ON INTEGERS WHOSE SUM IS THE REVERSE OF THEIR PRODUCT

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Abstract. We determine all pairs of positive integers (a, b) such that $a + b$ and $a \times b$ have the same decimal digits in reverse order:

$$(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), (2, 49997), \ldots$$

Our recursive procedure for constructing such pairs naturally extends to all numerical bases. We also investigate several phenomena related to the structure of the set of pairs that arise for a given base, and we give a visual interpretation of our construction in terms of deterministic finite automata.

1. Introduction

During a homeschool math lesson, the first author's children made the curious observation that $9 + 9 = 18$ and $9 \times 9 = 81$ — that is, the sum and product are the reverse of each other. A short computer search revealed the more interesting examples

$$2 + 47 = 49 \quad \text{and} \quad 2 \times 47 = 94$$
$$3 + 24 = 27 \quad \text{and} \quad 3 \times 24 = 72.$$

Are there other examples of integer pairs (a, b) for which the digits of $a + b$ are the reverse of the digits of ab?

To formalize the problem, we say that a base-β representation of a positive integer is in canonical form if it has no leading zero. A pair of positive integers $a \leq b$ will be called a **reversed sum-product pair for the base** β if the canonical representations of $a + b$ and ab in base β are the reverse of each other. We insist that all numbers be written in canonical form to avoid examples like $a = 15$ and $b = 624$ in base 10, for which $ab = 9360$ and $a + b = 0639$. (Allowing noncanonical representations is also interesting, but we do not address that problem in this paper.)

We will systematically develop an algorithm for determining all reversed sum-product pairs (a, b) for a given base β in Section 2–5. For example, in base 10 the complete list is

$$(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), (2, 49997), \ldots$$

For another example, consider the base 18, where we use the digits $0, 1, 2, \ldots, 9, A, B, C, \ldots, H$. The complete list of reversed sum-product pairs for the base 18 is

$$(2, 2), (H, H), (3, 37), (4, 25),$$
$$(7, 2483D8), (7, 2483D9E483D8), (7, 2483D9E483D9E483D8), \ldots$$
$$(B, 1961DC5), (B, 1961DBG461DC5), (B, 1961DBG461DBG461DC5), \ldots$$

We will break our discussion of the algorithm for constructing reversed sum-product pairs into three parts, corresponding to the relative sizes of a, b, and the base β:

- (small b) $a \leq b < \beta$
- (large a) $\beta < a \leq b$
- (small a, large b) $a < \beta < b$.

In Section 2, we show that there are only two reversed sum-product pairs when b is small: $(2, 2)$ and $(\beta - 1, \beta - 1)$. In Section 3, we will prove there is no reversed sum-product pair with large a. For the final case, it will be useful to know that we do not need to "carry" when computing the sum of a and b; this is proved in Section 4. When $a < \beta < b$, we proceed with a recursive construction. For a fixed choice of a, the technique builds two digits of b at a time. We describe this algorithm in Section 5, including a careful explanation for the base 10.

A convenient tool for visualizing our recursive construction of reversed sum-product pairs is the deterministic finite automaton (DFA). In Section 6, we explain how to produce a DFA for each base β and each positive integer $a < \beta$. The accepted strings for this DFA are in one-to-one correspondence with integers b such that (a, b) is a reversed sum-product pair. Python code for automating our recursive algorithm and for visualizing this DFA construction is available at

`https://github.com/RationalPoint/reverse`.

Next, we turn to a kind of opposite problem. Instead of fixing the numerical base, we fix a positive integer a and ask for which bases $\beta > a$ there exists a reversed sum-product pair containing a. Remarkably, this set has an enormous amount of structure. Let us say that $(2, 2)$ and $(\beta - 1, \beta - 1)$ are the **uninteresting** reversed sum-product pairs because they are present for all but the smallest bases β; any other pair is **interesting**. We prove the following structure result in Section 7:

Theorem 1.1. *Fix $a \geq 2$. The set of bases β for which there exists an interesting base-β reversed sum-product pair (a, b) is the union of a nonzero finite number of arithmetic progressions modulo $a^2 - 1$.*

In particular, for a fixed a, the set of bases β for which there exists an interesting base-β reversed sum-product pair containing a has positive density. We give more precise statements in Theorem 7.5 and Corollary 7.6, and we calculate this density for $a \leq 10$ at the end of Section 7.

Our investigation led to an intriguing phenomenon that we were unable to fully explain:

Conjecture 1.2. *The only bases for which there is no interesting reversed sum-product pair (a, b) are*

$$2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 21.$$

Using computer calculation and the tools in Section 7, we have verified that our conjecture holds for $\beta < 1,441,440$. We also prove that at least 99.3% of all bases admit an interesting reversed sum-product pair. This computation is explained in Section 8.

Finally, we note that a phenomenon closely related to reversed sum-product pairs can be found in "reverse multiples": integers whose digit reversals are multiples of themselves. For example, in base 10, the only four-digit reverse multiples are $9 \times 1089 = 9801$ and $4 \times 2178 = 8712$. Young found a construction of these numbers using special rooted trees in [6, 7], and Sloane reworked these trees into a DFA construction similar to ours [5], although the author refers to them as "Young graphs". See [2] for yet another variation on this theme: integers n for which some nontrivial multiple permutes the digits of n.

