SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES REVISITED

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Abstract. We explore four infinite sums involving a special class of gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $\Delta = \sqrt{x^2+4}$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ [4, 6, 7].

1.1. Fundamental Gibonacci Identities.

Using the Binet-like formulas, we can establish the following gibonacci identities [4, 6]:

$$g_{n+k}^2 - g_{n-k}^2 = \begin{cases} f_{2k}f_{2n}, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k}f_{2n}, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise.} \end{cases} \tag{2}$$

They play a pivotal role in our explorations.

2. Telescoping Gibonacci Sums

We will now investigate four telescoping sums in the following lemmas.

Lemma 1. *Let k and λ be positive integers. Then,*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g_{(4n-1)k}^{\lambda}} - \frac{1}{g_{(4n+3)k}^{\lambda}} \right] = \frac{1}{g_{3k}^{\lambda}}. \tag{3}$$

Proof. Using recursion [4, 6, 7], we will first establish that

$$\sum_{n=1}^{m} \left[\frac{1}{g_{(4n-1)k}^{\lambda}} - \frac{1}{g_{(4n+3)k}^{\lambda}} \right] = \frac{1}{g_{3k}^{\lambda}} - \frac{1}{g_{(4m+3)k}^{\lambda}}. \tag{4}$$

Letting A_m denote the left side (LHS) of this equation and B_m its right side (RHS), we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g^{\lambda}_{(4m-1)k}} - \frac{1}{g^{\lambda}_{(4m+3)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

establishing the validity of equation (4).

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, equation (4) yields the desired result. □

Lemma 2. *Let k and λ be positive integers. Then,*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g^{\lambda}_{(4n-3)k}} - \frac{1}{g^{\lambda}_{(4n+1)k}} \right] = \frac{1}{g^{\lambda}_{k}}. \tag{5}$$

Proof. By invoking recursion [4, 6, 7], we will first confirm that

$$\sum_{n=1}^{m} \left[\frac{1}{g^{\lambda}_{(4n-3)k}} - \frac{1}{g^{\lambda}_{(4n+1)k}} \right] = \frac{1}{g^{\lambda}_{k}} - \frac{1}{g^{\lambda}_{(4m+1)k}}. \tag{6}$$

Letting $A_m =$ LHS and $B_m =$ RHS of this equation, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g^{\lambda}_{(4m-3)k}} - \frac{1}{g^{\lambda}_{(4m+1)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

establishing the veracity of equation (6).

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, the given result now follows from equation (6), as desired. □

Lemma 3. *Let k and λ be positive integers. Then,*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g^{\lambda}_{4nk}} - \frac{1}{g^{\lambda}_{(4n+4)k}} \right] = \frac{1}{g^{\lambda}_{4k}}. \tag{7}$$

Proof. Using recursion [4, 6, 7], we will first verify that

$$\sum_{n=1}^{m} \left[\frac{1}{g^{\lambda}_{4nk}} - \frac{1}{g^{\lambda}_{(4n+4)k}} \right] = \frac{1}{g^{\lambda}_{4k}} - \frac{1}{g^{\lambda}_{(4m+4)k}}. \tag{8}$$

To this end, we let $A_m =$ LHS and $B_m =$ RHS of this equation. Then,

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g^{\lambda}_{4mk}} - \frac{1}{g^{\lambda}_{(4m+4)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

This establishes the validity of equation (8).

The given result now follows from it. □

Lemma 4. *Let k and λ be positive integers. Then,*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g^{\lambda}_{(4n-2)k}} - \frac{1}{g^{\lambda}_{(4n+2)k}} \right] = \frac{1}{g^{\lambda}_{2k}}. \tag{9}$$

Proof. Using recursion [4, 6, 7], we will first prove that

$$\sum_{n=1}^{m} \left[\frac{1}{g^{\lambda}_{(4n-2)k}} - \frac{1}{g^{\lambda}_{(4n+2)k}} \right] = \frac{1}{g^{\lambda}_{2k}} - \frac{1}{g^{\lambda}_{(4m+2)k}}. \tag{10}$$

Letting $A_m =$ LHS and $B_m =$ RHS of this equation, we then get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g^{\lambda}_{(4m-2)k}} - \frac{1}{g^{\lambda}_{(4m+2)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Using recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

This establishes equation (10).

