ADDITIONAL SUMS INVOLVING GIBONACCI POLYNOMIALS: GRAPH-THEORETIC CONFIRMATIONS

THOMAS KOSHY

Abstract. Using graph-theoretic techniques, we confirm four sums involving gibonacci polynomials.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $\Delta = \sqrt{x^2+4}$.

It follows by the Binet-like formulas that $\lim\limits_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 5, 6]:

$$f_{n+k} - f_{n-k} = \begin{cases} f_n l_k, & \text{if } k \text{ is odd;} \\ f_k l_n, & \text{otherwise;} \end{cases} \tag{1}$$

$$l_{n+k} - l_{n-k} = \begin{cases} l_k l_n, & \text{if } k \text{ is odd;} \\ \Delta^2 f_k f_n, & \text{otherwise;} \end{cases} \tag{2}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise.} \end{cases} \tag{3}$$

These properties can be established using the Binet-like formulas. They play a pivotal role in our discourse.

2. Additional Gibonacci Polynomial Sums

With the above tools, we established the following infinite sums in [6], where k is a positive integer:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}. \tag{4}$$

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}. \tag{5}$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}. \tag{6}$$

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}; \tag{7}$$

Our objective is to confirm these gibonacci sums using graph-theoretic techniques. To this end, we now present the needed graph-theoretic tools.

3. Graph-theoretic Tools

Consider the *Fibonacci digraph* in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [2, 3, 4].

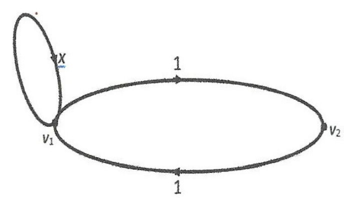

Figure 1. Weighted Fibonacci Digraph

It follows from its *weighted adjacency matrix* $M = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$M^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [2, 3, 4]. We can extend the exponent n to 0, which is consistent with the *Cassini-like formula* $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ [2].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; and *open*, otherwise. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

Let T_n^* denote the set of closed walks of length n originating at v_1 and U_n^* the set of *all* closed walks in the digraph. Correspondingly, we let T_n denote the sum of the weights of the elements in T_n^* and U_n that of those in U_n^*. Then, $T_n = f_{n+1}$ and $U_n = f_{n+1} + f_{n-1} = l_n$ [2, 3, 4].

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then, the sum of the weights of the elements (a, b) in the product set $A \times B$ is *defined* as the product of the sums of weights from each component. This definition can be extended to any finite number of such sets [2, 3, 4].

Let $C^* = \{u\}$, where u denotes the walk v_1-v_1, and $D^* = \{v\}$, where v denotes the walk v_1-v_2-v_1. Then, the weight of the element in the product set $C^* \times C^*$ is x^2 and that in $D^* \times D^*$ is 4. Consequently, the sum w of the weights of the elements in $(C^* \times C^*) \cup (D^* \times D^*)$ is given by $w = x^2 + 4 = \Delta^2$.

4. Graph-theoretic Confirmations

With the brief background above, we now present the desired graph-theoretic confirmations, beginning with the confirmation of sum (4). Throughout the discourse, k denotes a positive integer.

4.1. **Confirmation of Equation** (4).

Proof. Let k be odd and $m, n \geq (k+1)/2$.

Case 1. Suppose $g_n = f_n$. Then, $T_{2n-k-1} = f_{2n-k}$, $T_{2n+k-1} = f_{2n+k}$, and $T_{2r-2} = f_{2r-1}$, where $r \geq 1$.

Consequently, we have

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{T_{2n-k-1}} - \frac{1}{T_{2n+k-1}} \right) = \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{f_{2n-k}} - \frac{1}{f_{2n+k}} \right). \tag{8}$$

Case 2. Suppose $g_n = l_n$. With $U_{2n-k} = l_{2n-k}$ and $U_{2n+k} = l_{2n+k}$, we have

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{U_{2n-k}} - \frac{1}{U_{2n+k}} \right) = \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{l_{2n-k}} - \frac{1}{l_{2n+k}} \right). \tag{9}$$

Using the gibonacci pattern exhibited by the right-hand sides of equations (8) and (9), we conjecture that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-2r+k}}. \tag{10}$$

Using recursion [2, 5, 6], we will now establish this. To accomplish this, we let A_m denote the left-hand side (LHS) of this equation and B_m its right-hand side (RHS). Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-2(r+1)+k}} - \frac{1}{g_{2m-2r+k}} \right] \\ &= \frac{1}{g_{2m-k}} - \frac{1}{g_{2m+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this implies that

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} \\ &= \left(\frac{1}{g_1} - \frac{1}{g_{2k+1}} \right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-(2r-1)}} \right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$, as conjectured.

