ON ZECKENDORF RELATED PARTITIONS USING THE LUCAS SEQUENCE

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ABSTRACT. Zeckendorf proved that every positive integer has a unique partition as a sum of nonconsecutive Fibonacci numbers. Similarly, every natural number can be partitioned into a sum of nonconsecutive terms of the Lucas sequence, although such partitions need not be unique. In this paper, we
  (1) prove that a natural number can have at most two distinct nonconsecutive partitions in the Lucas sequence,
  (2) find all natural numbers with a fixed term in their partition, and
  (3) calculate the limiting value of the proportion of natural numbers that are not uniquely partitioned into the sum of nonconsecutive terms in the Lucas sequence.

1. INTRODUCTION

The Fibonacci numbers have fascinated mathematicians for centuries with many interesting properties. By convention, the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined as follows: let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. A theorem of Zeckendorf [31] states that every positive integer n can be uniquely written as a sum of nonconsecutive Fibonacci numbers. This gives the so-called Zeckendorf partition of n. A formal statement of Zeckendorf's theorem is as follows:

Theorem 1.1 (Zeckendorf). *For any $n \in \mathbb{N}$, there exists a unique increasing sequence of natural numbers $\{c_1, c_2, \ldots, c_k\}$ such that $c_1 \geq 2$, $c_i \geq c_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and $n = \sum_{i=1}^{k} F_{c_i}$.*

Much work has been done to understand the structure of Zeckendorf partitions and their applications (see [1, 2, 6, 8, 16, 18, 19, 20, 25, 29, 30]) and to generalize them (see [10, 12, 13, 14, 15, 17, 22, 23, 26, 27, 28]). In this paper, we study the partition of natural numbers into Lucas numbers. The Lucas sequence $\{L_n\}_{n=0}^{\infty}$ is defined as follows: let $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Because the Lucas sequence is closely related to the Fibonacci sequence, it is not surprising that we can also partition natural numbers using Lucas numbers.

Theorem 1.2 (Zeckendorf). *Every natural number can be partitioned into the sum of nonconsecutive terms of the Lucas sequence.*

Note that the distinction between Theorems 1.1 and 1.2 lies in the *uniqueness* property of such partitions of natural numbers in the Fibonacci and Lucas sequences. Although 5 is uniquely partitioned into $F_5 = 5$ in $\{F_2, F_3, \ldots\}$, its partition is not unique in the Lucas sequence because $5 = L_0 + L_2 = 2 + 3$ and $5 = L_1 + L_3 = 1 + 4$. In [7], Brown shows various ways to have a unique partition using Lucas sequence.[1] In this paper, we prove the following results.

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[1] For more on Brown's criteria, see [3, 4].

Theorem 1.3. *If we allow L_0 and L_2 to appear simultaneously in a partition, each natural number can have at most two distinct nonconsecutive partitions in the Lucas sequence.*

Theorem 1.4. *Suppose that we do not allow L_0 and L_2 to appear simultaneously in a partition. The set of all natural numbers having the summand L_k in their partition is given by*

$$Z(k) = \begin{cases} \left\{2 + 3n + \left\lfloor \frac{n+1}{\Phi} \right\rfloor \ : \ n \geq 0\right\}, & \text{if } k = 0; \\ \left\{3n + \left\lfloor \frac{n+\Phi^2}{\Phi} \right\rfloor \ : \ n \geq 0\right\}, & \text{if } k = 1; \\ \left\{L_k \left\lfloor \frac{n+\Phi^2}{\Phi} \right\rfloor + nL_{k+1} + j \ : \ n \geq 0 \text{ and } 0 \leq j \leq L_{k-1} - 1\right\}, & \text{if } k \geq 2. \end{cases}$$

Theorem 1.4 is an analogue of [18, Theorem 3.4]. For $k \geq 0$, we find all natural numbers having the summand L_k in their partition. We have a different formula when $k = 0$ instead of one formula for all values of k as in [18, Theorem 3.4].

Our next result is predicted by [9, Theorem 1], which addresses general recurrence relations; however, in the case of Lucas numbers, we can relate Lucas partitions to the golden string.

