A_k + b_k$ and $b_k = B_k/2$, we have the following corollary.

Corollary 4.1. *For any $\varepsilon > 0$, there exists an integer M_ε such that for all $k \geq M_\varepsilon$ we have*

$$\frac{a_k}{b_k} = \sqrt{5}\left(1 + O^*\left((D+\varepsilon)\left(\frac{\log\phi}{\pi}\right)^{k+1}\right)\right),$$

where $D = 4(1+\phi^2)/(\phi\sqrt{5}) = 4$. In particular, $\lim_{k\to\infty} a_k/b_k = \sqrt{5}$.

The sequences a_k, b_k, A_k, B_k are interesting in their own right. We will consider multiplicative properties of these sequences. For instance, the only values of $k \leq 1000$ for which A_k is prime are $k = 2, 3, 4, 5, 8, 81$. This data suggests that perhaps A_k is prime for only finitely many values of k. Initially, the A_k are mostly squarefree, however a computer search reveals that A_k is nonsquarefree for $k = 31$ and 50. Before studying the multiplicative properties of these sequences in more depth, we will provide a survey of known results.

4.3. **A Survey of Known Results Concerning $\mathbf{A_k}$ and $\mathbf{B_k}$.** The following result confirms that A_k and B_k are the sequences given in the On-line Encyclopedia of Integer Sequences, [4] and [5].

Proposition 4.9. *The exponential generating functions f_A and f_B of the sequences A_k and B_k are given by $f_A(x) = 1/(e^x - e^{2x} + 1) - 1$ and $f_B(x) = e^x/(e^x - e^{2x} + 1)$, respectively. In other words,*

$$f_A(x) = \frac{1}{e^x - e^{2x} + 1} - 1 = \sum_{k=0}^{\infty} A_k \frac{x^k}{k!}, \quad f_B(x) = \frac{e^x}{e^x - e^{2x} + 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Proof. We write $f_A(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{e^x - \overline{\phi}} - \frac{1}{e^x - \phi}\right) - 1$. Now for a given constant c,

$$\frac{1}{e^x - c} = -\frac{1}{c}\sum_{j=0}^{\infty}\left(\frac{e^x}{c}\right)^j = -\frac{1}{c}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(jx)^k}{c^j k!} = -\frac{1}{c-1} - \frac{1}{c}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{j^k}{c^j}\frac{x^k}{k!}$$
$$= -\frac{1}{c-1} - \sum_{k=1}^{\infty}\frac{S_k(c)}{c(c-1)^{k+1}}\frac{x^k}{k!},$$

where $S_k(c) = \sum_{j=0}^{k-1} \alpha_{kj} c^{j+1}$, so that $\mathrm{Li}_{-k}(1/c) = \sum_{j=1}^{\infty} j^k/c^j = S_k(c)/(c-1)^{k+1}$. It follows that

$$\begin{aligned} f_A(x) &= \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}\left(\frac{S_k(\phi)}{\phi(\phi-1)^{k+1}} - \frac{S_k(\overline{\phi})}{\overline{\phi}(\overline{\phi}-1)^{k+1}}\right)\frac{x^k}{k!} \\ &= \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}(S_k(\phi)\phi^k - S_k(\overline{\phi})\overline{\phi}^{\,k})\frac{x^k}{k!} \\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\alpha_{kj}\left(\frac{\phi^{k+j+1} - \overline{\phi}^{\,k+j+1}}{\sqrt{5}}\right)\frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} A_k \frac{x^k}{k!}. \end{aligned}$$

A similar argument establishes the claim for B_k. □

The sequences A_k and B_k also appear in a different kind of sum related to the Fibonacci sequence, given by $\sum_{k=1}^{n} k^m F_k$. This sum was studied by Ledin [3] and by Zeitlin [9]. The following proposition confirms that A_k and B_k are the same sequences appearing in these two papers.

Proposition 4.10. *For all $k \geq 1$, we have*

$$A_k = \sum_{n=0}^{k} n! \left\{ {k \atop n} \right\} F_{n+1}, \quad B_k = \sum_{n=0}^{k} n! \left\{ {k \atop n} \right\} F_{n+2},$$

where $\left\{ {k \atop n} \right\}$ *denotes a Stirling number of the second kind.*

Proof. We prove the claim for B_k. (A similar argument establishes the claim for A_k.) We have

$$\mathrm{Li}_{-k}(x) = \sum_{n=1}^{\infty} n^k x^n = \sum_{j=1}^{k+1} (-1)^{k+j+1} (j-1)! \left\{ {k+1 \atop j} \right\} \frac{1}{(1-x)^j} \quad (k \geq 1),$$

see for instance [8, p. 152]. Therefore, as in the proof of Proposition 4.9 and relying on Lemma 2.2, the generating function of B_k is

$$\begin{aligned} f_B(x) &= \frac{e^x}{1-e^{2x}+e^x} = \frac{1}{\sqrt{5}} \left(\frac{e^x}{e^x - \overline{\phi}} - \frac{e^x}{e^x - \phi} \right) \\ &= \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \sum_{j=1}^{k+1} (-1)^{j+1} (j-1)! \left\{ {k+1 \atop j} \right\} \left(\frac{1}{(1-\overline{\phi})^j} - \frac{1}{(1-\phi)^j} \right) \frac{x^k}{k!}. \end{aligned}$$