Because $\lim\limits_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$, it then follows that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}, \tag{11}$$

as desired. □

Thus, we have

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\left(\frac{1}{T_{2n-k-1}}-\frac{1}{T_{2n+k-1}}\right) = \sum_{r=1}^{k}\frac{1}{T_{2r-2}};$$

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\left(\frac{1}{U_{2n-k}}-\frac{1}{U_{2n+k}}\right) = \sum_{r=1}^{k}\frac{1}{U_{2r-1}}.$$

Interesting Consequences. It follows from equation (11) that

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\frac{g_{2n+k}-g_{2n-k}}{g_{2n+k}g_{2n-k}}=\sum_{r=1}^{k}\frac{1}{g_{2r-1}}.$$

With identities (1), (2), and (3), this yields [6]

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\frac{f_{2n}l_k}{f_{2n}^2+f_k^2} = \sum_{r=1}^{k}\frac{1}{f_{2r-1}}; \qquad \sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\frac{l_k l_{2n}}{l_{2n}^2-\Delta^2 f_k^2} = \sum_{r=1}^{k}\frac{1}{l_{2r-1}}.$$

Consequently, we have [6]

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{F_{2n}^2+1} = 1; \qquad \sum_{n=1}^{\infty}\frac{L_{2n}}{L_{2n}^2-5} = 1,$$

respectively.

4.2. **Confirmation of Equation** (5).

Proof. Let k be even and $m,n\geq \dfrac{k}{2}+1$. With T_n and U_n as before, we have

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{m}\left(\frac{1}{T_{2n-k-1}}-\frac{1}{T_{2n+k-1}}\right) = \sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{m}\left(\frac{1}{f_{2n-k}}-\frac{1}{f_{2n+k}}\right).$$

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{m}\left(\frac{1}{U_{2n-k}}-\frac{1}{U_{2n+k}}\right) = \sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{m}\left(\frac{1}{l_{2n-k}}-\frac{1}{l_{2n+k}}\right).$$

By invoking recursion [2, 6], we will now confirm that

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{m}\left(\frac{1}{g_{2n-k}}-\frac{1}{g_{2n+k}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}}-\sum_{r=0}^{k-1}\frac{1}{g_{2m-2r+k}}. \tag{12}$$

Letting A_m denote the LHS of this equation and B_m its RHS, we then get

$$\begin{aligned}
B_m-B_{m-1} &= \sum_{r=0}^{k-1}\left[\frac{1}{g_{2m-2(r+1)+k}}-\frac{1}{g_{2m-2r+k}}\right]\\
&= \frac{1}{g_{2m-k}}-\frac{1}{g_{2m+k}}\\
&= A_m-A_{m-1}.
\end{aligned}$$

Recursively, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{k/2+1} - B_{k/2+1} \\ &= \left(\frac{1}{g_2} - \frac{1}{g_{2k+2}}\right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r}} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-2(r-1)}}\right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$, as expected.

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, it then follows from equation (12) that

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}, \tag{13}$$

confirming equation (5). □

Consequently, we have

$$\begin{aligned} \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{T_{2n-k-1}} - \frac{1}{T_{2n+k-1}}\right) &= \sum_{r=1}^{k} \frac{1}{T_{2r-1}}; \\ \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{U_{2n-k}} - \frac{1}{U_{2n+k}}\right) &= \sum_{r=1}^{k} \frac{1}{U_{2r}}. \end{aligned}$$

Interesting Implications. It follows from equation (13) that

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{g_{2n+k} - g_{2n-k}}{g_{2n+k}\, g_{2n-k}} = \sum_{r=1}^{k} \frac{1}{g_{2r}}.$$

Using identities (1), (2), and (3), this yields

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{f_k l_{2n}}{f_{2n}^2 - f_k^2} = \sum_{r=1}^{k} \frac{1}{f_{2r}}; \qquad \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_k f_{2n}}{l_{2n}^2 + \Delta^2 f_k^2} = \sum_{r=1}^{k} \frac{1}{l_{2r}}.$$

It then follows that [6]

$$\sum_{n=2}^{\infty} \frac{l_{2n}}{f_{2n}^2 - x^2} = \frac{l_3}{f_4 f_2^2}; \qquad \sum_{n=2}^{\infty} \frac{f_{2n}}{l_{2n}^2 + \Delta^2 x^2} = \frac{f_3}{f_8},$$

respectively.