The given result now follows from it, as desired. □

3. Gibonacci Polynomial Sums

With the above identities and lemmas at our disposal, we will now explore four infinite sums involving a special class of gibonacci polynomial squares. We will restrict our discourse to the case $\lambda = 2$. In the interest of brevity, we let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} -1, & \text{if } g_n = f_n; \\ 1, & \text{otherwise.} \end{cases}$$

The following result invokes Lemma 1.

Theorem 1. *Let k be a positive integer. Then,*

$$\sum_{n=1}^{\infty} \frac{\mu f_{4k} f_{2(4n+1)k}}{\left[g^2_{(4n+1)k} + (-1)^k \mu\nu f^2_{2k} \right]^2} = \frac{1}{g^2_{3k}}. \tag{11}$$

Proof. It follows by identities (1) and (2) that

$$g^2_{(4n+3)k} - g^2_{(4n-1)k} = \begin{cases} f_{4k} f_{2(4n+1)k}, & \text{if } g_n = f_n; \\ \Delta^2 f_{4k} f_{2(4n+1)k}, & \text{otherwise;} \end{cases}$$

$$g_{(4n+3)k} g_{(4n-1)k} - g^2_{(4n+1)k} = \begin{cases} (-1)^{k+1} f^2_{2k}, & \text{if } g_n = f_n; \\ (-1)^k \Delta^2 f^2_{2k}, & \text{otherwise.} \end{cases}$$

Suppose $g_n = f_n$. With these two identities and Lemma 1, we then get

$$\begin{aligned}
\frac{f_{4k}f_{2(4n+1)k}}{\left[f^2_{(4n+1)k} - (-1)^k f^2_{2k}\right]^2} &= \frac{f^2_{(4n+3)k} - f^2_{(4n-1)k}}{f^2_{(4n+3)k}f^2_{(4n-1)k}};\\
\sum_{n=1}^{\infty} \frac{f_{4k}f_{2(4n+1)k}}{\left[f^2_{(4n+1)k} - (-1)^k f^2_{2k}\right]^2} &= \sum_{n=1}^{\infty}\left[\frac{1}{f^2_{(4n-1)k}} - \frac{1}{f^2_{(4n+3)k}}\right]\\
&= \frac{1}{f^2_{3k}}. \qquad (12)
\end{aligned}$$

On the other hand, let $g_n = l_n$. Then, using the two above identities, Lemma 1 yields

$$\begin{aligned}
\frac{\Delta^2 f_{4k}f_{2(4n+1)k}}{\left[l^2_{(4n+1)k} + (-1)^k \Delta^2 f^2_{2k}\right]^2} &= \frac{l^2_{(4n+3)k} - l^2_{(4n-1)k}}{l^2_{(4n+3)k}l^2_{(4n-1)k}};\\
\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k}f_{2(4n+1)k}}{\left[l^2_{(4n+1)k} + (-1)^k \Delta^2 f^2_{2k}\right]^2} &= \sum_{n=1}^{\infty}\left[\frac{1}{l^2_{(4n-1)k}} - \frac{1}{l^2_{(4n+3)k}}\right]\\
&= \frac{1}{l^2_{3k}}. \qquad (13)
\end{aligned}$$