Shifting the index in the sum on j, we write the kth term in the sum on k as

$$\frac{1}{\sqrt{5}} \sum_{j=0}^{k} (-1)^j j! \left\{ {k+1 \atop j+1} \right\} \left(\frac{1}{(1-\overline{\phi})^{j+1}} - \frac{1}{(1-\phi)^{j+1}} \right) \frac{x^k}{k!}.$$

Using the identities $1/(1-\phi) = -\phi$ and $1/(1-\overline{\phi}) = -\overline{\phi}$, this expression is equal to

$$\begin{aligned} \sum_{j=0}^{k} j! \left\{ {k+1 \atop j+1} \right\} F_{j+1} \frac{x^k}{k!} &= \sum_{j=0}^{k} j! \left((j+1) \left\{ {k \atop j+1} \right\} + \left\{ {k \atop j} \right\} \right) F_{j+1} \frac{x^k}{k!} \\ &= \left(\sum_{j=1}^{k+1} j! \left\{ {k \atop j} \right\} F_j + \sum_{j=0}^{k} j! \left\{ {k \atop j} \right\} F_{j+1} \right) \frac{x^k}{k!} \\ &= \sum_{j=0}^{k} j! \left\{ {k \atop j} \right\} F_{j+2} \frac{x^k}{k!}. \end{aligned}$$

Here we used the recurrence relations for Stirling numbers of the second kind and Fibonacci numbers. □

4.4. **Results on the Multiplicative Structure of $\mathbf{A_k}$ and $\mathbf{B_k}$.** We consider values of A_k and B_k modulo a given number. Table 2 suggests that the last digits of A_k, B_k, a_k, and b_k are periodic. We prove this in the following proposition.

Proposition 4.11. *For all $k \geq 1$, the terms A_k are odd and the terms B_k are even.*

Proof. This follows readily from Proposition 4.10. □

Proposition 4.12. *We have $a_k, b_k \in \mathbb{N}$ for all $k \geq 1$. Furthermore, a_k is even if and only if b_k is odd, if and only if k is odd.*

Proof. That $a_k, b_k \in \mathbb{N}$ have opposite parity for all $k \geq 1$ follows from Propositions 4.11 and 4.16 by the relations $a_k = A_k + B_k/2$, $b_k = B_k/2$. □

Proposition 4.13. *Let $k \geq 1$. We have:*

$$A_k \equiv \begin{cases} 1 \pmod 5, & \text{if } k \equiv 1 \pmod 4; \\ 0 \pmod 5, & \text{if } k \equiv 2 \pmod 4; \\ 1 \pmod 5, & \text{if } k \equiv 3 \pmod 4; \\ 2 \pmod 5, & \text{if } k \equiv 0 \pmod 4; \end{cases} \qquad B_k \equiv \begin{cases} 2 \pmod 5, & \text{if } k \equiv 1 \pmod 4; \\ 3 \pmod 5, & \text{if } k \equiv 2 \pmod 4; \\ 0 \pmod 5, & \text{if } k \equiv 3 \pmod 4; \\ 1 \pmod 5, & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

Proof. We prove the claim for A_k. (The same argument applies to B_k.) The claim clearly holds for $1 \le k \le 4$, so let $k \ge 5$. Because $5|n!$ for all $n \ge 5$, by Proposition 4.10, we have

$$\begin{aligned} A_k &\equiv \sum_{n=0}^{4} n! F_{n+1} \left\{ {k \atop n} \right\} \equiv \left\{ {k \atop 0} \right\} + \left\{ {k \atop 1} \right\} + 4\left\{ {k \atop 2} \right\} + 18\left\{ {k \atop 3} \right\} + 120\left\{ {k \atop 4} \right\} \pmod 5 \\ &\equiv 1 + 4\left\{ {k \atop 2} \right\} + 3\left\{ {k \atop 3} \right\} \pmod 5. \end{aligned}$$

The claim then follows from the formulas $\left\{ {k \atop 2} \right\} = 2^{k-1} - 1$ and $\left\{ {k \atop 3} \right\} = (3^{k-1}+1)/2 - 2^{k-1}$, and the periodicity of 2^k and 3^k modulo 5. □

We substantially generalize Proposition 4.13, proving that for all $m \ge 1$, the sequences $\{A_k \pmod m\}$ and $\{B_k \pmod m\}$ are periodic, and the period divides $\lambda(m)$, where λ denotes the Carmichael function. Note that $\lambda(m)|\varphi(m)$, where φ is the Euler totient function, so the period also divides $\varphi(m)$.