In particular, we then get [6]

$$\sum_{n=2}^{\infty} \frac{L_{2n}}{F_{2n}^2 - 1} = \frac{4}{3}; \qquad \sum_{n=2}^{\infty} \frac{F_{2n}}{L_{2n}^2 + 5} = \frac{2}{21},$$

again respectively.

4.3. **Confirmation of Equation** (6)**.**

Proof. Let k be odd and $m, n \geq (k+1)/2$. With T_n and U_n as before, we have

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{m} \left(\frac{1}{T_{2n-k}} - \frac{1}{T_{2n+k}}\right) = \sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{m} \left(\frac{1}{f_{2n+1-k}} - \frac{1}{f_{2n+1+k}}\right);$$

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{m} \left(\frac{1}{U_{2n+1-k}} - \frac{1}{U_{2n+1+k}}\right) = \sum_{\substack{n=(k+1)/2\\ k\geq,\, odd1}}^{m} \left(\frac{1}{l_{2n+1-k}} - \frac{1}{l_{2n+1+k}}\right).$$

Using the same gibonacci pattern displayed by the right-hand sides of these two equations, we conjecture that

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{m} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-(2r-1)+k}}. \tag{14}$$

Letting A_m denote the LHS of this equation and B_m its RHS, we can show that $B_m - A_m = A_m - A_{m-1}$ [6]. Recursively, this yields

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} = 0.$$

Thus, $A_m = B_m$, establishing the validity of the conjecture.

Because $\lim_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$, equation (14) yields

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}, \tag{15}$$

as desired. □

This implies

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{T_{2n-k}} - \frac{1}{T_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{T_{2r-1}};$$

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{U_{2n+1-k}} - \frac{1}{U_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{U_{2r}}.$$

Interesting Consequences. It follows from equation (15) that

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{g_{2n+1+k} - g_{2n+1-k}}{g_{2n+1+k}\, g_{2n+1-k}} = \sum_{r=1}^{k} \frac{1}{g_{2r}},$$

where k is odd.

With equations (1), (2), and (4), this yields [6]

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{2n+1} l_k}{f_{2n+1}^2 - f_k^2} = \sum_{r=1}^{k} \frac{1}{f_{2r}}; \tag{16}$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{l_{2n+1} l_k}{l_{2n+1}^2 + \Delta^2 f_k^2} = \sum_{r=1}^{k} \frac{1}{l_{2r}}. \tag{17}$$

In particular, we get [6]

$$\sum_{n=1}^{\infty} \frac{f_{2n+1}}{f_{2n+1}^2 - 1} = \frac{1}{f_2^2}; \qquad \sum_{n=1}^{\infty} \frac{l_{2n+1}}{l_{2n+1}^2 + \Delta^2} = \frac{1}{l_2 l_1},$$

respectively.

Consequently, we have [6]

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{F_{2n+1}^2 - 1} = 1; \qquad \sum_{n=1}^{\infty} \frac{L_{2n+1}}{L_{2n+1}^2 + 5} = \frac{1}{3},$$

again respectively.

Clearly, we can express equations (16) and (17) in terms of T_n and U_n:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{T_{2n} U_k}{T_{2n}^2 - T_{k-1}^2} = \sum_{r=1}^{k} \frac{1}{T_{2r-1}};$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{U_{2n+1} U_k}{U_{2n+1}^2 + wT_{k-1}^2} = \sum_{r=1}^{k} \frac{1}{U_{2r}},$$

respectively.

Using identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [2], we can rewrite equations (16) and (17) slightly differently:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{2n+1} l_k}{l_{2n+1}^2 - \Delta^2 f_k^2 + 4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{f_{2r}};$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{l_{2n+1} l_k}{\Delta^2 f_{2n+1}^2 + \Delta^2 f_k^2 - 4} = \sum_{r=1}^{k} \frac{1}{l_{2r}},$$

respectively.