Conventions. Throughout this article, we assume that $a \leq b$. If an integer n has base-β expansion

$$n = n_r\beta^r + n_{r-1}\beta^{r-1} + \cdots + n_1\beta + n_0,$$

we will say that n_r is the "first digit" of n and n_0 is the "last digit". If $a \leq b$ is a reversed sum-product pair for the base β, then neither one is divisible by β; indeed, the product would have trailing zeros.

2. Small b

Suppose that $a \leq b < \beta$ is a reversed sum-product pair for the base β. Then, we will show that exactly one of the following is true:

- $(a, b) = (2, 2)$ and $\beta \geq 5$; or
- $(a, b) = (\beta - 1, \beta - 1)$ and $\beta \geq 3$.

Suppose first that $a + b < \beta$. Then, ab and $a + b$ have a single digit in base β, so $ab = a + b$. Rearranging shows

$$(a-1)(b-1) = 1 \quad \implies \quad a = b = 2.$$

Our hypothesis that $a + b < \beta$ now becomes $\beta \geq 5$.

Now suppose that $a + b > \beta$. Since $a + b < 2\beta$, we can write

$$a + b = \beta + x, \quad 1 \leq x < \beta. \tag{1}$$

Since ab is the reverse of $a + b$, we have

$$ab = x\beta + 1. \tag{2}$$

Solving (1) for x and inserting into (2) gives

$$ab = 1 + a\beta + b\beta - \beta^2.$$

Rearranging yields

$$(\beta - a)(\beta - b) = 1.$$

Since $1 \leq (\beta - a), (\beta - b) < \beta$, we must have $a = b = \beta - 1$. Note that this means $x = \beta - 2$, which is a valid first digit only when $\beta \geq 3$.

3. Large a

If a is large, then we expect ab to have more digits than $a + b$. The next lemma uses this idea to produce a coarse upper bound for a.

Lemma 3.1. *Suppose that $a \leq b$ is a reversed sum-product pair with $a > \beta$. Then, $a < 2\beta$ and $b < \beta(\beta + 1)$.*

Proof. Note that $\beta(a+b)$ has one more digit than $a+b$ in base β. That $a+b$ and ab have the same number of digits implies that $ab < \beta(a + b)$. Solving for b gives

$$b < \frac{a\beta}{a - \beta}.$$

The right side is a decreasing function of a, so it is maximized when $a = \beta + 1$, which gives the inequality $b < \beta(\beta + 1)$. Since $a \leq b < \frac{a\beta}{a-\beta}$, we can solve for a to get $a < 2\beta$. □

We now refine the bound in the lemma and conclude there is no reversed sum-product pair with large a.

Proposition 3.2. *Suppose that $a \leq b$ is a reversed sum-product pair for the base β. Then, $a < \beta$.*

Proof. Suppose for the sake of a contradiction that $a > \beta$. By Lemma 3.1, we have $a < 2\beta$ and $b < \beta(\beta + 1)$. For $\beta \leq 5$, we can examine all $a \in (\beta, 2\beta)$ and $b \in [a, \beta(\beta + 1))$ and find there is no reversed sum-product pair with these constraints.

For the remainder of the proof, we may assume that

$$\beta < a < 2\beta, \quad a \leq b < \beta(\beta + 1), \quad \beta \geq 6.$$

With these assumptions, we find that $a + b < 3\beta + \beta^2$, so that $a + b$ has two or three digits. We write

$$a = \beta + a_0 \qquad \text{and} \qquad b = b_2\beta^2 + b_1\beta + b_0, \tag{3}$$

where $0 \leq a_i, b_i < \beta$ and $a_0 b_0 \neq 0$. Note that if $b_2 \neq 0$, then ab has four digits. So, we may further assume that $b_2 = 0$. Because $a \leq b$, it follows that $b_1 \geq 1$.

Case $a + b$ has two digits. From (3), we have

$$a + b = (1 + b_1)\beta + (a_0 + b_0),$$

so the first digit of $a + b$ is at least 2. Thus,

$$\frac{\beta}{2}(a + b) \geq \beta^2 > ab,$$

since ab must also have two digits. Solving for a gives

$$a < \frac{b\beta}{2b - \beta}.$$

For $b > \beta$, the right side is a decreasing function, so it is maximized by taking $b = \beta + 1$. We then have

$$a < \frac{b\beta}{2b - \beta} \leq \beta\frac{\beta + 1}{\beta + 2} < \beta,$$

a contradiction.

Case $a + b$ has three digits. From (3), we have

$$a + b = (1 + b_1)\beta + (a_0 + b_0).$$

As we are assuming $a + b$ has three digits, we have two subcases to consider:

(I) $1 + b_1 \geq \beta$, or

(II) $1 + b_1 = \beta - 1$ and $a_0 + b_0 \geq \beta$.

In (I) and (II), (3) shows that the product ab satisfies

$$ab = b_1\beta^2 + (a_0 b_1 + b_0)\beta + a_0 b_0. \tag{4}$$

In case (I), we must have $b_1 = \beta - 1$. Using $a_0, b_0 \geq 1$, we obtain the following estimate from (4):

$$ab \geq \beta^3 + 1.$$

But then ab has at least four digits, a contradiction.