Theorem 1.5. *If we allow L_0 and L_2 to appear simultaneously in a partition, the proportion of natural numbers that are not uniquely partitioned into the sum of nonconsecutive terms of the Lucas sequence converges to $\frac{1}{3\Phi+1}$, where Φ is the golden ratio.*

2. Preliminaries

2.1. Definitions.

Definition 2.1. *Let $A = \{a_0, a_1, \ldots, a_m\}$ be the set consisting of the first $m + 1$ terms of the sequence $\{a_k\}_{k=0}^{\infty}$. We say a proper subset B of A is a nonconsecutive subset of A if the elements of B are pairwise nonconsecutive in $\{a_k\}_{k=0}^{\infty}$. Furthermore, we say a sum S is a nonconsecutive sum of A if S is the sum of distinct elements of A that are pairwise nonconsecutive in $\{a_k\}_{k=0}^{\infty}$.*

Definition 2.2. *Let $A_m = \{L_0, L_1, \ldots, L_m\}$ denote the set consisting of the first $m + 1$ terms of the Lucas sequence.*

2.2. The Golden String.

The golden string $S = BABBABABBABBA\ldots$ is defined to be the infinite string of As and Bs constructed recursively as follows. Let $S_1 = A$ and $S_2 = B$, and then, for $k \geq 3$, S_k is the concatenation of S_{k-1} and S_{k-2}, which we denote by $S_{k-1} \circ S_{k-2}$. For example, $S_3 = S_2 \circ S_1 = a_2 \circ a_1 = BA$, $S_4 = S_3 \circ S_2 = a_2 a_1 \circ a_2 = BAB$, $S_5 = S_4 \circ S_3 = BABBA$, and so on. The golden string is highly connected to the Zeckendorf partition [19]. As we will see later, the string is also closely related to the partitions of natural numbers into Lucas numbers.

Remark 2.3. We mention two properties of the golden string that we will use.
   (1) For $j \geq 1$, the (F_{2j})th character of S is B and the (F_{2j+1})th character of S is A. This can be proved easily using induction.
   (2) The number of B's among the first n characters of S is given by $\left\lfloor \frac{n+1}{\Phi} \right\rfloor$, where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. For a proof of this result, see [19, Lemma 3.3].

3. At Most Two Partitions

In this section, we present our results that determine the maximum number of nonconsecutive partitions that a natural number can have in the Lucas sequence, the proofs of which are adapted from [21]. Before we prove Theorem 1.3, we introduce the following preliminary lemmas. For the proofs of Lemmas 3.1 and 3.2, see Appendix B of [11].

Lemma 3.1. *Let S be any nonconsecutive sum of A_m. Then*
   (1) *if m is odd, S assumes all values from 0 to $L_{m+1} - 1$ inclusive, and*
   (2) *if m is even, then S assumes all values from 0 to $L_{m+1} + 1$ inclusive, excluding L_{m+1}.*

Lemma 3.2. *If $m \geq 0$, then $L_{2m+1} + 1$ has exactly two nonconsecutive partitions in the Lucas sequence.*

Proof of Theorem 1.3. It suffices to show that for every nonnegative integer m, there is no natural number that is equal to three or more distinct nonconsecutive sums of A_m. We proceed by strong induction. No natural is equal to three or more distinct nonconsecutive sums of A_0 and A_1. This shows the base case. Assume Theorem 1.3 holds for all nonnegative integers less than or equal to $m = k$. In our first case, suppose that k is odd. From Lemma 3.1, the nonconsecutive sums that we can form from A_k are the values from 0 to $L_{k+1} - 1$ inclusive. Hence, when we add the term L_{k+1} to A_k, all new nonconsecutive sums that can be formed must be at least L_{k+1}. This implies there is no possible way in which we can form a third distinct nonconsecutive sum of A_{k+1} for any natural number because there is no intersection between the nonconsecutive sums in which we can form before and after the addition of the term L_{k+1}. When $k \geq 2$ is even, we have from Lemma 3.1 that all nonconsecutive sums we can form from A_k are the values from 0 to $L_{k+1} + 1$ inclusive, excluding L_{k+1}. When we add the term L_{k+1} to A_k, all new nonconsecutive sums that can be formed are at least L_{k+1} with $L_{k+1} + 1$ being the only nonconsecutive sum formed again, namely $L_{k+1} + L_1$. By Lemma 3.2, we know that $L_{k+1} + 1$ has exactly two distinct nonconsecutive partitions in the Lucas sequence. Therefore, there is no possible way in which we can form a third distinct nonconsecutive sum of A_{k+1} for any natural number. This completes the inductive step. □

4. Partitions with a Fixed Term

Let $\mathcal{X}_k$ denote the set of all natural numbers having L_k as the smallest summand in their partition. Let $\mathcal{Q}_k = \{q_k(j)\}_{j \geq 1}$ be the strictly increasing sequence obtained by rearranging the elements of $\mathcal{X}_k$ into ascending numerical order. We consider the cases $k = 0$ and $k \geq 1$ separately.