Combining equations (12) and (13), we get the desired result. □

It then follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{2(4n+1)}}{(F^2_{4n+1}+1)^2} &= \frac{1}{12}; & \sum_{n=1}^{\infty} \frac{F_{2(4n+1)}}{(L^2_{4n+1}-5)^2} &= \frac{1}{240};\\
\sum_{n=1}^{\infty} \frac{F_{4(4n+1)}}{[F^2_{2(4n+1)}-9]^2} &= \frac{1}{1,344}; & \sum_{n=1}^{\infty} \frac{F_{4(4n+1)}}{[L^2_{2(4n+1)}+45]^2} &= \frac{1}{34,020};\\
\sum_{n=1}^{\infty} \frac{F_{6(4n+1)}}{[F^2_{3(4n+1)}+64]^2} &= \frac{1}{166,464}; & \sum_{n=1}^{\infty} \frac{F_{6(4n+1)}}{[L^2_{3(4n+1)}-320]^2} &= \frac{1}{4,158,720}.
\end{aligned}$$

The next result is an application of Lemma 2.

Theorem 2. *Let k be a positive integer. Then,*

$$\sum_{n=1}^{\infty} \frac{\mu f_{4k}f_{2(4n-1)k}}{\left[g^2_{(4n-1)k} + (-1)^k \mu\nu f^2_{2k}\right]^2} = \frac{1}{g^2_k}. \qquad (14)$$

Proof. With identities (1) and (2), we get

$$\begin{aligned}
g^2_{(4n+1)k} - g^2_{(4n-3)k} &= \begin{cases} f_{4k}f_{2(4n-1)k}, & \text{if } g_n = f_n;\\ \Delta^2 f_{4k}f_{2(4n-1)k}, & \text{otherwise;}\end{cases}\\
g_{(4n+1)k}g_{(4n-3)k} - g^2_{(4n-1)k} &= \begin{cases} (-1)^{k+1} f^2_{2k}, & \text{if } g_n = f_n;\\ (-1)^k \Delta^2 f^2_{2k}, & \text{otherwise.}\end{cases}
\end{aligned}$$

Let $g_n = f_n$. Using these two identities, Lemma 1 then yields

$$\begin{aligned}
\frac{f_{4k} f_{2(4n-1)k}}{\left[f^2_{(4n-1)k} - (-1)^k f^2_{2k}\right]^2} &= \frac{f^2_{(4n+1)k} - f^2_{(4n-3)k}}{f^2_{(4n+1)k} f^2_{(4n-3)k}};\\
\sum_{n=1}^{\infty} \frac{f_{4k} f_{2(4n-1)k}}{\left[f^2_{(4n-1)k} - (-1)^k f^2_{2k}\right]^2} &= \sum_{n=1}^{\infty} \left[\frac{1}{f^2_{(4n-3)k}} - \frac{1}{f^2_{(4n+1)k}}\right]\\
&= \frac{1}{f^2_k}. \qquad (15)
\end{aligned}$$

Suppose $g_n = l_n$. Using Lemma 1 and the above identities, we then get

$$\begin{aligned}
\frac{\Delta^2 f_{4k} f_{2(4n-1)k}}{\left[l^2_{(4n-1)k} + (-1)^k \Delta^2 f^2_{2k}\right]^2} &= \frac{l^2_{(4n+1)k} - l^2_{(4n-3)k}}{l^2_{(4n+1)k} l^2_{(4n-3)k}};\\
\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(4n-1)k}}{\left[l^2_{(4n-1)k} + (-1)^k \Delta^2 f^2_{2k}\right]^2} &= \sum_{n=1}^{\infty} \left[\frac{1}{l^2_{(4n-3)k}} - \frac{1}{l^2_{(4n+1)k}}\right]\\
&= \frac{1}{l^2_k}. \qquad (16)
\end{aligned}$$

This, combined with equation (15), yields the desired result. □

It then follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{2(4n-1)}}{(F^2_{4n-1} + 1)^2} &= \frac{1}{3}; & \sum_{n=1}^{\infty} \frac{F_{2(4n-1)}}{(L^2_{4n-1} - 5)^2} &= \frac{1}{15};\\
\sum_{n=1}^{\infty} \frac{F_{4(4n-1)}}{[F^2_{2(4n-1)} - 9]^2} &= \frac{1}{21}; & \sum_{n=1}^{\infty} \frac{F_{4(4n-1)}}{[L^2_{2(4n-1)} + 45]^2} &= \frac{1}{945};\\
\sum_{n=1}^{\infty} \frac{F_{6(4n-1)}}{[F^2_{3(4n-1)} + 64]^2} &= \frac{1}{576}; & \sum_{n=1}^{\infty} \frac{F_{6(4n-1)}}{[L^2_{3(4n-1)} - 320]^2} &= \frac{1}{11,520}.
\end{aligned}$$

The next result invokes Lemma 4.