In case (II), we look at the coefficient on β in (4):

$$a_0 b_1 + b_0 = a_0(\beta - 2) + b_0 \geq a_0(\beta - 2) + \beta - a_0 = a_0(\beta - 3) + \beta.$$

This is an increasing function of a_0. If $a_0 \geq 2$, then this quantity is at least 2β since $\beta \geq 6$. As in case (I), we obtain the absurd conclusion that ab has four digits. So, $a_0 = 1$ and $b_0 = \beta - 1$. This completely determines a and b:

$$a = \beta + 1 \qquad \text{and} \qquad b = (\beta - 2)\beta + (\beta - 1).$$

As $a + b = \beta^2$, we do not obtain a reversed sum-product pair. This completes the proof. $\square$

4. Carries

From grade school arithmetic, we know about "carrying" when computing multidigit sums and products. Our primary goal for this section is to show that there is no carry when computing the sum $a+b$ for a reversed sum-product pair (a,b). (Typically there are carries in the product.) We begin with a careful definition of carry digits and a bound on how big they can be.

Suppose that a,d are two single-digit numbers, which means $0 \le a,d < \beta$. Their sum or product is at most two digits. If it is two digits, we call the leading digit the **carry digit**. Upon adding single-digit numbers, the resulting carry digit is at most 1: $a+d<2\beta$. If $a \ge 1$, then multiplying these two numbers produces a carry digit that is strictly less than a:

$$ad \le a(\beta-1) = (a-1)\beta + (\beta-a).$$

Now imagine that we are multiplying a single-digit number a by a multidigit number b. A particular digit of the product ab comes from multiplying a by a single digit of b and adding the previous carry digit. The new carry digit is still at most $a-1$:

$$ad + \text{previous carry} \le a(\beta-1) + (a-1) = (a-1)\beta + (\beta-1).$$

Next, we show that the first digit of $a+b$ does not arise from a carry.

Lemma 4.1. *If (a,b) is a reversed sum-product pair for the base β with $a<\beta<b$, then b and $a+b$ have the same number of digits.*

Proof. Assume the number of digits differs. Then, $a+b$ has one more digit than b, and $b=\beta^\ell - c$ for some $\ell \ge 2$ and c with $a > c \ge 1$. Then,

$$a+b = \beta^\ell + (a-c) \qquad \text{and} \qquad ab = a\beta^\ell - ac.$$

The above expression for $a+b$ has first digit 1, last digit $a-c$, and all other digits 0. Therefore, the reverse is true for ab:

$$ab = (a-c)\beta^\ell + 1.$$

Combining these two expressions for ab and rearranging, we get $c(\beta^\ell - a) = 1$. Since $\beta^\ell \ge \beta^2 > a+1$, this is a contradiction. □

Now, we improve the preceding lemma to show that the computation of $a+b$ involves no carry at all.

Proposition 4.2. *If (a,b) is a reversed sum-product pair for the base β with $a<\beta<b$, the last digit of b is strictly smaller than $\beta-a$.*

Proof. Write b_r for the first digit of b, and let d be the first digit of ab. Since ab has the same number of digits as $a+b$, which has the same number of digits as b (Lemma 4.1), we see that $d = ab_r + \lambda < \beta$, where $\lambda \le a-1$ is the carry from the $(r-1)$st place of the product ab. It follows that $d \ge ab_r \ge a$.

Write b_0 for the last digit of b. Since $a \le b$ is a reversed sum-product pair, the last digit of $a+b$ must be $d \equiv a + b_0 \pmod{\beta}$. That is, $b_0 \equiv d - a \pmod{\beta}$. Since $d \ge a$, we conclude that $b_0 = d-a < \beta - a$. □

In particular, the above proposition shows that $a+b$ and b have the same first digit, which we will capitalize on in the next section.

5. Small a, Large b

Suppose that (a, b) is a reversed sum-product pair for the base β, and that $a < \beta < b$. Write the base-β expansion of b as

$$b = b_r\beta^r + \cdots + b_0.$$

We begin by determining necessary—although not sufficient—conditions on b_0, b_r. From there, we will inductively determine two more digits (which may be the same digit if b has an odd number of digits), and so on. With careful bookkeeping, this procedure will result in only finitely many states, and we will be able to develop an algorithm for finding all valid reversed sum-product pairs.

5.1. **The Recursion.** To recap, we have now determined that if (a, b) is a reversed sum-product pair for the base β with $a < \beta < b$, then

- b and $a + b$ and ab have the same number of digits;
- b and $a + b$ have the same first digit; and
- the last digit of b is strictly smaller than $\beta - a$.

Recall that we write b_r, b_0 for the first and last digits of b, respectively. Then, we have $0 < b_0 < \beta - a$, and the last digit of $a + b$ is $a + b_0$. So, the first digit of ab is $ab_r + \lambda = a + b_0$ for some $0 \le \lambda < a$ corresponding to the carry from the $(r - 1)$st place of the product. (Here, λ stands for "left" carry.) The first digit of $a + b$ agrees with b_r. But, this is also equal to the last digit of ab, so we have $b_r \equiv ab_0 \pmod{\beta}$. Writing ρ for the carry from the units place—the "right" carry—we obtain the following constraints on the first and last digits of b:

$$\begin{aligned} a + b_0 &= ab_r + \lambda \text{ for some } 0 \le \lambda < a, \\ ab_0 &= b_r + \rho\beta \text{ for some } 0 \le \rho < a. \end{aligned} \tag{5}$$

Now, suppose that we have determined the first and last n digits of b for some $n > 0$. Write $b = \cdots xx' \cdots y'y \cdots$, where x, y have already been determined and we would like to find x' and y'. Assume further that we already know the carry into the x-column of the product ab; let us call it λ. (This will be part of our inductive information.) The product of a with the rightmost n digits of b determines a carry out of the y-column of ab; call it ρ, so that the product will have $ay' + \rho \pmod{\beta}$ in the next place to the left. See Figure 1(i).