4.1. When $k = 0$. Table 1 replaces each term $q_k(j)$ in $\mathcal{Q}_k$ with an ordered list of the summands in its partition.

Row					
1	L_0				
2	L_0	L_3			
3	L_0		L_4		
4	L_0			L_5	
5	L_0	L_3		L_5	
6	L_0				L_6
7	L_0	L_3			L_6
8	L_0		L_4		L_6

Table 1. The partitions of the natural numbers having L_0 as their smallest summand.

Lemma 4.1. *For $j \geq 3$, the rows of Table 1 for which L_j is the largest summand are those numbered from $F_{j-1} + 1$ to F_j inclusive.*

Proof. The proof is by induction.
<u>Base Cases</u>. It is easy to check that the statement of the lemma is true for $j = 3$ and $j = 4$.
<u>Inductive Hypothesis</u>. Assume that it is true for all j such that $3 \leq j \leq m$ for some $m \geq 4$.

By the inductive hypothesis, the number of rows such that their largest summands are no greater than L_{m-1} is

$$1 + \sum_{j=3}^{m-1} (F_j - F_{j-1}) = F_{m-1},$$

which is also the number of rows whose largest summand is L_{m+1}. Due to the inductive hypothesis, the rows whose largest summand is L_m are numbered from $F_{m-1} + 1$ to F_m inclusive. Therefore, the rows whose largest summand is L_{m+1} are numbered from $F_m + 1$ to F_{m+1}, as desired. This completes our proof. □

Lemma 4.2. *For $j \geq 1$, we have*

$$q_k(j+1) - q_k(j) = \begin{cases} L_2, & \text{if } A \text{ is the } j\text{th character of } S; \\ L_3, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases}$$

Proof. We prove by induction.
<u>Base Cases</u>. It is easy to check that the statement of the lemma is true for $1 \leq j \leq F_4 - 1$.
<u>Inductive Hypothesis</u>. Suppose that it is true for $1 \leq j \leq F_m - 1$ for some $m \geq 4$. By Lemma 4.1, the number of rows in Table 1 whose largest summand is no greater than L_{m-1} is

$$1 + \sum_{j=3}^{m-1} (F_j - F_{j-1}) = F_{m-1},$$

which is also the number of rows whose largest summand is L_{m+1}. Furthermore, the rows for which L_{m+1} is the largest summand are numbered from $F_m + 1$ to F_{m+1} inclusive. Therefore, the ordering of the rows in Table 1 implies that $q_k(i + F_m) = q_k(i) + L_{m+1}$ for $1 \leq i \leq F_{m-1}$. Hence, for $1 \leq i \leq F_{m-1} - 1$, we have

$$q_k(i + 1 + F_m) - q_k(i + F_m) = (q_k(i+1) + L_{m+1}) - (q_k(i) + L_{m+1}) = q_k(i+1) - q_k(i).$$

By the construction of S, the substring comprising of its first F_{m-1} characters is identical to the substring of its characters numbered from $F_m + 1$ to F_{m+1} inclusive. Thus, the lemma is true for $F_m + 1 \leq j \leq F_{m+1} - 1$. It remains to show that it is true for $j = F_m$. We have

$$q_k(F_m + 1) - q_k(F_m) = \begin{cases} L_{m+1} - (L_m + L_{m-2} + \cdots + L_4) = L_3, & \text{if } m \text{ is even}; \\ L_{m+1} - (L_m + L_{m-2} + \cdots + L_3) = L_2, & \text{if } m \text{ is odd}. \end{cases}$$

By Remark 2.3 item (1), we know that the lemma is true for $j = F_m$, completing the proof. □

4.2. **When $k \geq 1$.** Table 2 replaces each term $q_k(j)$ in $\mathcal{Q}_k$ with an ordered list of the summands in its partition.

Row					
1	L_k				
2	L_k	L_{k+2}			
3	L_k		L_{k+3}		
4	L_k			L_{k+4}	
5	L_k	L_{k+2}		L_{k+4}	
6	L_k				L_{k+5}
7	L_k	L_{k+2}			L_{k+5}
8	L_k		L_{k+3}		L_{k+5}

Table 2. The partitions of the natural numbers having L_k as their smallest summand.

Table 2 is similar to Table 1 in [18]. The next lemma follows from [18, Lemma 3.1].