Theorem 3. *Let k be a positive integer. Then,*

$$\sum_{n=1}^{\infty} \frac{\mu f_{4k} f_{8nk}}{\left(g^2_{4nk} + \mu\nu f^2_{2k}\right)^2} = \frac{1}{g^2_{2k}}. \qquad (17)$$

Proof. Using identities (1) and (2), we get

$$\begin{aligned}
g^2_{(4n+2)k} - g^2_{(4n-2)k} &= \begin{cases} f_{4k} f_{8nk}, & \text{if } g_n = f_n;\\ \Delta^2 f_{4k} f_{8nk}, & \text{otherwise;}\end{cases}\\
g_{(4n+2)k} g_{(4n-2)k} - g^2_{4nk} &= \begin{cases} -f^2_{2k}, & \text{if } g_n = f_n;\\ \Delta^2 f^2_{2k}, & \text{otherwise.}\end{cases}
\end{aligned}$$

Suppose $g_n = f_n$. Using these two identities, Lemma 4 then yields

$$\begin{aligned}
\frac{f_{4k}f_{8nk}}{\left(f_{4nk}^2 - f_{2k}^2\right)^2} &= \frac{f_{(4n+2)k}^2 - f_{(4n-2)k}^2}{f_{(4n+2)k}^2 f_{(4n-2)k}^2};\\
\sum_{n=1}^{\infty} \frac{f_{4k}f_{8nk}}{\left(f_{4nk}^2 - f_{2k}^2\right)^2} &= \sum_{n=1}^{\infty}\left[\frac{1}{f_{(4n-2)k}^2} - \frac{1}{f_{(4n+2)k}^2}\right]\\
&= \frac{1}{f_{2k}^2}. \qquad (18)
\end{aligned}$$

On the other hand, let $g_n = l_n$. With the above two identities, Lemma 4 then yields

$$\begin{aligned}
\frac{\Delta^2 f_{4k}f_{8nk}}{\left(l_{4nk}^2 + \Delta^2 f_{2k}^2\right)^2} &= \frac{l_{(4n+2)k}^2 - l_{(4n-2)k}^2}{l_{(4n+2)k}^2 l_{(4n-2)k}^2};\\
\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k}f_{8nk}}{\left(l_{4nk}^2 + \Delta^2 f_{2k}^2\right)^2} &= \sum_{n=1}^{\infty}\left[\frac{1}{l_{(4n-2)k}^2} - \frac{1}{l_{(4n+2)k}^2}\right]\\
&= \frac{1}{l_{2k}^2}. \qquad (19)
\end{aligned}$$

The given result now follows by equations (18) and (19). □

In particular, we then have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{8n}}{(F_{4n}^2 - 1)^2} &= \frac{1}{3}; & \sum_{n=1}^{\infty} \frac{F_{8n}}{(L_{4n}^2 + 5)^2} &= \frac{1}{135};\\
\sum_{n=1}^{\infty} \frac{F_{16n}}{(F_{8n}^2 - 9)^2} &= \frac{1}{189}; & \sum_{n=1}^{\infty} \frac{F_{16n}}{(L_{8n}^2 + 45)^2} &= \frac{1}{5,145};\\
\sum_{n=1}^{\infty} \frac{F_{24n}}{(F_{12n}^2 - 64)^2} &= \frac{1}{9,216}; & \sum_{n=1}^{\infty} \frac{F_{24n}}{(L_{12n}^2 + 320)^2} &= \frac{1}{233,280}.
\end{aligned}$$

The following result showcases an application of Lemma 3.