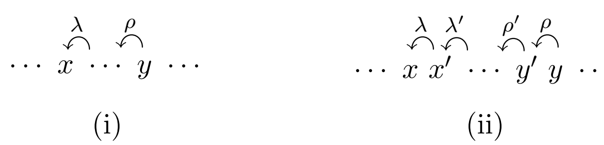

Figure 1. An illustration of the carries involved in the beginning and end of the recursion step of Section 5.1.

Since (a, b) is a reversed sum-product pair, and since the corresponding digit of $a + b$ is x', we find that

$$ay' + \rho = x' + \rho'\beta \text{ for some } 0 \le \rho' < a.$$

To determine the digit of the product ab arising from multiplication by x', we need to consider the unknown carry from the middle digits; call it λ'. See Figure 1(ii). The result is $ax' + \lambda'$, which must agree modulo β with the corresponding digit of $a + b$, namely y'. That is,

$$ax' + \lambda' = y' + \lambda\beta \text{ for some } 0 \le \lambda' < a.$$

We combine the recursion equations for future reference:

$$\begin{aligned} ax' + \lambda' &= y' + \lambda\beta \text{ for some } 0 \leq \lambda' < a, \\ ay' + \rho &= x' + \rho'\beta \text{ for some } 0 \leq \rho' < a. \end{aligned} \tag{6}$$

There are two ways for this construction to terminate: when the left and right sides may be concatenated, or when the left and right sides overlap in a digit. Suppose we know the first and last n digits of b for some $n \geq 1$. Write $b = \cdots x \cdots y \cdots$, where x, y have already been determined, and suppose that we know the carry λ into the x-column of the product ab and the carry ρ out of the y-column in the product. The left and right sides may be concatenated if the carries are compatible (i.e., if $\lambda = \rho$). In that case, a and $b = \cdots xy \cdots$ are a reversed sum-product pair.

To understand when the left and right sides may overlap in a digit, we must run the recursion one more step. With the setup of the previous paragraph, we solve the recursion equations (6) to obtain x', y', λ', ρ'. If $x' = y'$, then we claim that $\lambda' = \rho$ and $\lambda = \rho'$. To see it, set $y' = x'$ in (6) and subtract the two equations. We obtain

$$\lambda' - \rho = (\lambda - \rho')\beta.$$

Since $|\lambda' - \rho| < a < \beta$, we must have $\lambda' = \rho$ and $\lambda = \rho'$, as desired. It follows that the carries match up so that a and $b = \cdots xx'y \cdots$ are a reversed sum-product pair.

To summarize, we have shown that the above procedure can terminate in two ways:

- if $\lambda = \rho$ at any step, or
- if $x' = y'$ in the recursion step.

The recursive procedure begins by fixing a base β and an element $a < \beta$. We now give a useful criterion for detecting some values of a that will never participate in a reversed sum-product pair for the base β.

Proposition 5.1. *Fix $\beta \geq 2$. If (a, b) is a reversed sum-product pair for the base β, then*

$$\gcd(\beta - 1, a - 1) = \gcd(\beta - 1, b - 1) = 1.$$

Proof. Suppose the expansion of $a + b$ in base β is

$$a + b = n_r\beta^r + n_{r-1}\beta^{r-1} + \cdots + n_1\beta + n_0.$$

If (a, b) is a reversed sum-product pair, then the expansion of ab must be

$$ab = n_0\beta^r + n_1\beta^{r-1} + \cdots + n_{r-1}\beta + n_r.$$

Reducing modulo $\beta - 1$ gives

$$a + b \equiv n_r + n_{r-1} + \cdots + n_1 + n_0 \equiv ab \pmod{\beta - 1}.$$

Massaging this congruence, we find that

$$(a - 1)(b - 1) \equiv 1 \pmod{\beta - 1}.$$ □

We close this section by completing the promised description of all reversed sum-product pairs for the base 10:

Theorem 5.2. *If $a \leq b$ is a reversed sum-product pair for the base 10, then (a, b) is among*

$$(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), (2, 49997), \ldots$$

Proof. Section 2 shows that $(2,2)$ and $(9,9)$ are the only instances with $b < 10$. Section 3 shows that any remaining pair must have $a < 10 < b$.

For each $a < 10$, we begin by looking for all four-tuples $(b_r, b_0, \lambda, \rho)$ satisfying (5) with $0 < b_0 < 10 - a$ and $0 < b_r < 10$. As each of the variables b_0, b_r, ρ, λ is bounded, there are only finitely many solutions; we can obtain them by hand or computer calculation. Note that we do not have to try $a = 1, 4$, or 7, by Proposition 5.1. The result is given in Table 1.

a	b_r	b_0	λ	ρ
2	4	7	1	1
3	2	4	1	1

TABLE 1. The solutions to (5) for $\beta = 10$ with $0 < b_r < 10$ and $0 < b_0 < 10 - a$.

Let us look at $a = 3$ first. Any reversed sum-product pair (a, b) must have $b = 2 \cdots 4$ for some unknown (and possibly nonexistent) digits between the 2 and the 4. The equations (6) defining the recursion step have no solution, as one can check with a short calculation. Since $\lambda = \rho = 1$ in the first step, we obtain $b = 24$ as the only solution with $a = 3$.