Lemma 4.3. *For $j \geq 2$, the rows of Table 2 for which L_{k+j} is the largest summand are those numbered from $F_j + 1$ to F_{j+1} inclusive.*

Lemma 4.4. *For $j \geq 1$, we have*

$$q_k(j+1) - q_k(j) = \begin{cases} L_{k+1}, & \text{if } A \text{ is the } j\text{th character of } S; \\ L_{k+2}, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases}$$

Proof. The proof is by induction.

Base Cases. It is easy to check that the statement of the lemma is true for j such that $1 \leq j \leq F_4 - 1$.

Inductive Hypothesis. Assume that it is true for $1 \leq j \leq F_m - 1$ for some $m \geq 4$. From Lemma 4.3, the first F_{m-1} rows of Table 2 are those for which the largest summand is no greater than L_{k+m-2}. Also, the rows for which L_{k+m} is the largest summand are those numbered from $F_m + 1$ to F_{m+1} inclusive. Therefore, the ordering of the rows implies that $q_k(i + F_m) = q_k(i) + L_{k+m}$ for $i = 1, 2, \ldots, F_{m-1}$. Hence, for $i = 1, 2, \ldots, F_{m-1} - 1$, we have

$$q_k(i + 1 + F_m) - q_k(i + F_m) = (q_k(i+1) + L_{k+m}) - (q_k(i) + L_{k+m}) = q_k(i+1) - q_k(i).$$

By the construction of S, the substring comprising its first F_{m-1} characters is identical to the substring of its characters numbered from $F_m + 1$ to F_{m+1} inclusive. Thus, the lemma is true for $F_m + 1 \leq j \leq F_{m+1} - 1$. It remains to show that the lemma is true for $j = F_m$. We have

$$q_k(F_m + 1) - q_k(F_m) = \begin{cases} L_{k+m} - (L_{k+m-1} + L_{k+m-3} + \cdots + L_{k+3}) = L_{k+2}, & \text{if } m \text{ is even;} \\ L_{k+m} - (L_{k+m-1} + L_{k+m-3} + \cdots + L_{k+2}) = L_{k+1}, & \text{if } m \text{ is odd.} \end{cases}$$

By Remark 2.3 item (1), we know that the lemma is true for $j = F_m$, completing the proof. $\square$

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We consider three cases.

Case 1. $k = 0$. By Lemma 4.2, we have $\mathcal{X}_0 = \{2 + a(n)L_2 + b(n)L_3 : n \geq 0\}$, where $a(n)$ and $b(n)$ denote the number of As and Bs, respectively, among the first n characters in the golden string. Using Remark 2.3 item (2), we have

$$\mathcal{X}_0 = \left\{ 2 + 3\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + 4\left\lfloor \frac{n+1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ 2 + 3n + \left\lfloor \frac{n+1}{\Phi} \right\rfloor : n \geq 0 \right\}.$$

It is clear that $Z(0) = \mathcal{X}_0$; hence, the statement of the lemma is true when $k = 0$.

Case 2. $k = 1$. Using a similar reasoning as above, we have

$$\mathcal{X}_1 = \left\{ 1 + L_2\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + L_3\left\lfloor \frac{n+1}{\Phi} \right\rfloor : n \geq 0 \right\}$$

$$= \left\{ 1 + 3\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + 4\left\lfloor \frac{n+1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ 3n + \left\lfloor \frac{n+\Phi^2}{\Phi} \right\rfloor : n \geq 0 \right\}.$$

It is clear that $Z(1) = \mathcal{X}_1$; hence, the statement of the lemma is true when $k = 1$.

<u>Case 3</u>. $k \geq 2$. Using a similar reasoning as above, we have

$$
\begin{aligned}
\mathcal{X}_k &= \left\{ L_k + L_{k+1} \left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor \right) + L_{k+2} \left\lfloor \frac{n+1}{\Phi} \right\rfloor \ : \ n \geq 0 \right\} \\
&= \left\{ L_k \left(1 + \left\lfloor \frac{n+1}{\Phi} \right\rfloor \right) + n L_{k+1} \ : \ n \geq 0 \right\} = \left\{ L_k \left\lfloor \frac{n + \Phi^2}{\Phi} \right\rfloor + n L_{k+1} \ : \ n \geq 0 \right\}.
\end{aligned}
$$