Theorem 4. *Let k be a positive integer. Then,*

$$\sum_{n=1}^{\infty} \frac{\mu f_{4k} f_{2(4n+2)k}}{\left[g_{(4n+2)k}^2 + \mu\nu f_{2k}^2\right]^2} = \frac{1}{g_{4k}^2}. \qquad (20)$$

Proof. It follows by identities (1) and (2) that

$$\begin{aligned}
g_{(4n+4)k}^2 - g_{4nk}^2 &= \begin{cases} f_{4k}f_{2(4n+2)k}, & \text{if } g_n = f_n;\\ \Delta^2 f_{4k} f_{2(4n+2)k}, & \text{otherwise;}\end{cases}\\
g_{(4n+4)k}g_{4nk} - g_{(4n+2)k}^2 &= \begin{cases} -f_{2k}^2, & \text{if } g_n = f_n;\\ \Delta^2 f_{2k}^2, & \text{otherwise.}\end{cases}
\end{aligned}$$

Suppose $g_n = f_n$. Using these two identities and Lemma 3, we then get

$$\begin{aligned}\frac{f_{4k}f_{2(4n+2)k}}{\left[f^2_{(4n+2)k} - f^2_{2k}\right]^2} &= \frac{f^2_{(4n+4)k} - f^2_{4nk}}{f^2_{(4n+4)k}f^2_{4nk}};\\ \sum_{n=1}^{\infty}\frac{f_{4k}f_{2(4n+2)k}}{\left[f^2_{(4n+2)k} - f^2_{2k}\right]^2} &= \sum_{n=1}^{\infty}\left[\frac{1}{f^2_{4nk}} - \frac{1}{f^2_{(4n+4)k}}\right]\\ &= \frac{1}{f^2_{4k}}. \qquad (21)\end{aligned}$$

On the other hand, let $g_n = l_n$. Coupled the above two identities, Lemma 3 yields

$$\begin{aligned}\frac{\Delta^2 f_{4k}f_{2(4n+2)k}}{\left[l^2_{(4n+2)k} + \Delta^2 f^2_{2k}\right]^2} &= \frac{l^2_{(4n+4)k} - l^2_{4nk}}{l^2_{(4n+4)k}l^2_{4nk}};\\ \sum_{n=1}^{\infty}\frac{\Delta^2 f_{4k}f_{2(4n+2)k}}{\left[l^2_{(4n+2)k} + \Delta^2 f^2_{2k}\right]^2} &= \sum_{n=1}^{\infty}\left[\frac{1}{l^2_{4nk}} - \frac{1}{l^2_{(4n+4)k}}\right]\\ &= \frac{1}{l^2_{4k}}. \qquad (22)\end{aligned}$$

By combining equations (21) and (22), we get the desired result. □

It follows by this theorem that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{F_{2(4n+2)}}{(F^2_{4n+2} - 1)^2} &= \frac{1}{27}; & \sum_{n=1}^{\infty}\frac{F_{2(4n+2)}}{(L^2_{4n+2} + 5)^2} &= \frac{1}{735};\\ \sum_{n=1}^{\infty}\frac{F_{4(4n+2)}}{[F^2_{2(4n+2)} - 9]^2} &= \frac{1}{9,261}; & \sum_{n=1}^{\infty}\frac{F_{4(4n+2)}}{[L^2_{2(4n+2)} + 45]^2} &= \frac{1}{231,945};\\ \sum_{n=1}^{\infty}\frac{F_{6(4n+2)}}{[F^2_{3(4n+2)} - 64]^2} &= \frac{1}{2,985,984}; & \sum_{n=1}^{\infty}\frac{F_{6(4n+2)}}{[L^2_{3(4n+2)} + 320]^2} &= \frac{1}{74,652,480}.\end{aligned}$$

Finally, we explore some delightful byproducts of the theorems.