Next, we look at $a = 2$. Any reversed sum-product pair (a, b) must have $b = 4 \cdots 7$. Since $\lambda = \rho = 1$, we obtain a first solution $b = 47$. The recursion equations (6) have a single solution: $(x', y', \lambda', \rho') = (9, 9, 1, 1)$. Since $x' = y' = 9$, we obtain the value $b = 497$. Since $\lambda' = \rho'$, we also obtain the value $b = 4997$. Finally, note that the recursion equations depend only on λ, ρ; it follows that $(9, 9, 1, 1)$ is the unique solution obtained by running the recursion again. We obtain the solutions 49997 and 499997 from the next step of the recursion, and so on. □

6. DETERMINISTIC FINITE AUTOMATA

For a given base β and value $a < \beta$, the recursive procedure in Section 5.1 for constructing digits of the second member of a reversed sum-product pair (a, b) is best visualized using a DFA. Informally, a DFA is a directed graph with one vertex designated as the "initial state", one or more vertices that are "accepting states", and edge labels from some "alphabet". Starting at the initial state of a DFA, we can walk through the graph while writing down the edge labels we pass. If we stop at an accepting state, then the string of labels we have written is an "accepted string" for the DFA. See [3, Section 2.2] for the formal definition of a DFA and many more details. For additional connections between automata and number theory, we recommend [1] and [4].

Fix integers $1 \leq a < \beta$. We now describe a method for associating a DFA to this data. The allowable edge labels (i.e., the "alphabet" in DFA theory) are pairs (x, y) with $0 \leq x, y < \beta$ as well as singletons (x) for $0 \leq x < \beta$. We construct three types of states:

- An initial state s_i;
- An "odd state" s_o; and
- A "carry state" $s_{\lambda,\rho}$ for each pair of integers $0 \leq \lambda, \rho < a$.

The accepting states are $\{s_o\} \cup \{s_{\lambda,\lambda} \ : \ 0 \leq \lambda < a\}$. The transitions for our DFA are as follows:

- For each solution $(b_r, b_0, \lambda, \rho)$ to (5), we have a transition from the initial state s_i to the state $s_{\lambda,\rho}$ with label (b_r, b_0).
- For each carry state $s_{\lambda,\rho}$ and each solution $(x', y', \lambda', \rho')$ to the recursion equations (6), we have a transition from $s_{\lambda,\rho}$ to $s_{\lambda',\rho'}$ with label (x', y'). If $x' = y'$, we also include a transition from $s_{\lambda,\rho}$ to the odd state s_o with label (x').

- If $\beta \geq 5$ and $a = 2$, include a transition from s_i to s_o with label (2).
- If $\beta \geq 3$ and $a = \beta - 1$, include a transition from s_i to s_o with label $(\beta - 1)$.

Write $A_{\beta,a}$ for the DFA thus constructed.

The definition of our DFA captures the digit-construction process involved in the recursion in Section 5. We formalize this in the following statement:

Theorem 6.1. *For $\beta \geq 2$ and $a < \beta$, let $A = A_{\beta,a}$ be the DFA constructed above.*

- *Suppose that the string $(x_1, y_1)(x_2, y_2)\cdots(x_n, y_n)$ is accepted by A for some $n \geq 1$. Then, a and $b = x_1x_2\cdots x_ny_n\cdots y_2y_1$ are a reversed sum-product pair for the base β.*
- *Suppose that the string $(x_1, y_1)(x_2, y_2)\cdots(x_n, y_n)(z)$ is accepted by A for some $n \geq 0$. Then, a and $b = x_1x_2\cdots x_nzy_n\cdots y_2y_1$ are a reversed sum-product pair for the base β.*

If $a \leq b$ is a reversed sum-product pair for the base β, then b can be constructed from a string accepted by A in one of these two ways.

A priori, the number of states in $A_{\beta,a}$ is $a^2 + 2$. In practice, many of these states are unreachable by a path beginning at the initial state. To avoid these superfluous states, we will always "lazy construct" the DFA: beginning with solutions to (5), only construct states as needed to satisfy the recursion. Taking this approach does not affect the set of strings accepted by the DFA. For example, if (5) has no solution, then only the initial state needs to be constructed. See Figure 2 for the result of this construction in base 10.

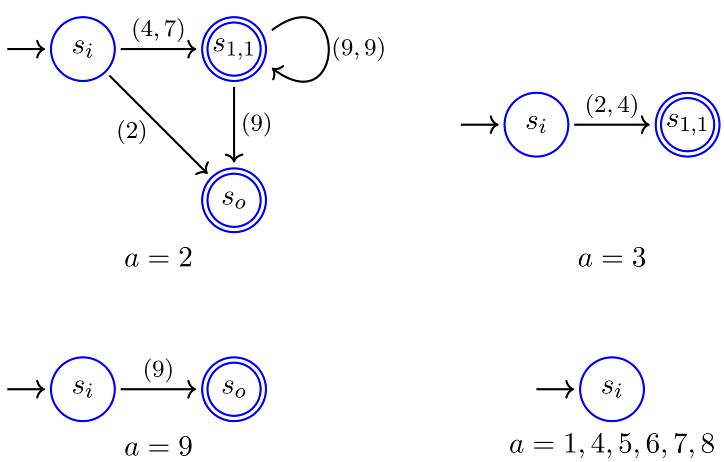

FIGURE 2. "Lazy constructed" DFAs $A_{10,a}$ for the base 10. It is customary to draw the initial state with a sourceless inward edge, and the accepting states are drawn with double circles.