If $k \geq 3$, the numbers in $\{L_0, L_1, \ldots, L_{k-2}\}$ are used to obtain the partitions of all integers for which the largest summand is no greater than L_{k-2}. In particular, such partitions generate all integers from 1 to $L_{k-1} - 1$ inclusive. Furthermore, such partitions can be appended to any partition having L_k as its smallest summand to produce another partition. Therefore,

$$
Z(k) = \left\{ L_k \left\lfloor \frac{n + \Phi^2}{\Phi} \right\rfloor + n L_{k+1} + j \ : \ n \geq 0 \text{ and } 0 \leq j \leq L_{k-1} - 1 \right\},
$$

as desired. It is easy to check that this formula is also true for $k = 2$. $\qquad \square$

5. Proportion of Nonunique Partitions

Let $c(N)$ count the number of numbers that are not uniquely represented in the Lucas sequence and are at most N. We want to show that $\lim\limits_{N \to \infty} \dfrac{c(N)}{N} = \dfrac{1}{1 + 3\Phi}$, where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Note that [7, Lemma 3] says we can make the Lucas partition unique by requiring that L_0 and L_2 do not both appear in the partition. Therefore, if a number has two partitions, then one of the partition starts with $L_0 + L_2$. If we can characterize all of these numbers and find a formula for $c(N)$ in terms of N, we are done. Call the set of these numbers K. Table 3 lists all of such numbers in increasing order. Let $q_k(j)$ be the jth smallest number in K.

Row				
1	$L_0 + L_2$			
2	$L_0 + L_2$	L_4		
3	$L_0 + L_2$		L_5	
4	$L_0 + L_2$			L_6
5	$L_0 + L_2$	L_4		L_6
6	$L_0 + L_2$			L_7
7	$L_0 + L_2$	L_4		L_7
8	$L_0 + L_2$		L_5	L_7

Table 3. The partitions of the natural numbers having L_0 and L_2 as their smallest summands.

Observe that Table 3 has the same structure as Table 1. Therefore, Lemma 4.2 applies with a change of index. In particular, we have the following.

Lemma 5.1. *For $j \geq 1$, we have*

$$
q_k(j+1) - q_k(j) = \begin{cases} L_3, & \text{if } A \text{ is the } j\text{th character of } S; \\ L_4, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases}
$$

Therefore, we can write

$$
K = \{L_0 + L_2 + a(n)L_3 + b(n)L_4 : n \geq 0\},
$$

where $a(n)$ and $b(n)$ denote the number of As and Bs, respectively, among the first n characters in the golden string. Hence,

$$K = \left\{ 5 + 4\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + 7 \left\lfloor \frac{n+1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ 5 + 4n + 3 \left\lfloor \frac{n+1}{\Phi} \right\rfloor : n \geq 0 \right\}.$$

Now, we are ready to compute the limit.

Proof of Theorem 1.5. The number of integers with two partitions up to a number N is exactly $\# \left\{ n \geq 0 : 5 + 4n + 3 \left\lfloor \frac{n+1}{\Phi} \right\rfloor \leq N \right\}$. The number is found to be $\frac{N-1}{4+\frac{3}{\Phi}}$ within an error of at most 1. Therefore, as claimed, the limit is

$$\lim_{N \to \infty} \frac{1}{N} \frac{N-1}{4 + \frac{3}{\Phi}} = \frac{1}{4 + \frac{3}{\Phi}} = \frac{1}{1 + 3\Phi}.$$

$\square$

Among the first N natural numbers, we see how $\alpha = \frac{1}{3\Phi+1} \approx 0.17082$ estimates the proportion of natural numbers within this range that do not have unique nonconsecutive partitions in the Lucas sequence. The data we collect is shown in Table 4.

N	$c(N)$	$\beta(N)$
10	1	10.000 %
100	17	17.000%
1,000	171	17.100%
10,000	1,708	17.080%
10^5	17,082	17.082%
10^6	170,820	17.082%

Table 4. Proportion $\beta(N)$ of the first N natural numbers that do not have unique nonconsecutive partitions in the Lucas sequence.

APPENDIX A. JAVA CODE

The following is our Java code for calculating nonconsecutive partitions of natural numbers in any infinite integer sequence given by a second-order linear recurrence. It is available on github at `https://github.com/dluo6745/Zeckendorf-Partitions/blob/master/ZP.java`. For each natural number n from 1 to N, the code returns the nonconsecutive partition(s) of n as a list of integers that correspond to the indices of the terms in the second-order linear recurrence sequence we are enumerating. Furthermore, the code also returns the number of natural numbers from 1 to N that do not have unique nonconsecutive partitions.

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ON ZECKENDORF RELATED PARTITIONS USING THE LUCAS SEQUENCE

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