Theorem 4.14. *Let m be a positive integer. Then, the sequences $\{A_k \pmod m\}$ and $\{B_k \pmod m\}$ eventually are periodic, with the period a divisor of $\lambda(m)$.*

Proof. We prove the theorem for A_k, and we note that an identical argument applies to B_k. We begin by establishing the theorem for $m = p^\alpha$, where p is prime. By Proposition 4.10, we have $A_k = \sum_{n=0}^{k} n! \left\{ {k \atop n} \right\} F_{n+1}$. Note that for all $n \ge \alpha p$, we have $n! \equiv 0 \pmod{p^\alpha}$. Thus for $k \ge \alpha p - 1$, we have

$$A_k \equiv \sum_{n=0}^{\alpha p - 1} n! \left\{ {k \atop n} \right\} F_{n+1} \pmod{p^\alpha}.$$

We now use the well-known explicit formula for Stirling numbers of the second kind,

$$n! \left\{ {k \atop n} \right\} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k. \tag{4.9}$$

Therefore,

$$A_k \equiv \sum_{n=0}^{\alpha p - 1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k F_{n+1} \pmod{p^\alpha}$$

and

$$A_{k+\lambda(p^\alpha)} \equiv \sum_{n=0}^{\alpha p - 1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^{k+\lambda(p^\alpha)} F_{n+1} \pmod{p^\alpha}.$$

Thus, it suffices to show that for all $0 \le n \le \alpha p - 1$ and $0 \le j \le n$, we have

$$(n-j)^{k+\lambda(p^\alpha)} \equiv (n-j)^k \pmod{p^\alpha}. \tag{4.10}$$

Congruence (4.10) holds if $p \nmid (n-j)$ by the definition of the Carmichael function as $\gcd(n-j, p^\alpha) = 1$, so that $(n-j)^{\lambda(p^\alpha)} \equiv 1 \pmod{p^\alpha}$. Also, if $p|(n-j)$, then both sides of congruence (4.10) are 0 mod p^α as $k \ge \alpha p - 1$. This establishes the theorem for $m = p^\alpha$.

Note that the theorem holds trivially for $m = 1$. The theorem holds for any integer $m \ge 2$ by an application of the Chinese remainder theorem, noting that $m \ge 2$ factors as $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where the $p_i^{\alpha_i}$ are pairwise coprime, and $\lambda(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \operatorname{lcm}\{\lambda(p_1^{\alpha_1}), \ldots, \lambda(p_r^{\alpha_r})\}$. □

Remark 4.15. *Note that $\{A_k \pmod{p^\alpha}\}$ and $\{B_k \pmod{p^\alpha}\}$ become periodic for $k \geq \alpha p - 1$, so $\{A_k \pmod m\}$ and $\{B_k \pmod m\}$ become periodic for $k \geq \max\{\alpha_i p_i - 1\}$. However, when m is a prime p, one can check that the assumption that $k \geq p-1$ can be dropped, so the periodicity applies for all $k \geq 1$.*

Define $P_A(m)$ to be the period of the sequence $\{A_k \pmod m\}$, and similarly define $P_B(m)$. Based on data including an exhaustive computer check in Pari/GP for all $m \leq 200$, we conjecture that $P_A(m) = \lambda(m)$ for all natural m. Although this was typically the case for $P_B(m)$, we found some counterexamples. We conjectured that for $m = 2^\alpha$ or $m = 3 \cdot 2^\alpha$, $\alpha \geq 4$, we have $P_B(m) = \lambda(m)/2$. Additionally, there are m not of this form for which $P_B(m) = \lambda(m)/2$. The exceptional cases up to 200 are $m = 112, 144, 160$, and 176.

We note that the periodicity of the sequences $\{A_k \pmod m\}$ and $\{B_k \pmod m\}$ is closely related to that of the ordered Bell numbers defined by $\omega(k) = \sum_{n=0}^{k} n! \left\{ {k \atop n} \right\}$, see for instance [7]. However, the extra factors of F_{n+1} or F_{n+2} in the summands pose additional difficulties in the study of A_k and B_k.

We next prove a result on the exact divisibility of the terms B_k by powers of 2.