Example 6.2. We look more carefully at the DFA $A_{10,2}$ in Figure 2 to see how it gives rise to reversed sum-product pairs. Integers b that make a base-10 reversed sum-product pair with $a = 2$ correspond to accepted strings in the DFA. The initial state is s_i. The accepting states are $s_{1,1}$ and s_o, drawn with double circles. As we walk through the DFA along directed edges, the edge labels describe how to build b; not from left-to-right, but from out-to-in. Consider the sequence of states $s_i \to s_{1,1} \to s_{1,1}$. The state $s_{1,1}$ is accepting, so we are allowed to stop there. The associated sequence of edge labels is $(4, 7), (9, 9)$. The first term tells us that $b = 4\cdots 7$;

the second term gives $b = 4997$. Similarly, the sequence of states $s_i \to s_{1,1} \to s_{1,1} \to s_o$ gives rise to $b = 49997$.

Example 6.3. In the introduction, we indicated that the complete list of reversed sum-product pairs for the base 18 is

$$(2,2),(H,H),(3,37),(4,25),$$
$$(7,2483D8),(7,2483D9E483D8),(7,2483D9E483D9E483D8),\ldots$$
$$(B,1961DC5),(B,1961DBG461DC5),(B,1961DBG461DBG461DC5),\ldots$$

One can prove this mechanically using the recursive procedure in Section 5.1. But, the patterns in the two infinite families become less mysterious when we examine the associated DFAs. For example, Figure 3 illustrates the DFA for $a = 7$.

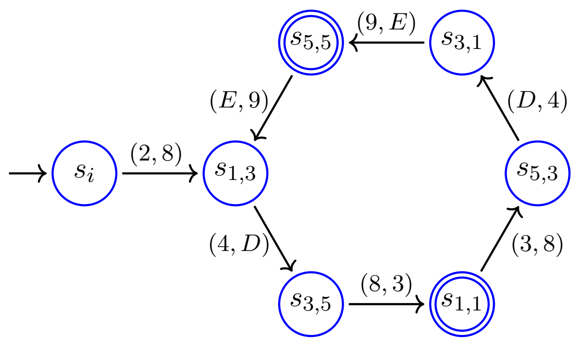

FIGURE 3. A deterministic finite automaton for the base 18 with $a = 7$.

Going around the cycle in the DFA gives an infinite family of values b such that $(7, b)$ is a reversed sum-product pair for the base 18, as detailed in the list above.

Remark 6.4. Now fix a base β and a positive integer $a < \beta$. After constructing the DFA $A_{\beta,a}$, we may find that there are states s that do not participate in any accepted string. We can "trim" all such s from the DFA. As an extreme example, the DFA $A_{150,31}$ has 13 states accessible from the initial state, but it has no accepting state. Consequently, we can trim all states but the initial one. Said another way, there is no base-150 reversed sum-product pair of the form $(31, b)$.

7. An Opposite Problem

Suppose that (a, b) is a base-β reversed sum-product pair with $b > \beta$; this hypothesis rules out the uninteresting pairs $(2, 2)$ and $(\beta - 1, \beta - 1)$. We will say that a **participates in a reversed sum-product pair** for the base β, or, for brevity, that a **participates** for β. So, $a = 2$ and $a = 3$ participate for the base 10, but $a = 4$ and $a = 9$ do not.

Example 7.1. For a given $a \geq 2$, we claim there are infinitely many bases β for which a participates. To see it, let T be a variable. Define

$$b = (T + 1)\beta + (aT + 1) \qquad \text{and} \qquad \beta = (a^2 - 1)T + a - 1.$$

Then, (a, b) is a formal reversed sum-product pair for the base β in the sense that

$$a + b = (T + 1)\beta + (aT + a + 1) \qquad \text{and} \qquad ab = (aT + a + 1)\beta + (T + 1).$$

By setting T equal to a positive integer, we get a reversed sum-product pair in the usual sense unless $a = 2$ and $T = 1$ or 2.

In the above example, we produced a set of bases β for which a participates; note that they all lie in the same congruence class modulo a^2-1. This is a general phenomenon: if a participates for one base in a congruence class, then it participates for all larger bases in the same class.

Lemma 7.2. *Let $a \geq 2$ be an integer, and suppose that a participates for the base $\beta > a$. Then, a also participates for the base $\hat{\beta} = \beta + a^2 - 1$.*

Proof. Let $b = b_r\beta^r + \cdots + b_0$ be such that (a, b) is a reversed sum-product pair for the base β. We will take the corresponding solutions to the recursion equations (5) and (6) and construct new solutions for the base $\hat{\beta}$.

The quadruple $(b_r, b_0, \lambda, \rho)$ satisfies the equations (5). Define

$$\hat{b}_r = b_r + \rho, \qquad \hat{b}_0 = b_0 + a\rho.$$

Since $\rho < a$, we see that $\hat{b}_r < \hat{\beta}$ and $\hat{b}_0 < \hat{\beta} - a$. One verifies immediately that the quadruple $(\hat{b}_r, \hat{b}_0, \lambda, \rho)$ satisfies the equations (5) with $\hat{\beta}$ in place of β. The carries λ, ρ did not change.