In particular, they yield [6]

$$\sum_{n=1}^{\infty} \frac{f_{2n+1}}{l_{2n+1}^2 - x^2} = \frac{1}{\Delta^2 f_2^2}; \qquad \sum_{n=1}^{\infty} \frac{l_{2n+1}}{\Delta^2 f_{2n+1}^2 + x^2} = \frac{1}{l_2 l_1};$$

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_{2n+1}^2 - 1} = \frac{1}{5}; \qquad \sum_{n=1}^{\infty} \frac{L_{2n+1}}{5F_{2n+1}^2 + 1} = \frac{1}{3}.$$

Finally, we confirm equation (7) using graph-theoretic techniques.

4.4. **Confirmation of Equation** (7).

Proof. With k even, $m, n \geq \frac{k}{2}$, and T_n and U_n as before, we have

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{m} \left(\frac{1}{T_{2n-k}} - \frac{1}{T_{2n+k}}\right) = \sum_{\substack{n=k/2\\k\geq 2,\,even}}^{m} \left(\frac{1}{f_{2n+1-k}} - \frac{1}{f_{2n+1+k}}\right);$$

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{m} \left(\frac{1}{U_{2n+1-k}} - \frac{1}{U_{2n+1+k}}\right) = \sum_{\substack{n=k/2\\k\geq 2,\,even}}^{m} \left(\frac{1}{l_{2n+1-k}} - \frac{1}{l_{2n+1+k}}\right).$$

As before, using the same gibonacci pattern on the right-hand sides of these two equations, we conjecture that

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{m} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-(2r-1)+k}}. \tag{18}$$

We can establish its validity using recursion [2, 6]. In the interest of brevity, we omit the proof [6].

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, equation (18) yields

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}, \tag{19}$$

as desired. □

This implies

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{\infty} \left(\frac{1}{T_{2n-k}} - \frac{1}{T_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{T_{2r-2}};$$

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{\infty} \left(\frac{1}{U_{2n+1-k}} - \frac{1}{U_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{U_{2r-1}}.$$

Interesting Implications. It follows from equation (19) that

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{\infty} \frac{g_{2n+1+k} - g_{2n+1-k}}{g_{2n+1+k}g_{2n+1-k}} = \sum_{r=1}^{k} \frac{1}{g_{2r-1}},$$

where k is even.

Using equations (1), (2), (4), and (7), this yields [6]

$$\sum_{\substack{n=k/2\\k\geq 2,\,even}}^{\infty} \frac{f_k l_{2n+1}}{f_{2n+1}^2 + f_k^2} = \sum_{r=1}^{k} \frac{1}{f_{2r-1}}; \qquad \sum_{\substack{n=k/2\\k\geq 2,\,even}}^{\infty} \frac{\Delta^2 f_k f_{2n+1}}{l_{2n+1}^2 - \Delta^2 f_k^2} = \sum_{r=1}^{k} \frac{1}{l_{2r-1}}.$$

In particular, using the identity $f_{n+1} + f_{n-1} = l_n$ [2], we then get [6]

$$\sum_{n=1}^{\infty} \frac{l_{2n+1}}{f_{2n+1}^2 + x^2} = \frac{l_2}{f_2 f_3}; \qquad \sum_{n=1}^{\infty} \frac{f_{2n+1}}{l_{2n+1}^2 - \Delta^2 x^2} = \frac{1}{l_1 l_3}.$$

Consequently, we have [6]

$$\sum_{n=1}^{\infty} \frac{L_{2n+1}}{F_{2n+1}^2 + 1} = \frac{3}{2}; \qquad \sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_{2n+1}^2 - 5} = \frac{1}{4},$$

respectively.

Finally, using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [2], we can rewrite the four infinite sums above involving gibonacci polynomials slightly differently. In the interest of brevity, we omit them; but we encourage gibonacci enthusiasts to pursue them as well as the numeric counterparts of the resulting sums.

5. Acknowledgment

The author thanks the reviewer for a careful reading of the article, and for encouraging and supporting words.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[3] T. Koshy, *A recurrence for gibonacci cubes with graph-theoretic confirmations*, The Fibonacci Quarterly, **57.2** (2019), 139–147.
[4] T. Koshy, *Graph-theoretic confirmations of four sums of gibonacci polynomial products of order 4*, The Fibonacci Quarterly, **59.2** (2021), 167–175.
[5] T. Koshy, *Sums involving gibonacci polynomials*, The Fibonacci Quarterly, **60.4** (2022), 344–351.
[6] T. Koshy, *Additional sums involving gibonacci polynomials*, The Fibonacci Quarterly, **61.1** (2023) 12–20.

MSC2020: Primary 05C20, 11B37, 11B39, 11C08

Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu