INFINITE SUMS INVOLVING GIBONACCI POLYNOMIALS

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Abstract. We explore infinite sums involving Fibonacci polynomials, and their Lucas, Pell, and Pell-Lucas implications.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \text{ and } l_n(x) = \alpha^n(x) + \beta^n(x),$$

where $2\alpha(x) = x + \Delta$, $2\beta(x) = x - \Delta$, and $\Delta = \sqrt{x^2+4}$. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [2, 3].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, and omit a lot of basic algebra.

1.1. Some Fundamental Identities.

Gibonacci polynomials g_n satisfy the following fundamental properties [3, p. 57]:

a) $l_n = f_{n+1} + f_{n-1}$;
b) $f_{2n} = f_n l_n$;
c) $xf_{2n} = f_{n+1}^2 - f_{n-1}^2$;
d) $(x^2+2)f_n = f_{n+2} + f_{n-2}$;
e) $xl_n = f_{n+2} - f_{n-2}$;
f) $(x^3+2x)f_{2n} = f_{n+2}^2 - f_{n-2}^2$;
g) $f_{n+k}f_{n-k} - f_n^2 = (-1)^{n+k-1}f_k^2$;
h) $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$.

Property (g) is the *Cassini-like* (or *Catalan-like*) identity for Fibonacci polynomials.

Property (c) implies that $xf_{2(2n+1)} = f_{2n+2}^2 - f_{2n}^2$ and $xf_{2(2n+2)} = f_{2n+3}^2 - f_{2n+1}^2$, and property (f) implies that $(x^3+2x)f_{2(2n+2)} = f_{2n+4}^2 - f_{2n}^2$. In addition, it follows by the Cassini-like identity that

$$\begin{aligned} f_{2n+1}f_{2n-1} &= f_{2n}^2 + 1; & f_{2n+2}f_{2n} &= f_{2n+1}^2 - 1; \\ f_{2n+3}f_{2n-1} &= f_{2n+1}^2 + x^2; & f_{2n+4}f_{2n} &= f_{2n+2}^2 - x^2. \end{aligned}$$

With this background, we now begin our explorations.

2. Fibonacci Polynomial Sums

We begin our discourse with a sum involving odd-numbered Fibonacci polynomials.

Theorem 2.1.

$$\sum_{n=1}^{\infty} \frac{f_{2(2n+1)}}{\left(f_{2n+1}^2 - 1\right)^2} = \frac{1}{x^3}. \tag{2.1}$$

Proof. Using recursion [3], we will first establish that

$$\sum_{n=1}^{m} \frac{x f_{2(2n+1)}}{\left(f_{2n+1}^2 - 1\right)^2} = \frac{1}{x^2} - \frac{1}{f_{2m+2}^2}. \tag{2.2}$$

To this end, we let A_m and B_m be the left and right side of (2.2), respectively. Then,

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{f_{2m}^2} - \frac{1}{f_{2m+2}^2} \\ &= \frac{f_{2m+2}^2 - f_{2m}^2}{f_{2m+2}^2 f_{2m}^2} \\ &= \frac{x f_{2(2m+1)}}{\left(f_{2m+1}^2 - 1\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Consequently,

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 = \frac{x f_6}{(f_3^2 - 1)^2} - \left(\frac{1}{x^2} - \frac{1}{f_4^2}\right) = 0.$$

Thus, $A_m = B_m$, as expected.

Because $\lim_{m\to\infty} \frac{1}{f_m} = 0$, the given result now follows from equation (2.2). □

It follows from equation (2.2) that

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_{2(2n+1)}}{\left(F_{2n+1}^2 - 1\right)^2} &= 1 - \frac{1}{F_{2m+2}^2}; \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(F_{2n+1}^2 - 1\right)^2} &= 1. \end{aligned} \tag{2.3}$$

Next, we explore the counterpart of Theorem 2.1 for even-numbered Fibonacci polynomials.

Theorem 2.2.

$$\sum_{n=1}^{\infty} \frac{(x^3 + 2x) f_{2(2n+2)}}{\left(f_{2n+2}^2 - x^2\right)^2} = \frac{1}{x^2} + \frac{1}{(x^3 + 2x)^2}. \tag{2.4}$$

Proof. Using recursion [3], we will first confirm that

$$\sum_{n=1}^{m} \frac{(x^3 + 2x) f_{2(2n+2)}}{\left(f_{2n+2}^2 - x^2\right)^2} = \frac{1}{x^2} + \frac{1}{(x^3 + 2x)^2} - \frac{1}{f_{2m+2}^2} - \frac{1}{f_{2m+4}^2}. \tag{2.5}$$

Letting A_m and B_m be the left and right side of (2.5), respectively, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{f_{2m}^2} - \frac{1}{f_{2m+4}^2} \\ &= \frac{f_{2m+4}^2 - f_{2m}^2}{f_{2m+4}^2 f_{2m}^2} \\ &= \frac{(x^3+2x) f_{2(2m+2)}}{\left(f_{2m+2}^2 - x^2\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

This implies

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 = \frac{(x^3+2x)f_8}{(f_4^2 - x^2)^2} - \left(\frac{1}{x^2} + \frac{1}{f_4^2} - \frac{1}{f_4^2} - \frac{1}{f_6^2}\right) = 0.$$

Thus, $A_m = B_m$, as desired.

The given result now follows from equation (2.5). □

Equation (2.5) yields

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2 - 1\right)^2} &= \frac{10}{27} - \frac{1}{3F_{2m+2}^2} - \frac{1}{3F_{2m+4}^2}; \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2 - 1\right)^2} &= \frac{10}{27}, \end{aligned} \tag{2.6}$$

respectively.

A Fibonacci Delight: Equation (2.3), coupled with (2.6), yields an interesting consequence:

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{F_{2n}}{(F_n^2 - 1)^2} &= \sum_{n=1}^{\infty} \left[\frac{F_{2(2n+1)}}{\left(F_{2n+1}^2 - 1\right)^2} + \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2 - 1\right)^2}\right] \\ &= \frac{37}{27} \\ &= 1.\overline{370}. \end{aligned}$$

Next, we investigate two additional sums involving Fibonacci polynomials.

Theorem 2.3.

$$\sum_{n=1}^{\infty} \frac{(x^3+2x) f_{2(2n+1)}}{\left(f_{2n+1}^2 + x^2\right)^2} = 1 + \frac{1}{(x^2+1)^2}. \tag{2.7}$$

Proof. Using recursion [3], we will first prove that

$$\sum_{n=1}^{m} \frac{(x^3+2x) f_{2(2n+1)}}{\left(f_{2n+1}^2 + x^2\right)^2} = 1 + \frac{1}{(x^2+1)^2} - \frac{1}{f_{2m+1}^2} - \frac{1}{f_{2m+3}^2}. \tag{2.8}$$

Again, we let A_m and B_m be the left and right side of (2.8), respectively. Then,

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{f_{2m-1}^2} - \frac{1}{f_{2m+3}^2} \\ &= \frac{f_{2m+3}^2 - f_{2m-1}^2}{f_{2m+3}^2 f_{2m-1}^2} \\ &= \frac{(x^3+2x) f_{2(2m+1)}}{\left(f_{2m+1}^2 + x^2\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Consequently, $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 = \dfrac{f_6 f_4}{(f_3^2 + x^2)^2} - \left(1 - \dfrac{1}{f_5^2}\right) = 0.$ So, $A_m = B_m$, as expected.

The given result now follows from equation (2.8). □

Equations (2.8) and (2.7) yield

$$\begin{aligned} \sum_{n=1}^{m} \frac{3F_{2(2n+1)}}{\left(F_{2n+1}^2 + 1\right)^2} &= \frac{5}{4} - \frac{1}{F_{2m+1}^2} - \frac{1}{F_{2m+3}^2}; \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(F_{2n+1}^2 + 1\right)^2} &= \frac{5}{12}, \end{aligned} \tag{2.9}$$

respectively.

Theorem 2.4.

$$\sum_{n=1}^{\infty} \frac{x f_{2(2n+2)}}{\left(f_{2n+2}^2 + 1\right)^2} = \frac{1}{(x^2+1)^2}. \tag{2.10}$$

Proof. Using recursion [3], we will first establish that

$$\sum_{n=1}^{m} \frac{x f_{2(2n+2)}}{\left(f_{2n+2}^2 + 1\right)^2} = \frac{1}{(x^2+1)^2} - \frac{1}{f_{2m+3}^2}. \tag{2.11}$$

As before, letting A_m and B_m be the left and right side of (2.11), respectively, yields

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{f_{2m+1}^2} - \frac{1}{f_{2m+3}^2} \\ &= \frac{f_{2m+3}^2 - f_{2m+1}^2}{f_{2m+3}^2 f_{2m+1}^2} \\ &= \frac{x f_{2(2m+2)}}{\left(f_{2m+2}^2 + 1\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Then, $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 = \dfrac{x f_8}{(f_4^2+1)^2} - \left(\dfrac{1}{f_3^2} - \dfrac{1}{f_5^2}\right) = 0.$ Consequently, $A_m = B_m$, as expected.

Clearly, equation (2.11) yields the given result, as desired. □

It follows from equation (2.11) that

$$\begin{aligned}\sum_{n=1}^{m} \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2+1\right)^2} &= \frac{1}{4} - \frac{1}{F_{2m+3}^2};\\ \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2+1\right)^2} &= \frac{1}{4}. \end{aligned} \tag{2.12}$$

Another Fibonacci Delight: Equation (2.9), coupled with equation (2.12), yields an interesting consequence:

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{\left(F_n^2+1\right)^2} &= \sum_{n=1}^{\infty} \left[\frac{F_{2(2n+1)}}{\left(F_{2n+1}^2+1\right)^2} + \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2+1\right)^2}\right]\\ &= \frac{2}{3};\\ \sum_{n=1}^{\infty} \frac{F_{2n}}{\left(F_n^2+1\right)^2} &= \frac{5}{3},\end{aligned}$$

as in [1, 4].

3. Pell Versions

Equations (2.1) through (2.12) have Pell implications. In the interest of brevity, we now explore four of them. Because $b_n(x) = g_n(2x)$, equations (2.1), (2.4), (2.7), and (2.10) yield the following results:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{p_{2(2n+1)}}{\left(p_{2n+1}^2-1\right)^2} &= \frac{1}{8x^3};\\ \sum_{n=1}^{\infty} \frac{(2x^3+x)p_{2(2n+2)}}{\left(p_{2n+2}^2-4x^2\right)^2} &= \frac{1}{16x^2} + \frac{1}{64(2x^3+x)^2};\\ \sum_{n=1}^{\infty} \frac{(2x^3+x)p_{2(2n+1)}}{\left(p_{2n+1}^2+4x^2\right)^2} &= \frac{1}{4} + \frac{1}{4(4x^2+1)^2};\\ \sum_{n=1}^{\infty} \frac{xp_{2(2n+2)}}{\left(p_{2n+2}^2+1\right)^2} &= \frac{1}{2(4x^2+1)^2},\end{aligned}$$

respectively.

They imply that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{\left(P_{2n+1}^2-1\right)^2} &= \frac{1}{8};\\ \sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{\left(P_{2n+2}^2-4\right)^2} &= \frac{37}{1728};\\ \sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{\left(P_{2n+1}^2+4\right)^2} &= \frac{13}{150};\\ \sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{\left(P_{2n+2}^2+1\right)^2} &= \frac{1}{50},\end{aligned}$$

respectively.

4. Lucas Versions

Equations (2.1) through (2.12) also have Lucas implications. Again, in the interest of brevity, we focus on equations (2.1), (2.4), (2.7), and (2.10) only; using property (h), they yield the following results:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\Delta^4 f_{2(2n+1)}}{\left(l_{2n+1}^2 - x^2\right)^2} &= \frac{1}{x^3};\\
\sum_{n=1}^{\infty} \frac{\Delta^4 (x^3+2x) f_{2(2n+2)}}{\left[l_{2n+2}^2 - (x^2+2)^2\right]^2} &= \frac{1}{x^2} + \frac{1}{(x^3+2x)^2};\\
\sum_{n=1}^{\infty} \frac{\Delta^4 (x^3+2x) f_{2(2n+1)}}{\left[l_{2n+1}^2 + (x^2+2)^2\right]^2} &= 1 + \frac{1}{(x^2+1)^2};\\
\sum_{n=1}^{\infty} \frac{\Delta^4 x f_{2(2n+2)}}{\left(l_{2n+2}^2 + x^2\right)^2} &= \frac{1}{(x^2+1)^2},
\end{aligned}$$

respectively.

Consequently, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 - 1\right)^2} &= \frac{1}{25};\\
\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 - 9\right)^2} &= \frac{2}{135};\\
\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 + 9\right)^2} &= \frac{1}{60};\\
\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 + 1\right)^2} &= \frac{1}{100},
\end{aligned}$$

respectively.

5. Pell-Lucas Versions

Using the relationships $b_n(x) = g_n(2x)$ and $q_n(1) = 2Q_n$, we can find the Pell-Lucas versions of equations (2.1) through (2.12). But, for convenience, we focus on equations (2.1), (2.4), (2.7), and (2.10) only:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(x^2+1)^2 p_{2(2n+1)}}{\left(q_{2n+1}^2 - 4x^2\right)^2} &= \frac{1}{128x^3};\\
\sum_{n=1}^{\infty} \frac{(x^2+1)^2 (2x^3+x) p_{2(2n+2)}}{\left[q_{2n+2}^2 - 4(2x^2+1)^2\right]^2} &= \frac{1}{256x^2} + \frac{1}{1024(2x^3+x)^2};\\
\sum_{n=1}^{\infty} \frac{(x^2+1)^2 (2x^3+x) p_{2(2n+1)}}{\left[q_{2n+1}^2 + 4(2x^2+1)^2\right]^2} &= \frac{1}{64} + \frac{1}{64(4x^2+1)^2};
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{x(x^2+1)^2 p_{2(2n+2)}}{\left(q_{2n+2}^2 + 4x^2\right)^2} = \frac{1}{32(4x^2+1)^2},$$

respectively.

They yield

$$\sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{\left(Q_{2n+1}^2 - 1\right)^2} = \frac{1}{32};$$
$$\sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{\left(Q_{2n+2}^2 - 9\right)^2} = \frac{37}{6912};$$
$$\sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{\left(Q_{2n+1}^2 + 9\right)^2} = \frac{13}{600};$$
$$\sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{\left(Q_{2n+2}^2 + 1\right)^2} = \frac{1}{200},$$

respectively.

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