Now suppose that the quadruple $(x', y', \lambda', \rho')$ satisfies (6). Define

$$\hat{x}' = x' + \lambda a + \rho', \qquad \hat{y}' = y' + \rho' a + \lambda.$$

Again, we see that $\hat{x}', \hat{y}' < \hat{\beta}$ and the quadruple $(\hat{x}', \hat{y}', \lambda', \rho')$ satisfies (6) with $\hat{\beta}$ in place of β.

To complete the proof, we will show that the termination conditions agree; that is, we get an integer $\hat{b}$ with the same number of digits as b, and such that $(a, \hat{b})$ is a reversed sum-product pair for the base $\hat{\beta}$. If the recursion for b terminates because $\lambda = \rho$, then clearly the same is true for $\hat{b}$ because we used all of the same carries. If instead the recursion terminates because $x' = y'$ at some stage, then we claim that $\hat{x}' = \hat{y}'$. Indeed, we saw in Section 5.1 that $x' = y'$ implies that $\lambda = \rho'$ and $\lambda' = \rho$. It follows that

$$\hat{x}' = x' + \lambda a + \rho' = y' + \rho' a + \lambda = \hat{y}'.$$

It follows that the recursion terminates for $\hat{b}$, as desired. □

Example 7.3. The proof of Lemma 7.2 is constructive. For example, the pair $(3, 24)$ is a reversed sum-product pair for the base $\beta = 10$. Looking back at the proof of Theorem 5.2, we see that this pair arose from the four-tuple $(b_r, b_0, \lambda, \rho) = (2, 4, 1, 1)$. Set $\hat{\beta} = \beta + a^2 - 1 = 18$. The proof of the lemma gives the new four-tuple $(\hat{b}_r, \hat{b}_0, \hat{\lambda}, \hat{\rho}) = (3, 7, 1, 1)$, from which we obtain the new reversed sum-product pair $(3, 37)$ for the base 18.

There is at least one congruence class that contains no base for which a participates.

Lemma 7.4. *Suppose there exists a reversed sum-product pair (a, b) for the base β with $b > \beta$. Then $\beta \not\equiv 0 \pmod{a^2-1}$.*

Proof. We can write $b = b_r\beta^r + \cdots + b_0$, with $r \geq 1$. Consequently, b_r and b_0 must satisfy the equations (5) for some choice of λ, ρ. Eliminating b_0 from (5) shows that

$$(a^2 - 1)b_r = a^2 - a\lambda + \rho\beta.$$

If $\beta \equiv 0 \pmod{a^2-1}$, then reducing this equation modulo a^2-1 yields $a^2 \equiv a\lambda \pmod{a^2-1}$. As a is coprime to a^2-1, we conclude that $\lambda \equiv a \pmod{a^2-1}$. Since $a < a^2-1$, we must have $\lambda = a$, a contradiction. □

For a positive integer q and an integer $0 \leq r < q$, let us agree to write $q\mathbb{N} + r$ for the arithmetic progression $\{qn + r \ : \ n = 0, 1, 2, \ldots\}$.

Theorem 7.5. *For $a \geq 2$, there exists a nonempty set of arithmetic progressions*

$$(a^2 - 1)\mathbb{N} + v_1, \ (a^2 - 1)\mathbb{N} + v_2, \ \ldots, \ (a^2 - 1)\mathbb{N} + v_\ell$$

such that

- $0 < v_i < a^2 - 1$ *for all* i*;*
- *If* a *participates for* β*, then* $\beta \in (a^2 - 1)\mathbb{N} + v_i$ *for some* i*; and*
- *For each* i *and each sufficiently large* $\beta \in (a^2 - 1)\mathbb{N} + v_i$*,* a *participates for* β*.*

Proof. Fix $a \geq 2$ and let B be the set of all $\beta \geq 2$ for which a participates. Define $v_1, \ldots, v_\ell$ to be the set of integers in the interval $[0, a^2 - 1)$ that are congruent to some element of B. Example 7.1 shows that the set $\{v_1, \ldots, v_\ell\}$ is nonempty. Lemma 7.4 shows that no $v_i = 0$. The final statement is immediate from Lemma 7.2. □

We can now address the question, "How big is the set of bases for which a given a participates in a reversed sum-product pair?" To that end, define the limit

$$\Omega(a) = \lim_{N \to \infty} \frac{1}{N} \left| \{2 \leq \beta \leq N \ : \ a \text{ participates for } \beta\} \right| .$$

Corollary 7.6. *For each $a \geq 2$, the limit defining $\Omega(a)$ exists and is of the form $\ell/(a^2 - 1)$ for some integer $1 \leq \ell < a^2 - 1$ that depends on a. In particular, $0 < \Omega(a) < 1$.*

Proof. Let $v_1, \ldots, v_\ell$ be as in the theorem. Then, $1 \leq \ell < a^2 - 1$. For N sufficiently large, we have

$$\begin{aligned} &|\{2 \leq \beta \leq N \ : \ a \text{ participates for } \beta\}| \\ &\qquad = \left| \bigcup_{i=1}^{\ell} \left\{ (a^2 - 1)n + v_i \ : \ 1 \leq n \leq \frac{N}{a^2 - 1} \right\} \right| + O(1) \\ &\qquad = \sum_{i=1}^{\ell} \frac{N}{a^2 - 1} + O(1) = \frac{\ell N}{a^2 - 1} + O(1). \end{aligned}$$