Proposition 4.16. *If k is odd, then 2 exactly divides B_k. If $k > 0$ is even, then 2^{k+1} exactly divides B_k. It follows that if $k > 0$ is even, then 2^k exactly divides b_k.*

Proof. Let $D(n,k) = \frac{d^k}{dx^k} \sinh^n x\Big|_{x=0}$. By Proposition 4.9, $B_k = f_B^{(k)}(0)$, where $f_B(x) = 1/(1 - 2\sinh x) = \sum_{n=0}^{\infty} 2^n \sinh^n x$. Thus, $B_k \equiv \sum_{n=0}^{1} 2^n D(n,k) \pmod 4$. For odd k, we thus have $B_k \equiv 2D(1,k) \pmod 4$. Because the kth derivative of $\sinh x$ is $\cosh x$ if k is odd, we have $B_k \equiv 2 \pmod 4$ for odd k. This establishes the first assertion.

We now prove the second assertion. Note that $D(n,k) = 0$ if $n > k$. To see this, after k applications of the product rule, the power of $\sinh x$ in each term is at least $n - k$. Thus, $B_k = \sum_{n=0}^{k} 2^n D(n,k)$. It suffices to show that if $k \geq 2$ is even and $0 \leq n \leq k$ is even, then $2^{k+1-n} | D(n,k)$, with exact divisibility if and only if $n = 2$, whereas if k is even and n is odd, then $D(n,k) = 0$. This holds by Lemma 4.17 below.

Then, the last assertion holds because $b_k = B_k/2$. □

Lemma 4.17. *If $k \geq 2$ is even and $0 \leq n \leq k$ is even, then $2^{k+1-n} | D(n,k)$, and the divisibility is exact if and only if $n = 2$. If k is even and n is odd, then $D(n,k) = 0$.*

Proof. Suppose first that n is odd. We show that $D(n,k) = 0$ for all even k by induction on n. We have $D(1,k) = \sinh 0 = 0$ for all even k. Suppose that for some odd number $n \geq 1$, we have $D(n,k) = 0$ for all even k. Then by the Leibniz rule,

$$D(n+2,k) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{dx^j} \sinh^n x \frac{d^{k-j}}{dx^{k-j}} \sinh^2 x\Big|_{x=0} = \sum_{j=0}^{k} \binom{k}{j} D(n,j) D(2,k-j).$$

By the induction hypothesis, if j is even, then $D(n,j) = 0$. Therefore,

$$D(n+2,k) = \sum_{\substack{j=0 \\ j \text{ odd}}}^{k} \binom{k}{j} D(n,j) D(2,k-j).$$

By the assumption that k is even, we have that $k - j$ is odd for all odd j. Thus, it suffices to show that $D(2,m) = 0$ for all odd m. This follows from the identity $\frac{d}{dx} \sinh^2 x = \sinh 2x$. Thus, if n is odd and k is even, then $D(n,k) = 0$.

We next prove the assertion that if $k \geq 2$ is even and $0 \leq n \leq k$ is even, then $2^{k+1-n} | D(n,k)$, with the divisibility being exact if and only if $n = 2$. We again proceed by induction on n. When $n = 0$, the claim holds trivially. When $n = 2$,

$$D(2,k) = \frac{d^k}{dx^k} \sinh^2 x \Big|_{x=0} = \frac{d^{k-1}}{dx^{k-1}} \sinh 2x \Big|_{x=0} = 2^{k-1} \cosh 2x \Big|_{x=0} = 2^{k+1-n}.$$

Suppose that the claim holds for some even $n \geq 2$. Then proceeding as above,

$$D(n+2,k) = \sum_{j=0}^{k} \binom{k}{j} D(n,j) D(2,k-j).$$

Again using the identity $\frac{d}{dx} \sinh^2 x = \sinh 2x$, we have for even k and odd j that $k-j$ is odd and hence $D(2,k-j) = 0$, whereas for even k and even j, we have that $k-j$ is even and $2^{k-j-1} | D(2,k-j)$. Also by the induction hypothesis, we have $2^{j+1-n} | D(n,j)$ for n and j even. Therefore, $D(n+2,k)$ is divisible by $2^{j+1-n} 2^{k-j-1} = 2^{k-n}$, so $2^{k+1-(n+2)} | D(n+2,k)$ but not exactly. $\square$

4.5. **A Few Notes on Reciprocal Sums of $\mathbf{A_k}$ and $\mathbf{B_k}$.** By either Theorem 4.6 or 4.7, the sequences A_k and B_k have asymptotic density zero, as well as a bounded sum of reciprocals. From computation in Pari/GP, we find the following approximations for $\sum_{k=1}^{\infty} 1/A_k$:

```
1.236555727473873167020242334503562260601000019599907162770024
8492738789649192905339500692557197368122
```

and $\sum_{k=1}^{\infty} 1/B_k$:

```
0.647655134257726353153264534639207422518167551452237278775600
8972531819787914850128505070375311744529
```

These computations are completed using the formulas for A_k and B_k of Proposition 4.10. Computing larger sums does not change the results of Pari/GP for the selected real precision of 115 significant digits. We note that $A_k, B_k > k!$ for all $k > 1$, so the error terms are easily less than $2/201!$, although the actual error term is much smaller.

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