3.1. Gibonacci Delights.

Theorem 1, coupled with Theorem 2, yields:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{F_{2(2n+1)}}{(F^2_{2n+1} + 1)^2} &= \sum_{n=1}^{\infty}\frac{F_{2(4n-1)}}{(F^2_{4n-1} + 1)^2} + \sum_{n=1}^{\infty}\frac{F_{2(4n+1)}}{(F^2_{4n+1} + 1)^2}\\ &= \frac{1}{3} + \frac{1}{12}\\ &= \frac{5}{12}\ [5,\ 7];\\ \sum_{n=1}^{\infty}\frac{F_{4(2n+1)}}{[F^2_{2(2n+1)} - 9]^2} &= \sum_{n=1}^{\infty}\frac{F_{4(4n-1)}}{[F^2_{2(4n-1)} - 9]^2} + \sum_{n=1}^{\infty}\frac{F_{4(4n+1)}}{[F^2_{2(4n+1)} - 9]^2}\\ &= \frac{65}{1,344};\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(L_{2n+1}^2 - 5)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(4n-1)}}{(L_{4n-1}^2 - 5)^2} + \sum_{n=1}^{\infty} \frac{F_{2(4n+1)}}{(L_{4n+1}^2 - 5)^2} \\
&= \frac{17}{240} \text{ [6]}; \\
\sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{(L_{2(2n+1)}^2 + 45)^2} &= \sum_{n=1}^{\infty} \frac{F_{4(4n-1)}}{(L_{2(4n-1)}^2 + 45)^2} + \sum_{n=1}^{\infty} \frac{F_{4(4n+1)}}{(L_{2(4n+1)}^2 + 45)^2} \\
&= \frac{37}{34,020};
\end{aligned}$$

Likewise, Theorems 3 and 4 yield additional byproducts:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{F_{4n}}{(F_{2n}^2 - 1)^2} &= \sum_{n=1}^{\infty} \frac{F_{4(2n)}}{[F_{2(2n)}^2 - 1]^2} + \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[F_{2(2n+1)}^2 - 1]^2} \\
&= \frac{10}{27} \text{ [7]}; \\
\sum_{n=2}^{\infty} \frac{F_{8n}}{(F_{4n}^2 - 9)^2} &= \sum_{n=1}^{\infty} \frac{F_{8(2n)}}{[F_{4(2n)}^2 - 9]^2} + \sum_{n=1}^{\infty} \frac{F_{8(2n+1)}}{[F_{4(2n+1)}^2 - 9]^2} \\
&= \frac{50}{9,261}; \\
\sum_{n=2}^{\infty} \frac{F_{4n}}{(L_{2n}^2 + 5)^2} &= \sum_{n=1}^{\infty} \frac{F_{4(2n)}}{[L_{2(2n)}^2 + 5]^2} + \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[L_{2(2n+1)}^2 + 5]^2} \\
&= \frac{58}{6,615} \text{ [7]}; \\
\sum_{n=1}^{\infty} \frac{F_{4n}}{(L_{2n}^2 + 5)^2} &= \frac{13}{540}; \\
\sum_{n=2}^{\infty} \frac{F_{8n}}{(L_{4n}^2 + 45)^2} &= \sum_{n=1}^{\infty} \frac{F_{8(2n)}}{[L_{4(2n)}^2 + 45]^2} + \sum_{n=1}^{\infty} \frac{F_{8(2n+1)}}{[L_{4(2n+1)}^2 + 45]^2} \\
&= \frac{2,258}{11,365,305}; \\
\sum_{n=1}^{\infty} \frac{F_{8n}}{(L_{4n}^2 + 45)^2} &= \frac{117,077}{45,461,220}.
\end{aligned}$$

4. Pell, Chebyshev, and Vieta Implications

Finally, Pell polynomials $b_n(x)$, Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $b_n(x) = g_n(2x)$, $V_n(x) = i^{n-1}f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$, where $i = \sqrt{-1}$ [2, 3, 4]. They can be employed to find the Pell, Chebyshev, and Vieta versions of the theorems. In the interest of brevity, we omit them; but we encourage gibonacci enthusiasts to pursue them.

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