Dividing by N and passing to the limit gives the result. □

Example 7.7. We claim that $\Omega(2) = \frac{1}{3}$. The theorem shows that all bases β for which $a = 2$ participates lie in the arithmetic progressions $3\mathbb{Z} + 1$ or $3\mathbb{Z} + 2$. We will now argue that if $\beta \in 3\mathbb{Z} + 2$, then there is no reversed sum-product pair for the base β other than the uninteresting pair $(2, 2)$. Write $\beta = 3n + 2$. Solving the recursion equations (5) for b_r shows that $3b_r = 4 - 2\lambda + \rho\beta$. Reducing modulo 3 and simplifying gives $\rho \equiv \lambda + 1 \pmod 3$. The only solution to this congruence with $0 \leq \lambda, \rho < 2$ is $\lambda = 0$ and $\rho = 1$. As $\lambda \neq \rho$, we must continue the recursion. A similar analysis applied to (6) shows that $\rho' = 0$ and $\lambda' = 1$. But, then the equations (6) become

$$\begin{aligned} 2x' + 1 &= y', \\ 2y' + 1 &= x', \end{aligned}$$

which has $x' = y' = -1$ as their unique solution. These are not valid digits in any base. This contradiction shows that the only reversed sum-product pair for a base $\beta \in 3\mathbb{Z} + 2$ is $(2, 2)$.

A similar strategy to the one in Example 7.7 allows us to compute the ratio $\Omega(a)$ for any a. Table 2 gives the first few values. (Given a, the procedure for determining which arithmetic progressions actually contain bases β for which a participates is implemented in the function `construct_generic_automata` in our Python module.)

a	$\Omega(a)$	a	$\Omega(a)$
2	$1/3 \approx 0.333$	7	$4/48 \approx 0.083$
3	$1/8 = 0.125$	8	$22/63 \approx 0.349$
4	$4/15 \approx 0.267$	9	$12/80 = 0.15$
5	$3/24 = 0.125$	10	$26/99 \approx 0.263$
6	$13/35 \approx 0.371$		

Table 2. The value of the ratio $\Omega(a)$ for $a \leq 10$.

8. Existence of Interesting Pairs

Recall that a reversed sum-product pair (a, b) for the base β is deemed to be **interesting** if it is not one of the pairs $(2, 2)$ or $(\beta - 1, \beta - 1)$. For a given base β, do we expect to find any interesting reversed sum-product pair at all? The following propositions provide support for Conjecture 1.2 in the introduction.

Proposition 8.1. *Among all bases $\beta \geq 2$, at least 99.3% of them admit an interesting reversed sum product-pair.*

Proposition 8.2. *The only bases $\beta < 1,441,440$ for which there is no interesting reversed sum-product pair are*

$$2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 21.$$

We describe a sieving procedure that will allow us to prove both propositions simultaneously:

(1) Set $B = 1,441,440$, and consider an array of integers from 0 to $B - 1$.
(2) For each $a < 100$ such that $a^2 - 1$ divides B, do the following:
   (a) Compute $v_1, \ldots, v_\ell$ as described by Theorem 7.5.
   (b) For each arithmetic progression $(a^2 - 1)\mathbb{N} + v_i$, let $T_i \geq 0$ be minimal such that $\beta = (a^2 - 1)T_i + v_i$ admits an interesting reversed sum-product pair.
   (c) For each $t \geq T_i$ such that $\beta = (a^2 - 1)t + v_i$ lies in the interval $[0, B)$, cross β off of our array.
(3) For each $\beta \in [0, B)$ that we have not crossed off yet, run the recursion from Section 5 for each value $2 \leq a < \beta$. If we find an interesting reversed sum-product pair, cross β off of our array.

Step (2a) can be accomplished by a "generic" version of the construction in Section 5. Set $\beta = (a^2 - 1)T + v$ for some fixed $0 < v < a^2 - 1$. We solve the recursion equations (5) for integers $0 \leq \lambda, \rho < a$ and with b_0, b_r being linear polynomials in T. Then, we solve (6) for $0 \leq \lambda', \rho' < a$ and with x', y' being linear polynomials in T. This is implemented in our Python module in the function `construct_generic_automata` (using the language of DFAs).

To determine T_i as in Step (2b), we run the recursion from Section 5 with $\beta = (a^2 - 1)t + v_i$ for each $t = 0, 1, 2, \ldots$ until we find a base for which a participates. By Theorem 7.5, we eventually find such a t; in practice, some $t \leq 3$ always worked. At the end of Step (2), we find that $1,431,542$ of the elements of our array have been crossed off. Since $a^2 - 1$ divides

B for each a used in the computation, Lemma 7.2 tells us that all integers β in each of these residue classes modulo B admit an interesting reversed sum-product pair. Thus, at least $\frac{1,431,542}{1,441,440} \approx 99.31\%$ of all bases admit an interesting pair, which proves Proposition 8.1. This part of the computation took approximately 5.5 minutes on a Xeon(R) E5-2699 processor (2.30GHz with 500GB memory).

The construction in Step (3) is implemented in the function `construct_automata` in our Python module. Applying it, we cross off all remaining entries in the array except for

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 21.$$

Since $\beta = 0, 1$ are not valid numerical bases, we drop these from consideration, thus proving Proposition 8.2. This step required an additional 1.5 hours of compute time.

Remark 8.3. Trying Step (2) again with the larger parameter $B = 53,542,288,800$ took 42 hours and gives the improved lower bound of $\frac{53497192379}{53542288800} \approx 99.92\%$ in Proposition 8.1. We did not extend the computation in Step (3) to this larger value of B.

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